GENERALIZED JACOBI WEIGHTS, CHRISTOFFEL FUNCTIONS, AND JACOBI POLYNOMIALS

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Dedicated to Dick Askey on the occasion of his 60^{th} *birthday*

Abstract. We obtain such upper bounds for Jacobi polynomials which are uniform in all the parameters involved and which contain explicit constants. This is done by a combination of some results on generalized Christoffel functions and some estimates of Jacobi polynomials in terms of Christoffel functions.

§1. Introduction

Orthogonal Polynomials. Given $w(\geq 0) \in L^1(\mathbb{R})$, $p_n(w)$ denotes the corresponding orthonormal polynomial of precise degree n with leading coefficient $\gamma_n(w) > 0$.

Jacobi Weights and Jacobi Polynomials. Given $\alpha > -1$ and $\beta > -1$, w is called a Jacobi weight if $\operatorname{supp}(w) = [-1, 1]$ and $w(x) = (1-x)^\alpha(1+x)^\beta$ for $x \in [-1, 1]$. The corresponding orthogonal polynomials (for historical reasons with various normalizations) are called Jacobi polynomials.

For a wide class of orthogonal polynomials associated with weight functions supported in $[-1, 1]$, the expression $\sqrt{\sqrt{1-x^2}w(x)}p_n(w, x)$ asymptotically equioscillates between $\pm\sqrt{\frac{2}{\pi}}$ for $x \in (-1, 1)$ when n tends to ∞ (cf. [22, Chapters VIII and X–XII]). Therefore, it is natural to seek inequalities for $\sqrt{1-x^2}w(x)p_n^2(w, x)$ for $x \in [-1, 1]$.

Such inequalities for Jacobi polynomials involving with optimal constants are truly fascinating. They are easy to prove for the first and second (and third and fourth) kind

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Chebysev polynomials since they are related to simple trigonometric functions. For Legendre polynomials this is somewhat more complicated, and the appropriate inequality was proved by S. Bernstein (cf. [22, (7.3.8), p. 165] for Bernstein's result and [1, 14] for a sharper version of it). Bernstein's results can be extended to Jacobi (i.e., ultraspherical or Gegenbauer) polynomials with parameters $-\frac{1}{2} < \alpha = \beta < \frac{1}{2}$ (cf. [22, (7.33.4) and (7.33.5), p. 171] and also [15] for a refinement). In addition, for a wider range of the parameters, similar inequalities have been proved in [13] ($\alpha = \beta > -\frac{1}{2}$) and [7] ($\alpha = \beta > \frac{1}{2}$).

For instance, L. Lorch [15, formula (10), p. 115] proved[1]

$$\max_{x\in[-1,1]} \left|(1-x^2)^{\frac{\lambda}{2}} P_n^{(\lambda)}(x)\right| \le \frac{2^{1-\lambda}(n+\lambda)^{\lambda-1}}{\Gamma(\lambda)}$$

for $n = 0, 1, \dots$ and $0 < \lambda < 1$, which, in terms of the orthonormal Jacobi polynomials, can be stated as

$$\max_{x\in[-1,1]} \left|\sqrt{\sqrt{1-x^2}w(x)}\, p_n(w,x)\right| \le \sqrt{\frac{2\,\Gamma(n+1)}{\pi\,\Gamma(n+2\alpha+1)}} \left(n+\alpha+\frac{1}{2}\right)^{\alpha}$$

for $n = 0, 1, \dots,$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$ where $w(x) = (1-x^2)^\alpha$.

For nonsymmetric Jacobi weights much less is known. In 1988, L. Gatteschi [10] extended Bernstein's results to Jacobi polynomials with $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$. For instance, he proved that if $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ and $\alpha + \beta > 0$ then[2]

$$\max_{\theta\in\left[0,\frac{\pi}{2}\right]} \left|(\sin\theta/2)^{\alpha+\frac{1}{2}}\,(\cos\theta/2)^{\beta+\frac{1}{2}}\, P_n^{(\alpha,\beta)}(\cos\theta)\right| \le \frac{\Gamma(\beta+1)}{\Gamma\left(\frac{1}{2}\right)\left(n+\frac{\alpha+\beta+1}{2}\right)^{\beta+\frac{1}{2}}} \binom{n+\beta}{n}$$

for $n = 0, 1, \dots$.[3] Again, in terms of the the orthonormal Jacobi polynomials, this can be stated as

$$\max_{x\in[0,1]} \left|\sqrt{\sqrt{1-x^2}w(x)}\, p_n(w,x)\right| \le \sqrt{\frac{2^{2\beta+1}\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)}{\pi\Gamma(n+1)\Gamma(n+\alpha+1)}}\,(2n+\alpha+\beta+1)^{-\beta}$$

for $n = 0, 1, \dots,$ where $w(x) = (1-x)^\alpha(1+x)^\beta$ with $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ and $\alpha + \beta > 0$.

In a sense our goal is less ambitious than the previously mentioned inequalities in that we do not expect to be able to obtain sharp constants with our techniques. On the other hand, our techniques enable us to extend these Jacobi polynomial inequalities with very explicit constants for all parameters $\alpha \ge -\frac{1}{2}$ and $\beta \ge -\frac{1}{2}$.

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[1] Here $P_n^{(\lambda)}$ is the standard normalization of the Gegenbauer polynomials, that is, $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$ and Γ denotes the gamma function.

[2] Here $P_n^{(\alpha,\beta)}$ is the standard normalization of the Jacobi polynomials, that is, $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ and Γ again denotes the gamma function.

[3] For $\theta \in \left[\frac{\pi}{2}, \pi\right]$, one can use $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ to obtain an analogous inequality.

Generalized Polynomials. The function f given by

$$f(z) \stackrel{\text{def}}{=} |\omega| \prod_{j=1}^{k} |z - z_j|^{r_j}, \qquad \omega \neq 0,\ z_j \in \mathbb{C},\ z \in \mathbb{C},\ r_j > 0,$$

is called a *generalized (non–negative) algebraic polynomial* of (*generalized*) degree

$$N \stackrel{\text{def}}{=} \sum_{j=1}^{k} r_j\ ,$$

and we will write $f \in |\,\mathrm{GCAP}\,|_N$.

If $w(x) = (1-x)^\alpha(1+x)^\beta$ is a Jacobi weight then $\sqrt{1-x^2}w(x)p_n^2(w,x)$ is a generalized polynomial (of degree $2n+\alpha+\beta+1$), and as such the framework of generalized polynomials is (one of) the perfect setting for studying Jacobi polynomials. As a matter of fact, this was the primary reason for introducing generalized polynomials in the first place (cf. [6, 5]).

This paper is a modest attempt to demonstrate the applicability of generalized polynomials to problems which have not yet been settled in a satisfactory way despite more than a hundred years of undiminished interest in them.

Our method consists of two steps. First, in §2, we use *generalized polynomials* to estimate the *Christoffel function* $\sum_{k=0}^n p_k^2(w)$, and then, in §3, we obtain a Riccati equation which yields estimates for the ratio $\frac{p_n^2(w)}{\sum_{k=0}^n p_k^2(w)}$. The reason that we have to limit ourselves to considering $\alpha \geq -\frac{1}{2}$ and $\beta \geq -\frac{1}{2}$ is that the function $\sqrt{1-x^2}w(x)$ for either $\alpha < -\frac{1}{2}$ or $\beta < -\frac{1}{2}$ is no longer a generalized polynomial.

Our main result is the following

Theorem 1. *For all Jacobi weight functions $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha \geq -\frac{1}{2}$ and $\beta \geq -\frac{1}{2}$, the inequalities*

$$\max_{x\in[-1,1]} \frac{p_n^2(w,x)}{\sum_{k=0}^n p_k^2(w,x)} \leq \frac{4\left(2+\sqrt{\alpha^2+\beta^2}\right)}{2n+\alpha+\beta+2} \tag{1}$$

and

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)p_n^2(w,x) \leq \frac{2e\left(2+\sqrt{\alpha^2+\beta^2}\right)}{\pi} \tag{2}$$

hold for $n = 0, 1, \ldots$.

Our method is therefore able to give $O((\alpha^2+\beta^2)^{1/2})$ estimates for large $\alpha^2+\beta^2$. It is natural to ask how a sharp bound should behave. Numerically computed examples of the actual maximum of $\sqrt{1-x^2}w(x)p_n^2(w,x)$ suggest that small values of n give relevant information. For instance, with $\alpha = 10$ and $\beta = 2$,

n	0	1	2	3	4	5	6	7	8	9	10
max	1.478	1.251	1.191	1.161	0.845	0.747	1.123	0.727	1.112	0.703	0.685

For $n = 0$, explicit calculation yields

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)p_0^2(w,x) =$$

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \max_{x\in[-1,1]} (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2},$$

that is,[4]

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)p_0^2(w,x) = \frac{(\alpha+1/2)^{\alpha+1/2}(\beta+1/2)^{\beta+1/2}\Gamma(\alpha+\beta+2)}{(\alpha+\beta+1)^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$$

which behaves like $[(\alpha+\beta)/(2\pi)]^{1/2}$ for α and β large. We expect a $O((\alpha^2+\beta^2)^{1/4})$ bound to be valid for all $n \geq 1$.

§2. Generalized Christoffel Functions and Generalized Polynomials

Generalized Christoffel Functions. Given $w(\geq 0) \in L^1(\mathbb{R})$ and $p \in (0,\infty)$,

$$\lambda_n^*(w,p,z) \stackrel{\text{def}}{=} \inf_{f\in|\,\mathrm{GCAP}\,|_{n-1}} \int_{\mathbb{R}} \frac{f^p(t)}{f^p(z)}\, w(t)\, dt, \qquad z \in \mathbb{C}, \tag{3}$$

where $n \geq 1$ is *real*, that is, n is not necessarily an integer.

Remark 2. Of course, $\lambda_n^*(w,p) \equiv \lambda_{np-p+1}^*(w,1)$. As a matter of fact, this is one of the underlying reasons for the usefulness of the concept of *generalized polynomials*. The notation $\lambda_n^*(w,p)$ was kept for historical reasons. Eventually, the parameter p may disappear from it.

Generalized Jacobi Weights. Given a non–negative integer m, the function w satisfying $\mathrm{supp}(w) = [-1,1]$ and

$$w(x) \stackrel{\text{def}}{=} (1-x)^{\tau_0} \prod_{k=1}^{m} |x-a_k|^{\tau_k}(1+x)^{\tau_{m+1}}, \qquad a_k \in \mathbb{R}, \quad \tau_k \in \mathbb{R}, \tag{4}$$

for $x \in [-1,1]$, is called a *generalized Jacobi weight*, and its *degree* is denoted by $\deg(w) \stackrel{\text{def}}{=} \sum_{k=0}^{m+1} \tau_k$.

We start with

Theorem 3. *Let w be a generalized Jacobi weight of the form* (4) *such that $a_k \neq a_j$ for $k \neq j$, $a_k \neq \pm 1$ for $k = 1,2,\ldots,m$, $\tau_0 \geq -\frac{1}{2}$, $\tau_k > 0$ for $k = 1,2,\ldots,m$, and $\tau_{m+1} \geq -\frac{1}{2}$. Then, for all $0 < p < \infty$ and $n \geq 1$, the generalized Christoffel functions $\lambda_n^*(w,p)$ satisfy the inequality*

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)\left[\lambda_n^*(w,p,x)\right]^{-1} \leq \frac{(2+pn-p+\deg(w))\,e}{2\pi}\,.$$

[4] The maximum is taken at $x = \frac{\beta-\alpha}{\alpha+\beta+1}$.

Since the reciprocal of $\sum_{k=0}^{n-1} p_k^2(w,z)$ equals the right–hand side of (3) with the infimum (that is, *minimum*) taken for all ordinary polynomials of degree at most $n-1$,

$$\sum_{k=0}^{n} p_k^2(w,x) \le \left[\lambda_{n+1}^*(w,2,x)\right]^{-1}, \qquad n=0,1,\dots,$$

and, thus, we have

Corollary 4. *For all Jacobi weights $w(x)=(1-x)^\alpha(1+x)^\beta$ with $\alpha \ge -\frac{1}{2}$ and $\beta \ge -\frac{1}{2}$,*

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)\sum_{k=0}^{n} p_k^2(w,x) \le \frac{(2n+\alpha+\beta+2)\,e}{2\pi}, \qquad n=0,1,\dots, \tag{5}$$

holds.

Remark 5. We point out the uniformity of (5) in all parameters.

Remark 6. The corresponding lower estimates are essentially the same with a proper interpretation of the word "essentially" (cf. [6, Theorems 2.1 and 2.2, p. 113]).

Remark 7. Of course, given $\epsilon > 0$, for all Jacobi weights we have

$$\lim_{n\to\infty} \frac{\sqrt{1-x^2}w(x)\sum_{k=0}^{n} p_k^2(w,x)}{n} = \frac{1}{\pi}$$

uniformly for $-1+\epsilon \le x \le 1-\epsilon$ (cf. [19, Theorem 6.2.35, p. 94]).

Question 8. It remains to be seen how to extend (5) for all Jacobi weights, with parameters $\alpha > -1$ and $\beta > -1$.

Proof of Theorem 3. We start out as in the proof of [16, Theorem 6, p. 149], and we closely follow the proof of [6, Theorem 3.2, p. 126]. If h is analytic in the unit disk then

$$(1-|rz|^2)h(rz) = \frac{1}{2\pi i}\int_{|u|=1} h(u)\frac{1-r\bar{z}u}{u-rz}du, \qquad |z|\le 1, \quad 0\le r<1.$$

Hence, if P is a polynomial and $0<p<\infty$ then

$$(1-|r|^2)|P^*(rz)|^p \le \frac{1}{2\pi}\int_{|u|=1} |P^*(u)|^p|du|, \qquad |z|=1, \quad 0\le r\le 1,$$

where P^* is obtained from P by replacing all the zeros z^* of P which are inside the unit disk by $\bar{z^*}^{-1}$.

Since

$$\frac{1+r}{2}|z-\sigma| \le |rz-\sigma|, \qquad |\sigma|\ge 1, \quad |z|=1, \quad 0\le r\le 1,$$

we have

$$(1-|r|^2)\left(\frac{1+r}{2}\right)^{p\deg(P)} |P^*(z)|^p \le \frac{1}{2\pi}\int_{|u|=1} |P^*(u)|^p|du|, \qquad |z|=1, \quad 0\le r\le 1.$$

Maximizing the left–hand side here for $0 \le r \le 1$ and using $|P^*(z)| = |P(z)|$ for $|z| = 1$, the inequality

$$|P(z)|^p \le \frac{(2 + p\deg(P))\, e}{8\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^p \, d\theta, \qquad |z| = 1,$$

follows.

For every real trigonometric polynomial R_n of degree at most n there is an algebraic polynomial $P_{2n} \in \Pi_{2n}$ such that $R_n^2(\theta) = |P_{2n}(e^{i\theta})|^2$. Therefore,

$$\|R_n\|_{L^\infty(\mathbb{R})}^p \le \frac{(1 + pn)\, e}{4\pi} \int_{-\pi}^{\pi} |R_n(\theta)|^p \, d\theta \tag{6}$$

for every such trigonometric polynomial R_n.

If the multiplicity of each zero of $g \in |\,\mathrm{GCAP}\,|_N$ is rational, then there is $q > 0$ such that $g^q(\cos\cdot)$ is a non–negative trigonometric polynomial so that applying (6) with $R_{Nq} = g^q$ and $p = \frac{1}{q}$ yields

$$\|g(\cos\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{(1 + N)\, e}{4\pi} \int_{-\pi}^{\pi} g(\cos\theta)\, d\theta. \tag{7}$$

Once (7) holds for all $g \in |\,\mathrm{GCAP}\,|_N$ such that the multiplicity of each zero of g is rational, by continuity it remains valid for all $g \in |\,\mathrm{GCAP}\,|_N$. Hence,

$$\|G(\cos\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{(1 + N)\, e}{4\pi} \int_{-\pi}^{\pi} G(\cos\theta)\, d\theta, \qquad \forall\, G \in |\,\mathrm{GCAP}\,|_N, \tag{8}$$

for every $N \ge 0$.

Thus,

$$\left\| \sqrt{1 - (\cdot)^2} F \right\|_{L^\infty([-1,1])} \le \frac{(2 + N)\, e}{2\pi} \int_{-1}^{1} F(t)\, dt, \qquad \forall\, \sqrt{1 - (\cdot)^2} F \in |\,\mathrm{GCAP}\,|_{N+1},$$

$N \ge 0$. Applying this inequality with $F = f^p w$, Theorem 3 follows immediately. $\square$

§3. Christoffel Functions and Jacobi Polynomials

I've tried A! I've tried B! I've tried C!
Tom Wolfe, *The Right Stuff*

If we want to find upper bounds for p_n^2 from upper bounds for $\sum_0^n p_k^2$, then we must have upper bounds for $p_n^2 / \sum_0^n p_k^2$, and that is precisely what is attempted here.

Theorem 9. *Given $n = 1, 2, \dots,$ and a Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha > -1$ and $\beta > -1$, let $x_{nn}(w) < x_{1n}(w)$ be the extreme zeros of the corresponding n^{th}–degree Jacobi polynomial. Then the inequalities*

$$\frac{p_n^2(w,x)}{\sum_{k=0}^n p_k^2(w,x)} \le \begin{cases} \frac{(2n+\alpha+\beta+1)(\beta+1)}{(n+\alpha+\beta+1)(n+\beta+1)}\,, & -1 \le x \le 2x_{nn}(w)+1\,, \\ \frac{4(2n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}}\,, & \xi_1 \le x \le \xi_2\,, \\ \frac{(2n+\alpha+\beta+1)(\alpha+1)}{(n+\alpha+\beta+1)(n+\alpha+1)}\,, & 2x_{1n}(w)-1 \le x \le 1\,, \end{cases} \tag{9}$$

hold provided $-1 < \xi_1 \le \xi_2 < 1$ are such that $(2n+\alpha+\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}$ on the right–hand side of the second inequality of (9) is positive in $[\xi_1, \xi_2]$.

Remark 10. The inequality

$$\frac{p_n^2(w,x)}{\sum_{k=0}^n p_k^2(w,x)} \le \frac{\mathrm{const}}{n}\,, \qquad -1 \le x \le 1,$$

is well–known [19, Lemma 6.2.17, p. 82] (see [18, Lemma 2.1, p. 336] for the necessary Christoffel function estimates) but its proof is rather cumbersome. Theorem 1 of the present paper yields a new proof with an explicit formula for the constant which depends on α and β.

Proof of Theorem 9. By Christoffel–Darboux's formula

$$\begin{aligned}\sum_{k=0}^n p_k^2(w,x) &= \frac{\gamma_n(w)}{\gamma_{n+1}(w)}\left[p'_{n+1}(w,x)p_n(w,x) - p'_n(w,x)p_{n+1}(w,x)\right] = \\ &= \frac{\gamma_n(w)}{\gamma_{n+1}(w)} p_n^2(w,x)\frac{d}{dx}\left[\frac{p_{n+1}(w,x)}{p_n(w,x)}\right].\end{aligned}$$

Hence, we need to find an appropriate lower bound for r' in $[-1,1]$, where

$$r(x) = \frac{\gamma_n(w)}{\gamma_{n+1}(w)}\,\frac{p_{n+1}(w,x)}{p_n(w,x)}\,.$$

Here r is a rational function with simple poles at the zeros $\{x_{kn}(w)\}_{k=1}^n$ of $p_n(w)$. It has an asymptotic behavior $x+c$ for $x \to \infty$, where c is a constant. Since r' is positive everywhere, r must have negative residues at its poles, so that we obtain

$$r(x) = x + c - \sum_{k=1}^n \frac{A_k}{x - x_{kn}(w)} \qquad \text{and} \qquad r'(x) = 1 + \sum_{k=1}^n \frac{A_k}{(x - x_{kn}(w))^2}$$

with $A_k > 0$, $k = 1, 2, \dots, n$. See the graphs of r and r' (solid thick line) on upper and lower parts of Figure 1 as an example. When $-1 \le x < x_{nn}(w)$, r' is the sum of

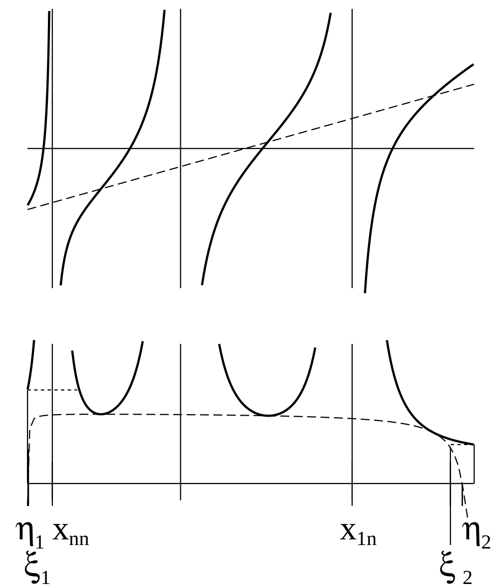

Figure 1

increasing functions of x, and, therefore it is greater than $r'(-1)$. When x is slightly greater than $x_{1n}(w)$, r' is decreasing but it is still greater or equal than $r'(-1)$ as long as each term $\frac{A_k}{(x-x_{kn}(w))^2}$ is greater or equal than the corresponding term at -1, that is, $(x - x_{kn}(w))^2 \le (-1 - x_{kn}(w))^2$ for $k = 1, \dots, n$, or $(x+1)(x - 2x_{kn}(w) - 1) \le 0$ which holds if $-1 \le x \le 2\min x_{kn}(w) + 1 = 2x_{nn}(w) + 1$. A similar argument shows that $r'(x) \ge r'(1)$ if $x \ge 2\max x_{kn}(w) - 1 = 2x_{1n}(w) - 1$. This will prove the first and third inequalities of (9) as soon as we get the actual values of $r'(-1)$ and $r'(1)$.

The bottom part of Figure 1 shows a graph of r' for $-1 \le x \le 1$ when $\alpha = 3/2$, $\beta = -3/10$, and $n = 3$. A short–dashed horizontal line has been drawn between -1 and $2x_{nn}(w) + 1$ at the ordinate $r'(-1)$. One can see that this horizontal line segment lies indeed under the graph of r'. As $r'(1)$ is a lower bound of r' on the whole interval $[-1, 1]$ in the present case, only a part of the horizontal line of the ordinate $r'(1)$ has been drawn. Other features of this figure will be explained later.

In order to establish the second inequality of (9) and to compute $r'(\pm 1)$, we need the following formulas for the orthonormal Jacobi polynomials.
(i) The recurrence relation:

$$\frac{\gamma_n(w)}{\gamma_{n+1}(w)} p_{n+1}(w,x) = (x - b_n(w)) p_n(w,x) - \frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_{n-1}(w,x), \tag{10}$$

where

$$b_n(w) = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \,,$$

and

$$\frac{\gamma_n(w)}{\gamma_{n+1}(w)} = 2 \times \sqrt{\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}} \,,$$

$n = 1, 2, \ldots$ (cf. [22, formula (4.5.1), p. 71] or [3, Table III.11, p. 220]).
(ii) The differential relation:

$$\begin{aligned}(1-x^2)p_n'(w,x) = \\ & n\left(\frac{\alpha-\beta}{2n+\alpha+\beta}-x\right)p_n(w,x) + (2n+\alpha+\beta+1)\frac{\gamma_{n-1}(w)}{\gamma_n(w)}p_{n-1}(w,x) = \\ & (n+\alpha+\beta+1)\left(\frac{\alpha-\beta}{2n+\alpha+\beta+2}+x\right)p_n(w,x) - \\ & (2n+\alpha+\beta+1)\frac{\gamma_n(w)}{\gamma_{n+1}(w)}p_{n+1}(w,x),\end{aligned} \tag{11}$$

$n = 1, 2, \ldots$ (cf. [22, formula (4.5.7), p. 72]). Of course, each one of these formulas can be deduced from the other one by the three–term recurrence formula.

Combination of (i) and (ii) yields
(iii) The differential equation:

$$(1-x^2)p_n''(w,x) + [\beta-\alpha-(\alpha+\beta+2)x]p_n'(w,x) + n(n+\alpha+\beta+1)p_n(w,x) = 0 \tag{12}$$

(cf. [22, formula (4.2.1), p. 60] or [3, formula (2.20), p. 149]).

In order to compute $r'(\pm 1)$, we proceed as follows. From (11),

$$\frac{\gamma_n(w)}{\gamma_{n+1}(w)}\frac{p_{n+1}(w,\pm 1)}{p_n(w,\pm 1)} = r(\pm 1) = \frac{(n+\alpha+\beta+1)[\alpha-\beta\pm(2n+\alpha+\beta+2)]}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

and from (12),

$$\frac{p_n'(w,\pm 1)}{p_n(w,\pm 1)} = \frac{n(n+\alpha+\beta+1)}{\pm(\alpha+\beta+2)+\alpha-\beta},$$

so that we have

$$r'(\pm 1) = \frac{\gamma_n}{\gamma_{n+1}}\frac{p_{n+1}(w,\pm 1)}{p_n(w,\pm 1)}\left(\frac{p_{n+1}'(w,\pm 1)}{p_{n+1}(w,\pm 1)} - \frac{p_n'(w,\pm 1)}{p_n(w,\pm 1)}\right),$$

which allows the computation of the requested special values as given in the following table.

$\boxed{f}$	$\boxed{f(-1)}$	$\boxed{f(1)}$
r	$-\frac{2(n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$	$\frac{2(n+\alpha+\beta+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$
p_n'/p_n	$-\frac{n(n+\alpha+\beta+1)}{2(\beta+1)}$	$\frac{n(n+\alpha+\beta+1)}{2(\alpha+1)}$
r'	$\frac{(n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(\beta+1)}$	$\frac{(n+\alpha+\beta+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(\alpha+1)}$

(13)

The values of $1/r'(-1)$ and $1/r'(1)$ are used in the right–hand sides of the first and third inequalities of (9).

Now, we come to the second inequality. This one will be established through a *Riccati* equation for r. Use the differential relations (11) for eliminating p'_n and p'_{n+1} in $p'_{n+1}p_n - p'_n p_{n+1}$. This gives an equation in terms of p^2_{n+1}, $p_n p_{n+1}$, and p^2_n, so that after some rather tedious calculations,

$$r'(x) = \frac{A + B(x)r(x) + Cr(x)^2}{1-x^2}$$

with

$$\begin{aligned} A &= (2n+\alpha+\beta+3)\left(\frac{\gamma_n(w)}{\gamma_{n+1}(w)}\right)^2 \\ &= 4\,\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2}\,, \\ B(x) &= -(2n+\alpha+\beta+2)x - \frac{\alpha^2-\beta^2}{2n+\alpha+\beta+2}\,, \qquad C = 2n+\alpha+\beta+1. \end{aligned}$$

The idea is that whatever the actual value of r is, $A + Br + Cr^2$ will always be greater than the absolute minimum of this trinomial, that is,

$$r'(x) \ge \frac{4AC - B(x)^2}{4C(1-x^2)}\,.$$

Equality will occur whenever $r(x)$ is equal to $-B(x)/(2C)$ which happens once between any pair of consecutive zeros of p_n, as can be seen in Figure 1, where the graphs of r and $-B/(2C)$ are shown in the upper part, and the graphs of r' and its lower bound (dashed line) in the lower part. Working out the numerator yields

$$r'(x) \ge \frac{(2n+\alpha+\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}}{4(2n+\alpha+\beta+1)}\,, \qquad -1 < x < 1,$$

and, thus, the theorem follows, as long as x is restricted to an interval $[\xi_1, \xi_2]$ where the above lower bound is positive. $\square$

Combining Theorem 3 (that is, Corollary 4) and Theorem 9, we obtain the following pointwise estimate for the Jacobi polynomials.

Theorem 11. *For all Jacobi weight functions $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha \ge -\frac{1}{2}$ and $\beta \ge -\frac{1}{2}$, we have*

$$p^2_n(w,x) \le \frac{2}{\pi\sqrt{1-x^2}w(x)}\,\frac{e\,(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}}\,, \qquad n = 1, 2, \ldots,$$

for $-1 < x < 1$, as long as the denominator $(2n+\alpha+\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}$ on the right–hand side is positive. In particular, given $0 < \epsilon < 1$,

$$p^2_n(w,x) \le \frac{2}{\pi\sqrt{1-x^2}w(x)}\,\frac{e}{1 - \frac{2(\alpha^2+\beta^2)}{(2n+\alpha+\beta+2)^2\,\epsilon}}\,,$$

for $-1+\epsilon \le x \le 1-\epsilon$ *and* $n > \sqrt{2(\alpha^2+\beta^2)/\epsilon} - (\alpha+\beta)/2 - 1$.

For fixed $x \in (-1,1)$ and $n \to \infty$, this is no more than $e \times (1+o(1))$ times worse than an optimal inequality could be. However, when x is close to ± 1, the parameter n needs to be sufficiently large so that the estimate would become useful. The quest for estimates valid for every $n > 0$ is the subject of the following investigations.

First, we deal with the first and third inequalities in (9).

Lemma 12. *When* $\alpha \ge -1/2$, $\beta \ge -1/2$, *and* $n > 0$, *we have*

$$\max\left[\frac{(2n+\alpha+\beta+1)(\beta+1)}{(n+\alpha+\beta+1)(n+\beta+1)}\,,\ \frac{(2n+\alpha+\beta+1)(\alpha+1)}{(n+\alpha+\beta+1)(n+\alpha+1)}\right] \le \frac{4(1+\max(\alpha,\beta))}{2n+\alpha+\beta+2}\,. \tag{14}$$

This is the first instance showing how the right–hand sides of (9) behaves. The factor $2n+\alpha+\beta+2$ has been chosen because it will reappear when (5) is used.

Proof of Lemma 12. First of all, since $\frac{t}{n+t}$ is an increasing function for $t > -n$, we only have to consider

$$\frac{(2n+\alpha+\beta+1)(\gamma+1)}{(n+\alpha+\beta+1)(n+\gamma+1)}\,,$$

where $\gamma = \max(\alpha,\beta)$. Let $\delta = \min(\alpha,\beta)$. Then, we have to show

$$(2n+\gamma+\delta+2)(2n+\gamma+\delta+1) \le 4(n+\gamma+1)(n+\gamma+\delta+1),$$

when $\gamma \ge \delta \ge -1/2$ and $n > 0$. This is quite elementary and amounts to

$$2n(2\gamma+1) + (\gamma+\delta+1)(3\gamma-\delta+2) \ge 0,$$

which holds since $\gamma \ge \delta$ and $2\gamma+1 \ge 0$. $\square$

In order to use the second inequality of (9), we must find a valid interval $[\xi_1,\xi_2]$ containing $[2x_{nn}(w)+1, 2x_{1n}(w)-1]$, so that then we would have upper bounds of $p_n^2(w)/\sum_0^n p_k^2(w)$ in the whole interval $[-1,1]$. Since no simple formulas for $x_{nn}(w)$ and $x_{1n}(w)$ are known, we will now find a lower bound η_1 for $x_{nn}(w)$ large enough for allowing $\xi_1 = 2\eta_1+1$ to be a valid choice in (9), and, similarly, a sufficiently small upper bound η_2 for $x_{1n}(w)$.

There is much literature on bounds for the zeros of Jacobi polynomials (see e.g., [22, Sections 6.2 and 6.21, p. 116–123]), but most are useful only when α and β are between $-1/2$ and $1/2$. For large n, the extreme zeros behave like $-1+j_\beta^2/(2n^2)$ and $1-j_\alpha^2/(2n^2)$, where j_κ denotes the smallest positive zero of the Bessel function J_κ [22, § 8.1, p. 192].

The next theorem gives reasonably satisfactory lower and upper estimates for the zeros of the Jacobi polynomials.

Theorem 13. *Given $n = 1, 2, \ldots,$ the zeros $\{x_{kn}(w)\}_{k=1}^{n}$ of the n^{th}–degree Jacobi polynomial corresponding to a Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha \ge -1/2$ and $\beta \ge -1/2$ satisfy*

$$\eta_1 = -1 + \frac{2\beta^2}{N^2} \le x_{kn}(w) \le \eta_2 = 1 - \frac{2\alpha^2}{N^2}\,, \tag{15}$$

$k = 1, 2, \ldots, n$, where $N = 2n + \alpha + \beta + 1$.

The proof of this theorem requires the following lemma on oscillations of solutions of differential equations.

Lemma 14. *Let Y, Z, Y', Z', Y'', Z'', K, and L be continuous functions in the open interval (a, b), with $Y \not\equiv 0$, such that*

$$Y''(x) + K(x)Y(x) = 0, \qquad Z''(x) + L(x)Z(x) = 0, \qquad x \in (a, b).$$

If

$$\begin{array}{lll} \text{i)} & K(x) \le L(x), & x \in (a, b), \\ \text{ii)} & Y'(x)Z(x) - Y(x)Z'(x) \to 0, & x \to x_0, \end{array}$$

where x_0 is one of the endpoints of (a, b), and if $Z(x)$ has no zero in (a, b), then Y has no zero in (a, b) either.

Proof of Lemma 14. The lemma is a variant of the Sturmian comparison theorems for solutions of second order linear differential equations. It is almost the same as "*Szegő's comparison theorem*" in #16.626 of the 1980 edition of Gradshteyn and Ryzhik's book [11], coming from Theorem 1.82.1 of Szegő [22 Section 1.82 p.19], known as "Sturm's theorem for open intervals", see also the introduction of [9].[5] Here is a self–contained proof.

Suppose that $Y(x_1) = 0$ for some $x_1 \in (a, b)$. Since the equations are homogeneous, we may assume that $Z(x) > 0$ on (a, b), and $(x_0 - x_1)Y'(x_1) > 0$.[6] Therefore, $Y(x) = \int_{x_1}^{x} Y'(t)dt > 0$ when x is between x_1 and x_0 and it is sufficiently close to x_1, and

$$Y'(x)Z(x) - Y(x)Z'(x) = Y'(x_1)Z(x_1) + \int_{x_1}^{x} [L(t) - K(t)]Y(t)Z(t)\,dt$$

keeps the sign of $x_0 - x_1$ with an increasing absolute value when x varies from x_1 to x_0 (since $Y(x) = Z(x)\int_{x_1}^{x}(Z(t))^{-2}[Y'(t)Z(t) - Y(t)Z'(t)]dt$ keeps its positive sign), and it cannot vanish when $x \to x_0$. $\square$

Proof of Theorem 13. First, one uses the fact that if all the zeros of a polynomial p_n are real and are contained in an interval (a, b), a smaller interval containing all the zeros is $a - p_n(a)/p_n'(a), b - p_n(b)/p_n'(b)$. This is a well known theorem of the numerical analysis of

[5] We thank our dear friend Luigi Gatteschi for drawing our attention to [9].

[6] N.B. $Y'(x_1)$ must be different from zero, otherwise, $Y \equiv 0$.

the Newton–Raphson iteration method (see, for instance, [21, Chapter 9, p. 55]). Hence, by (13),

$$-1+\frac{2(\beta+1)}{n(n+\alpha+\beta+1)} \le x_{kn}(w) \le 1-\frac{2(\alpha+1)}{n(n+\alpha+\beta+1)}\ , \tag{16}$$

$k=1,2,\ldots,n$, is valid for every $\alpha>-1$, $\beta>-1$, and $n\ge 1$ (and is *exact* if $n=1$). However, considering only the upper bound, it behaves like $1-2(\alpha+1)/n^2$ for large n, instead of $1-j_\alpha^2/(2n^2)$, and j_α behaves like $\alpha+(1.855757\ldots)\alpha^{1/3}$ for large α (Tricomi's formula, see [4, p. 60]). Thus, we need better estimates when either n, α, or β are large.

Since $Y(x)=(1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}p_n(w,x)$ is a solution of $Y''+KY=0$ with

$$K(x)=\frac{1-\alpha^2}{4(1-x)^2}+\frac{1-\beta^2}{4(1+x)^2}+\frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-x^2)}\ ,$$

(cf. [22, formula (4.24.1), p. 67]), we take $Z(x)=(1-x)^{(\widetilde\alpha+1)/2}(1+x)^{(\widetilde\beta+1)/2}$ (cf. Lemma 14), so that

$$L(x)=\frac{1-\widetilde\alpha^2}{4(1-x)^2}+\frac{1-\widetilde\beta^2}{4(1+x)^2}+\frac{(\widetilde\alpha+1)(\widetilde\beta+1)}{2(1-x^2)}\ ,$$

and

$$L(x)-K(x)=$$
$$\frac{\alpha^2-\widetilde\alpha^2}{4(1-x)^2}+\frac{\beta^2-\widetilde\beta^2}{4(1+x)^2}+\frac{(\widetilde\alpha+1)(\widetilde\beta+1)-(\alpha+1)(\beta+1)-2n(n+\alpha+\beta+1)}{2(1-x^2)}\ .$$

Now we turn to the upper bound in (15). Since $L-K$ must be positive in a neighborhood of 1, $\widetilde\alpha^2<\alpha^2$, and since $Y'Z-YZ'$ behaves like $(1-x)^{(\alpha+\widetilde\alpha)/2}$ near 1, one must have $\widetilde\alpha>-\alpha$. This implies that the method will work only when $\alpha>0$. Let us choose $\widetilde\alpha=0$ and $\widetilde\beta=\beta$ so that

$$L(x)-K(x)=\frac{1}{4(1-x^2)}\left[\alpha^2\frac{1+x}{1-x}-2\alpha(\beta+1)-4n(n+\alpha+\beta+1)\right],$$

which is positive between $1-\frac{2\alpha^2}{(2n+\alpha)(2n+\alpha+2\beta+2)}<1-\frac{2\alpha^2}{N^2}$ and 1. Finally, considering that the first upper bound in (16) satisfies $1-\frac{2(\alpha+1)}{n(n+\alpha+\beta+1)}\le 1-\frac{8(\alpha+1)}{N^2}$ when α and $\beta\ge -1/2$, and that $8(\alpha+1)>2\alpha^2$ when $-1/2\le\alpha\le 0$, we conclude that $1-\frac{2\alpha^2}{N^2}$ is a valid upper bound.

For the lower bound in (15) one can use the symmetry property of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ given by $P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x)$. $\square$

It is interesting to compare (15) with formulas given in [17, p. 160] (which is also quoted in [2, p. 1448]) stating that, when α, β, and n tend to ∞ in such a way that $\lim_{n\to\infty}\alpha/N=A$ and $\lim_{n\to\infty}\beta/N=B$, then the zeros remain smaller than $B^2-A^2+[(1-A^2-B^2)^2-4A^2B^2]^{1/2}$ which is indeed smaller than $1-2A^2$ (for an alternative proof see [12, Theorem 8, p. 137]).

We will also need the following estimates for the second inequality of (9).

Lemma 15. *For real α, β, and $N \geq [2(\alpha^2+\beta^2)]^{1/2}$, the inequality*

$$(N+1)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x} \geq CN(N+1), \qquad -1+\frac{4\beta^2}{N^2} \leq x \leq 1-\frac{4\alpha^2}{N^2},$$

holds with $C = \min\left(\frac{1}{2}, \frac{3}{\sqrt{8(\alpha^2+\beta^2)}}\right)$.

Proof of Lemma 15. First, since $(N+1)^2 - 2\alpha^2/(1-x) - 2\beta^2/(1+x)$ is a concave function of x in $[-1,1]$, we only have to check its values at $-1+4\beta^2/N^2$ and $1-4\alpha^2/N^2$ which are $(N+1)^2 - \alpha^2/(1-2\beta^2/N^2) - N^2/2$ and $(N+1)^2 - N^2/2 - \beta^2/(1-2\alpha^2/N^2)$, respectively. Let $\gamma = \max(|\alpha|,|\beta|)$ and $\delta = \min(|\alpha|,|\beta|)$.[7] Since $\gamma^2/(1-2\delta^2/N^2) \geq \delta^2/(1-2\gamma^2/N^2)$, we have to find a lower bound for

$$F(N) \stackrel{\text{def}}{=} \frac{(N+1)^2 - \dfrac{N^2}{2} - \dfrac{\gamma^2 N^2}{N^2-2\delta^2}}{N(N+1)}$$

when $N \geq [2(\alpha^2+\beta^2)]^{1/2}$. Note that

$$F(N) \geq \frac{\dfrac{N^2}{2} + 2N - \dfrac{\gamma^2 N^2}{N^2-2\delta^2}}{N(N+1)} = \frac{1}{2} + \frac{\dfrac{3}{2} - \dfrac{\gamma^2 N}{N^2-2\delta^2}}{N+1},$$

so that $F(N) \geq 1/2$ when $N \to \infty$. $F(N)$ is greater than $1/2$ for all $N \geq [2(\gamma^2+\delta^2)]^{1/2}$ if

$$G(N) \stackrel{\text{def}}{=} \frac{3}{2} - \frac{\gamma^2 N}{N^2-2\delta^2} \geq 0$$

for all these values of N, that is, if γ^2 is smaller than the values of the increasing function $3N/2 - 3\delta^2/N$, so that the least value is taken at $N = [2(\gamma^2+\delta^2)]^{1/2}$. This happens when $\alpha^2+\beta^2 = \gamma^2+\delta^2 \leq 9/2$, and, hence, the minimum of $F(N)$ is therefore $1/2$ in this case.

When $\alpha^2+\beta^2 = \gamma^2+\delta^2 > 9/2$, we only have to search for the negative values of $G(N)$ in $F(N) \geq 1/2 + G(N)/(N+1)$. We will show that $G(N)/(N+1)$ is an increasing function of N, that is, $-G(N)/(N+1)$ is a positive decreasing function of N. Indeed, $-G(N) = \gamma^2/(N-2\delta^2/N) - 3/2$ is a decreasing function itself, as $N - 2\delta^2/N$ is increasing. Thus, the minimum of $F(N)$ is not smaller than $1/2 + G(N)/(N+1) \geq 1/2 + G(N)/N$ at $N = [2(\gamma^2+\delta^2)]^{1/2}$. This gives $G(N) = 3/2 - N/2$ and $F(N) \geq 3/\{2[2(\gamma^2+\delta^2)]^{1/2}\} = 3/[8(\alpha^2+\beta^2)]^{1/2}$. $\square$

Now we are ready for the

Proof of Theorem 1. First we prove formula (1). Since it is obvious for $n=0$, we can assume that $n \geq 1$. Let $N = 2n+\alpha+\beta+1$. If $x \in [-1,-1+4\beta^2/N^2] \cup [1-4\alpha^2/N^2, 1]$, then (1) follows from Theorem 9, Lemma 12, and Theorem 13. If $N < [2(\alpha^2+\beta^2)]^{1/2}$, then $[-1+4\beta^2/N^2, 1-4\alpha^2/N^2]$ is empty. If $N \geq [2(\alpha^2+\beta^2)]^{1/2}$ and $x \in [-1+4\beta^2/N^2, 1-4\alpha^2/N^2]$, then (1) follows from Theorem 9 and Lemma 15. Finally, Corollary 4 and (1) yield (2). $\square$

[7]N.B. these values γ and δ are not the same as in Lemma 12 if α or β is negative.

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