10th International Symposium on Orthogonal Polynomials, Special Functions and Applications, Leuven, July 20 to July 25, 2009.

Rational Freud equations

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This version: July 17, 2009 (incomplete and unfinished)

The present file is http://perso.uclouvain.be/alphonse.magnus/Freud2009s.pdf

Mon centre cède, ma droite recule, situation excellente, j'attaque. [1]
G al Foch

[1] My centre is giving way, my right is retreating, situation excellent, I am attacking.

Yiannis Gabriel, Freud and society , Routledge, 1983

http://books.google.com/books?id=suI9AAAAIAAJ&pg=PA201

being. The equations of tension with life with pain on the one hand, and inertia with pleasure and death on the other, are broken or, at least, they become mere descriptions of the present oppressive reality principle. Freud himself had not been perfectly happy with these equations, and it may be remembered that in The Economic Problem of Masochism (1924c) he had felt

G. Freud found a way in 1976 to investigate orthogonal pol. p_n with respect to an $\exp(-P(x))$ weight function through equations for recurrence coefficients

$$a_{n+1}p_{n+1}(x) = xp_n(x) - a_n p_{n-1}(x) \Rightarrow$$

$$F_n(\boldsymbol{a}) := a_n \left(P'(\boldsymbol{J})\right)_{n,n-1} = n, \qquad n = 1, 2, \ldots \quad (1)$$

if P is an even polynomial, and where $\boldsymbol{J} = \begin{bmatrix} 0 & a_1 & & \\ a_1 & 0 & a_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$ ([*polynomial*] ***Freud equations***).

Proof. $0 = \int_{-\infty}^{\infty} d[p_n(x)p_{n-1}(x)\exp(-P(x))] =$

$$\int_{-\infty}^{\infty} \underbrace{p_n'(x)}_{\frac{n}{a_n}p_{n-1}(x)+\cdots} p_{n-1}(x)\exp(-P(x))dx + \underbrace{\int_{-\infty}^{\infty} p_n(x)p_{n-1}'(x)\exp(-P(x))dx}_{0}$$

$$-\underbrace{\int_{-\infty}^{\infty} p_n(x)p_{n-1}(x)P'(x)\exp(-P(x))dx}_{(P'(\boldsymbol{J}))_{n,n-1}\text{(Grenander \& Szegő)}}$$

Example $P(x) = x^4 : 4a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) = n$ (a lot of people)

Interesting relations are not limited to $w'/w =$ a polynomial, but something can be done when the result is a rational function too. Going back to Laguerre, if $w'/w = A/B$, equations are found from

$$0 = \int_{\text{support}} d[B(x)p_n(x)p_{n-1}(x)w(x)], \tag{2}$$

provided Bw vanishes at the support endpoints.

This is useful in investigating generalized Jacobi polynomials $w(x) = \prod_k |x - x_k|^{\alpha_k}$ (**new** results in sight! A. Foulquie Moreno, A. Martinez-Finkelshtein, V. L. Sousa, On a conjecture of A. Magnus concerning the asymptotic behavior of the recurrence coefficients of the generalized Jacobi polynomials, `http://front.math.ucdavis.edu/0905.2753`), Pollaczek-like weights (Y. Chen), some orthogonal polynomials on the unit circle (N.S. Witte).

What if we keep (1) $F_n(\boldsymbol{a}) := -a_n \left(\dfrac{w'}{w}(\boldsymbol{J})\right)_{n,n-1} = n$ instead? Then the actual contents of the formula $F_n(\boldsymbol{a}) = -a_n \left(\dfrac{A(\boldsymbol{J})}{B(\boldsymbol{J})}\right)_{n,n-1} = n$ involves now

$\int_{\text{support}} p_n(x)p_{n-1}(x)\frac{A(x)}{B(x)}w(x)dx$. Considering only simple fractions, this asks for terms $\int_{\text{support}} \frac{p_n(x)p_m(x)\ w(x)dx}{x-\zeta} = ((\boldsymbol{J}-\zeta\boldsymbol{I})^{-1})_{m,n} = -p_m(\zeta)q_n(\zeta)$ if $n \geqslant m$ (Wall).

$$\frac{q_{n+1}(z)}{q_n(z)} = \cfrac{a_{n+1}}{z - \cfrac{a_{n+2}^2}{z - \cfrac{a_{n+3}^2}{\cdots}}}$$

comfortably convergent if $z \notin$ support (Gautschi).

Example: particle distribution in high atmosphere (Lemaire, Pierrard, Shizgal) $w(x) = |x|^{2\alpha+1}(1+x^4/\lambda^2)^{-\beta-1}\exp(-\varepsilon x^4)$, so $\frac{A(x)}{B(x)} = \frac{2\alpha+1}{x} - \frac{4(\beta+1)x^3}{x^4+\lambda^2} - 4\varepsilon x^3$.

Szegő: on $[-1,1]$, $a_n \sim 1/2$,

$$(\zeta \boldsymbol{I} - \boldsymbol{J})^{-1} \sim \begin{bmatrix} & \ddots & \ddots & \ddots & \\ \frac{\rho^2}{\sqrt{\zeta^2 - 4a_n^2}} & \frac{\rho}{\sqrt{\zeta^2 - 4a_n^2}} & \frac{1}{\sqrt{\zeta^2 - 4a_n^2}} & \frac{\rho}{\sqrt{\zeta^2 - 4a_n^2}} & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

with $\rho = \frac{\zeta}{2a_n} - \sqrt{\frac{\zeta^2}{4a_n^2} - 1}$, $|\rho| < 1$.

Gegenbauer-like: $A(x)/B(x) = \alpha/(x-1) + \alpha/(x+1)$, so

$$\frac{n}{a_n} \sim \alpha \left[\frac{1}{\sqrt{1 - 4a_n^2}} + \frac{1}{\sqrt{1 - 4a_n^2}} \right],$$

or $a_n \sim \frac{1}{2} - \frac{\alpha^2}{4n^2}$. Wrong! $4a_n^2 = \frac{n(n+2\alpha)}{(n+\alpha)^2 - 1/4} = 1 - \frac{\alpha^2 - 1/4}{(n+\alpha)^2 - 1/4} =$ $4\left[\frac{1}{2} - \frac{\alpha^2 - 1/4}{4n^2} + \cdots\right]^2$. Excellent!

Pollaczek-like: $w(x) = \exp\left(-\frac{C}{(1-x^2)^\alpha}\right), \quad \frac{A(x)}{B(x)} = \frac{C\alpha\, x}{(1-x^2)^{\alpha+1}}$

Multiple pole: $\int_{\text{support}} \frac{p_n(x)p_m(x)\ dx}{(x-\zeta)^{\alpha+1}} = -\frac{1}{\alpha!}\frac{d^\alpha}{d\zeta^\alpha}[p_m(\zeta)q_n(\zeta)]$

Freud eq. $n \sim$ const. $(1-4a_n^2)^{-1/2-\alpha}$, so $a_n - 1/2 \sim$ const. $n^{-1/(1/2+\alpha)}$, confirmed by Y. Chen, June 26, 2009, thanks!

Historical Pollaczek: $\alpha = 1/2 \Rightarrow n^{-1}$ behaviour, right!

$1/(1/2 - a_n)^{3/2}$ of $\exp(-1/(1-x^2))$:

10	20	30	40	50	60	70	80	90	100
176.341	339.320	501.523	663.320	824.860	986.219	1147.44	1308.56	1469.59	1630.56

behaves like $16n + O(?)\ldots$

$\exp(-x^2 - C/x^2)$ Y.Chen & A.Its [5]

a_n^2 close to $n/2$. Here are $a_n^2 - n/2$, $n = 25, 50, 75, \ldots, 125$

	25	50	75	100	125
$C = 1$	1.63406	-1.70609	2.22593	-2.21181	2.59849
$C = 10^{-3}$	0.11295	-0.14903	0.17977	-0.20008	0.22131

could a_n^2 be $n/2 - (-1)^n$ ... ?

$$2a_n^2 - 2Ca_n(\boldsymbol{J}^{-3})_{n,n-1} = n \tag{3}$$

If $\{a_n\} \sim \{\cdots, \alpha, \beta, \alpha, \beta, \cdots\}$, with, say, $\beta > \alpha$, spectrum of $\boldsymbol{J}$ is $\pm\left[\frac{\beta-\alpha}{\sqrt{\alpha\beta}}, \frac{\alpha+\beta}{\sqrt{\alpha\beta}}\right]$, $p_n(z)$ and $q_n(z)$ satisfy $p_{n+2}(z) = \frac{z^2-\alpha^2-\beta^2}{\alpha\beta}p_n(z) - p_{n-2}(z)$, also, the elements $u_{m,n}(z)$ of $(z\boldsymbol{I} - \boldsymbol{J})^{-1}$ satisfy $zu_{m,n} - \beta(\alpha u_{m+2,n} + \beta u_{m,n})/z - \alpha(\alpha u_{m,n} + \beta u_{m-2,n})/z = \delta_{m,n}$, so $u_{n+2k,n} = \rho^{|k|}u_{n,n}$, where ρ is the root of $\alpha\beta(\rho^2+1) = (z^2-\alpha^2-\beta^2)\rho$, and

$|\rho| < 1$ outside the spectrum of $\boldsymbol{J}$. Then, $u_{n,n}(z) = \dfrac{z}{\alpha\beta(1/\rho - \rho)} = \dfrac{z}{\sqrt{[z^2-(\alpha-\beta)^2][z^2-(\alpha+\beta)^2]}}$, and $u_{n,n-1}$ takes alternatively the values $(\alpha\rho+\beta)u_{n,n}/z$ and $(\beta\rho+\alpha)u_{n,n}/z = \dfrac{\alpha\rho+\beta}{\alpha\beta(1/\rho-\rho)}$ and $\dfrac{\beta\rho+\alpha}{\alpha\beta(1/\rho-\rho)}$. Second derivative in $z = 2(z\boldsymbol{I} - \boldsymbol{J})^{-3}_{n,n-1}$ at $z = 0$, i.e., $\rho = -\alpha/\beta$, so, if $\rho = -\alpha/\beta + \varepsilon$, $z^2 = \alpha^2 + \beta^2 + \alpha\beta(\rho + 1/\rho) = -\beta(\beta^2 - \alpha^2)\varepsilon/\alpha + \cdots$, $\left[\dfrac{\alpha\rho+\beta}{\alpha\beta(1/\rho-\rho)}, \dfrac{\beta\rho+\alpha}{\alpha\beta(1/\rho-\rho)}\right] =$ $\left[\dfrac{(\beta^2-\alpha^2)/\beta + \alpha\varepsilon}{-(\beta^2-\alpha^2) - \beta(\alpha^2+\beta^2)\varepsilon/\alpha + \cdots}, \dfrac{\beta\varepsilon}{-(\beta^2-\alpha^2) - \beta(\alpha^2+\beta^2)\varepsilon/\alpha + \cdots}\right] =$ $\left[-\dfrac{1}{\beta} + \dfrac{\beta^2/\alpha}{\beta^2-\alpha^2}\varepsilon + \cdots, -\dfrac{\beta\varepsilon}{\beta^2-\alpha^2} + \cdots\right] = \left[-\dfrac{1}{\beta} - \dfrac{\beta}{(\beta^2-\alpha^2)^2}z^2 + \cdots, \dfrac{\alpha}{(\beta^2-\alpha^2)^2}z^2 + \cdots\right]$

The approximation to (3) is therefore

$$2\alpha^2 + 2C\alpha\frac{\alpha}{(\beta^2-\alpha^2)^2} = n, 2\beta^2 - 2C\beta\frac{\beta}{(\beta^2-\alpha^2)^2} = n \pm 1,$$

whence $\beta^2 - \alpha^2 = (Cn)^{1/3}$, $a_n^2 \sim n/2 \pm (Cn)^{1/3}/2$

By expanding (2) with $A(x)/B(x) = (-2x^3 + 2C/x)/x^2$,

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} d[x^2 p_n(x) p_{n-1}(x) w(x)] = \int_{-\infty}^{\infty} [p_n(x) \underbrace{2xp_{n-1}(x)}_{2a_n p_n(x) + \cdots} p_n(x) + \\
&+ \underbrace{\underbrace{x^2 p_n'(x)}_{x[np_n(x) + 2(a_1^2 + \cdots + a_{n-1}^2) p_{n-2}(x)/(a_{n-1}a_n) + \cdots]}}_{n a_{n+1} p_{n+1}(x) + n a_n p_{n-1}(x) + 2(a_1^2 + \cdots + a_{n-1}^2) p_{n-1}(x)/a_n + \cdots} p_{n-1}(x) \\
&+ \underbrace{x^2 p_{n-1}'(x)}_{(n-1)xp_{n-1}(x) + \cdots = (n-1)a_n p_n(x) + \cdots} p_n(x) \\
&+ \underbrace{[-2x^3 + 2C/x] p_n(x) p_{n-1}(x)}_{-2a_n(a_{n-1}^2 + a_n^2 + a_{n+1}^2) + C(1 - (-1)^n)/a_n}] w(x) dx
\end{aligned}
$$

$$
(2n+1)a_n^2 + 2(a_1^2 + \cdots + a_{n-1}^2) - 2a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) + C(1 - (-1)^n) = 0
$$

1^{st} correction.

Let the neighboorhood of the n^{th} row of $\boldsymbol{J}$ be

$$\boldsymbol{J} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a_n & 0 & a_{n+1} & & \\ & & a_{n+1} & 0 & a_{n+2} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a_n & 0 & a_n & & \\ & & a_n & 0 & a_n & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$+ \begin{bmatrix} -\varepsilon_n & 0 & 0 & & & \\ & 0 & 0 & \varepsilon_n & & \\ & & \varepsilon_n & 0 & 2\varepsilon_n & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$ so, $a_{n+p} \sim a_n + p\varepsilon_n$.

Then, $(\boldsymbol{J})^{2k+1}_{n,n-1} \sim -(-4)^k \binom{-1/2}{k+1} a_n^{2k+1} + (-4)^k k \binom{-5/2}{k-1} \varepsilon_n^2 a_n^{2k-1}$,

$(z\boldsymbol{I} - \boldsymbol{J})^{-1}_{n,n-1} \sim [(1-4a_n^2)^{-1/2} - 1]/(2a_n) + a_n[5(1-4a_n^2)^{-7/2} - 3(1-4a_n^2)^{-5/2}]\varepsilon_n^2/2$

Fréchet derivative. [19]

$F_n(\boldsymbol{a} + da_m\delta_{i,m}) - F_n(\boldsymbol{a}) = -(a_n + da_m\delta_{n,m})\left(\frac{w' + dw'}{w + dw}(\boldsymbol{J} + d\boldsymbol{J})\right)_{n,n-1} + a_n\left(\frac{w'}{w}(\boldsymbol{J})\right)_{n,n-1} = -(a_n + da_m\delta_{n,m})\int_{\mathbb{R}}(w'(x) + dw'(x))(p_n(x) + dp_n(x))(p_{n-1}(x) + dp_{n-1}(x)) + a_n\int_{\mathbb{R}} w'(x)p_n(x)p_{n-1}(x)dx = \ldots = \int_{\mathbb{R}}\int_{\mathbb{R}} a_n\frac{w'(y)w(x) - w'(x)w(y)}{y - x}p_n(y)p_{n-1}(x)p_m(y)p_{m-1}(x)dxdy$

(For instance, $\partial p_n(x)/\partial a_m = 0$ if $n \leqslant m$; $-p_m(x)[p_n(x)q_{m-1}(x) - q_n(x)p_{m-1}(x)]/a_m = -p_m(x)\int_{\mathbb{R}}\frac{p_n(x)p_{m-1}(y) - p_n(y)p_{m-1}(x)}{a_m(x-y)}w(y)dy$ if $n \geqslant m$, etc.)

From M.Ismail [10]

3.2 Differential Equations

Define $A_n(x)$ and $B_n(x)$ via

$$\frac{A_n(x)}{a_n} = \left.\frac{w(y)\,p_n^2(y)}{y-x}\right|_a^b + \int_a^b \frac{v'(x)-v'(y)}{x-y}\, p_n^2(y)\, w(y)\, dy, \tag{3.2.1}$$

$$\begin{aligned}\frac{B_n(x)}{a_n} &= \left.\frac{w(y)\,p_n(y)\,p_{n-1}(y)}{y-x}\right|_a^b \\ &\quad + \int_a^b \frac{v'(x)-v'(y)}{x-y}\, p_n(y)p_{n-1}(y)\, w(y)\, dy.\end{aligned} \tag{3.2.2}$$

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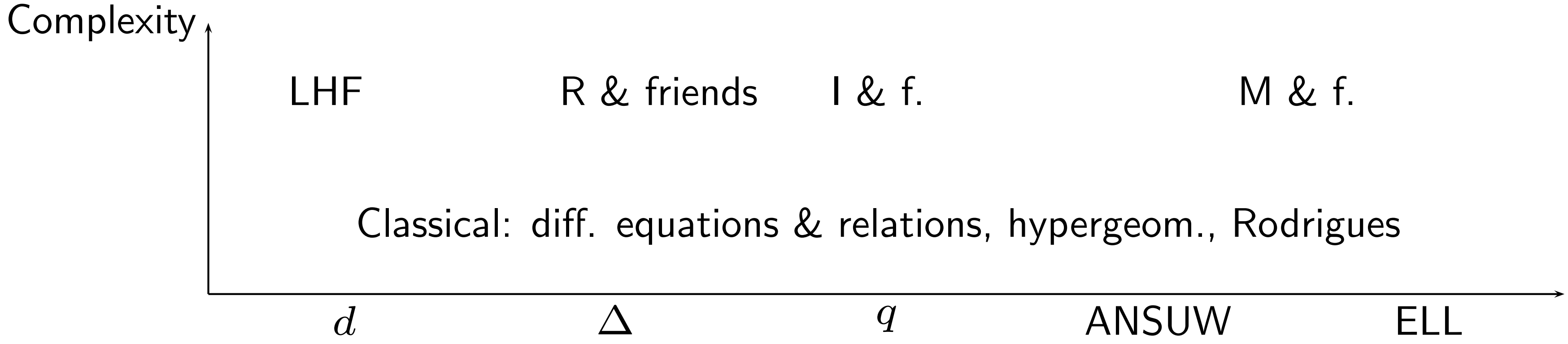
We shall tacitly assume that the boundary terms in (3.2.1) and (3.2.2) exist.

Theorem 3.2.1 *Let $v(x)$ be a twice continuously differential function on $[a, b]$. Then the polynomials $\{p_n(x)\}$ orthonormal with respect to $w(x)dx$, $w(x) = \exp(-v(x))$ satisfy*

$$p_n'(x) = -B_n(x)p_n(x) + A_n(x)p_{n-1}(x), \tag{3.2.3}$$

where A_n and B_n are given by (3.2.1) and (3.2.2).

Various difference lattices.



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