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L.L.N. , 18 June 1986

Tentative asymptotics with Freud's equations .

Problem : get the a_n's of $\exp(-f(x))$ (f supposed even for simplicity) . Suggested method : try a sequence $\{ \ldots\ a_{n-1}, a_n, a_{n+1}, \ldots \}$ and look if $F_n(a)$ is close to n ...

Here , $F_n(a) = a_n \int q_n q_{n-1} f' d\sigma$, where the q_n's and $d\sigma$ come from the solution of the moment problem for the proposed a_n's . It is assumed that $F_n(a)$ depends mostly on the a_m's with m near n .

Simplest example : all the a_m's for m near n are equal to a_n . Then , $d\sigma(x) = 1/[\rho(x)(4a_n^2 - x^2)^{1/2}]$, $q_n(x) \sim (2\rho(x)/\pi)^{1/2} \cos(n\theta + \varphi(\theta))$ (Bernstein-Szegö) $x = 2a_n \cos\theta$

$$F_n(a) \sim \frac{1}{\pi} a_n \int_{-2a_n}^{2a_n} (4a_n^2 - x^2)^{-1/2} \, [\cos\theta + \cos((2n-1)\theta + 2\varphi)] f'(x) dx$$

The first part is nothing else than the Mhaskar and Saff function ! So we see how this function appears as a first approximation .

I hoped that the second part should give the oscillatory behaviour of a_n for $f(x) = |x|^\alpha$ when α is not an even integer . Well , working the integrals (replacing φ by $\varphi(\pi/2) = 0$: the Van Hove singularity at $x=0$ creates the damped oscillation) .

$$F_n(a) \sim \frac{\alpha}{\pi} (2a_n)^\alpha \left(\int_0^{\pi/2} (\cos\theta)^\alpha d\theta + \int_0^{\pi/2} \cos((2n-1)\theta (\cos\theta)^{\alpha-1} d\theta \right) \sim n ,$$

$$\sim C(\alpha)(a_n)^\alpha \left(1 + (-1)^{n-1} \Gamma(n-\alpha/2)\Gamma(1+\alpha/2)/(\Gamma(n+\alpha/2)\Gamma(1-\alpha/2)) \right)$$

i.e. $C(\alpha)(a_n)^\alpha \left(1 + (-1)^{n-1} A(\alpha)/n^\alpha \right) \sim n$

$C(\alpha)(a_n)^\alpha - n \sim (-1)^n A(\alpha) n^{1-\alpha}$, with $A(\alpha) = \Gamma(\alpha/2)\Gamma(1+\alpha/2)\sin(\pi\alpha/2)/\pi$

For $\alpha > 1$, the oscillatory part of $C(\alpha)(a_n)^\alpha - n$ behaves indeed like $\text{const.} n^{1-\alpha}$, but the constant is not the same , it is about two times $A(\alpha)$ for large α , and the behaviour when α approaches 1 is quite interesting : I find the numerical formula

$$C(\alpha)(a_n)^\alpha - n = (\alpha-2)/(24n) - 13(\alpha-2)(3\alpha-2)/(11520 n^3) \ldots$$

$\alpha - 1/\alpha$

$$+ (-1)^n B(\alpha) \frac{n}{n^{\alpha}-n^{1/\alpha}} \ldots$$

α	$B(\alpha)$	$(\alpha-1/\alpha)B(\alpha)$	$A(\alpha)$
7	-3.5 ?	-24 ?	-12.3
5	0.39	1.9	1.40
3	-0.085	-0.23	-0.375
1.5	0.030		
1	0.035 for $\alpha=1$, the perturbation is $(-1)^n B(1)/\log n$		
0.75	0.029		
0.66666	0.026		
0.5	0.016		
0.33333	0.0003 ?		

the non oscillatory part comes from Nevai et al. ($\alpha=4$) and Sheen ($\alpha=6$).

$$+\frac{(-1)^n \beta}{n^{\alpha}-n^{1/\alpha}}$$

$$\frac{(C)^{2/\alpha} a_n^2}{n^{2/\alpha}} = 1 + \frac{\alpha-2}{12\alpha n^2} - \frac{(\alpha-2)(2\alpha-1)(3\alpha-2)}{720\alpha^2 n^4} + \frac{(\alpha-2)(2\alpha-1)(5\alpha-2)(4\alpha^3+11\alpha^2+72\alpha+20)}{181440\alpha^4 n^6} \ldots$$

$$F_n(\tilde{a}) = \tilde{a}_n \int_{-\infty}^{\infty} \tilde{p}_n \tilde{p}_{n-1} f' \, d\tilde{\mu} \qquad \tilde{\mu} \leftrightarrow \tilde{a}$$

$$J(\tilde{a}) = \left\{ \frac{\partial F_i(\tilde{a})}{\partial \log a_j} \right\} \quad \text{sym.}$$

forme quad. $\sigma' J \sigma = \iint \left(\sum_1^{\infty} \tilde{a}_n \sigma_n \tilde{p}_n(u) \tilde{p}_{n-1}(t) \right)^2 \frac{f'(u)-f'(t)}{u-t} \, d\tilde{\mu}(u) \, d\tilde{\mu}(t)$

él. ij : $\iint \tilde{a}_i \tilde{p}_i(u) \tilde{p}_{i-1}(t) \tilde{a}_j \tilde{p}_j(u) \tilde{p}_{j-1}(t) \frac{f'(u)-f'(t)}{u-t} \, d\tilde{\mu}(u) \, d\tilde{\mu}(t)$

$i \sim j$ $\quad d\tilde{\mu} \sim \frac{\pi}{\rho\sqrt{\ }} \quad \tilde{p}_i(u)\tilde{p}_j(u) \approx \frac{2\rho}{\pi} \frac{1}{2} \cos(i-j)\theta_u$

$$J_{ij} \sim \frac{1}{\pi^2} \iint \tilde{a}_i \tilde{a}_j \cos(i-j)\theta_u \cos(i-j)\theta_t \frac{f'(u)-f'(t)}{u-t} \frac{du\,dt}{\sqrt{\ }\sqrt{\ }}$$

$u = 2a\cos$

$2\times$

$$\sim \frac{a_i^2}{\pi^2} \int_0^{\pi}\!\!\int_0^{\pi} \cos(i-j)\theta_u \cos(i-j)\theta_t \frac{f'(2a\cos\theta_u) - f'(2a\cos\theta_t)}{2a(\cos\theta_u - \cos\theta_t)} \, d\theta_u \, d\theta_t$$

Toeplitz $\quad$ ex: $f = x^2 \quad 0: 4a^2$ $\qquad$ symbole $4a^2$

$f = x^4 \quad 2\left(4\frac{a^2}{\pi^2}\right)\iint \cos\cos\left(\cos^2\theta_1 + \cos\theta_1\cos\theta_2 + \cos^2\theta_2\right) d\theta_1 d\theta_2$

$F = 4a_i^2(a_{i-1}^2 + a_i^2 + a_{i+1}^2)$ $\quad \pm 1: \frac{(4a^2)^2}{\pi^2} \iint \cos^2\cos^2 = 8a^4 \quad 0: 2\frac{(4a^2)^2}{\pi^2} \cdot 2 \iint \cos^2 = 32a^4$ $\quad | \; 32a^4 + 16a^4\cos 2$

α pair: $\frac{1^2}{\pi^2} \cdot 2\alpha (2a)^{\alpha-2} \iint \sum_{p=0}^{\alpha-2} \cos\cos \left(\cos^p\theta_1 \cos^{\alpha-2-p}\theta_2 \right)$

en $i-j = \frac{\alpha-2}{2}$: $\frac{1^2}{\pi^2} \cdot 2\alpha (2a)^{\alpha-2} \left(\int_0^{\pi} \cos\frac{\alpha-2}{2}\theta \, (\cos\theta)^{\frac{\alpha-2}{2}} \right)^2 = \frac{a^{\alpha}}{\pi^2} \alpha 2^{\alpha-1} \cdot \frac{\pi^2}{4} \frac{1}{2^{\alpha-4}}$

$\frac{1}{2^{\frac{\alpha-2}{2}-1}} \cos\frac{\alpha-2}{2}\theta + \ldots \qquad = 2\alpha a^{\alpha}$

REFINED ASYMPTOTICS FOR FREUD'S RECURRENCE COEFFICIENTS

A. P. MAGNUS

1.Freud's weights and coefficients.

Let $\{p_n(x) = p_n(x; d\mu(x))\}_{n=0}^{\infty}$ be the orthonormal polynomials with respect to an even measure (i.e., $\int_{-\infty}^{\infty} x^{2m-1} d\mu(x) = 0, m = 1, 2, \ldots$) on $\mathbf{R}$. These polynomials then satisfy the recurrence relations

$$a_{n+1}p_{n+1}(x) = xp_n(x) - a_n p_{n-1}(x), \qquad n = 0, 1, \ldots \; (a_0 = 0) \tag{1}$$

One wishes to relate the behaviour of $d\mu(x)$ for large x to the behaviour of the recurrence coefficients a_n for large n (Freud's programme [2], see also § 4.18 of [11]). Quite a number of dramatic achievements have been made recently, using advanced orthogonal polynomials theory (Christoffel functions), functional spaces theory and potential theory, see [3,4,5(Appendix),6,7,11,13,14(Chap.4)] as landmarks and surveys.

Freud remarked in [2] how one can use the identity

$$\frac{n}{a_n} = \int_{-\infty}^{\infty} p_n'(x)p_{n-1}(x)e^{-Q(x)}dx = \int_{-\infty}^{\infty} p_n(x)p_{n-1}(x)Q'(x)e^{-Q(x)}dx, \; n = 1, 2, \ldots \tag{2}$$

when the polynomials p_n are orthonormal with respect to $\exp(-Q(x))dx$ (*Freud's weight*). Indeed, if Q is a polynomial, repeated applications of (1) in the right-hand side of (2) yields a polynomial in $a_n, a_{n\pm1}, \ldots$, so that (2) turns as a set of equations for the recurrence coefficients $\{a_n\}$. For instance, $Q(x) = x^4$ gives

$$4a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) = n, \qquad n = 1, 2, \ldots, \tag{3}$$

a well worked example ([1,2,8,9,10,11,12(pp.470-471)]). Moreover, such equations allowed Freud and followers to establish the asymptotic behaviour of a_n for various exponential weights (exponentials of polynomials [1,2,8,9,10]), and to arrive naturally to a conjecture for non polynomial exponentials:

$$\text{if } d\mu(x) = \exp(-|x|^\alpha)dx, \quad \text{then} \quad a_n \sim \left(\frac{n}{C(\alpha)}\right)^{1/\alpha}, \tag{4}$$

when $n \to \infty$, with $C(\alpha) = \dfrac{2^\alpha\Gamma((\alpha+1)/2)}{\Gamma(1/2)\Gamma(\alpha/2)}$, for $\alpha > 0$. The proof of (4) appears as a special case of very powerful investigations which led to

$$\text{if } d\mu(x) = \exp(-Q(x))dx, \quad \text{then} \quad a_n \sim \tilde{a}_n, \text{ with } \tilde{a}_n \int_0^\pi Q'(2\tilde{a}_n\cos\theta)\cos\theta \, d\theta = n\pi \tag{4'}$$

where Q is even, continuous and convex on $\mathbf{R}$, $Q'(x) > 0$ for $x > 0$, among other conditions. The (positive) root $\tilde{a}_n$ of the equation of (4') is called the *Lubinsky-Mhaskar-Rahmanov-Saff's number*. The method of proof of (4') can even give asymptotic estimates of $a_1a_2\ldots a_n$([3,4,5,6,7,13]). Now, let us try to explore the subject further with Freud's equations.

2.Freud's functionals and equations.

Let us consider only positive sequences $\mathbf{a} = \{a_n\}_1^\infty$ with $\sum_1^\infty 1/a_n = \infty$ so as to be sure that the related moment problem, amounting to finding $d\mu$ such that

$$\cfrac{1}{z - \cfrac{a_1^2}{z - \cfrac{a_2^2}{z - \cdots}}} = \int_{-\infty}^{\infty} (z-x)^{-1}\, d\mu(x)\,, \qquad \forall z \notin \mathbf{R} \tag{5}$$

has a unique solution. Then the *Freud's functional* related to Q and $\mathbf{a}$ is defined as

$$F_n(Q;\mathbf{a}) = a_n \int_{-\infty}^{\infty} p_n(x)p_{n-1}(x)Q'(x)\, d\mu(x), \tag{6}$$

where the $p_n(x) = p_n(x; d\mu(x))$'s are the orthonormal polynomials related to the measure $d\mu$ solving the moment problem (5). Thus, Freud's remark becomes: $\mathbf{a}$ is truly the sequence of recurrence coefficients related to the measure $\exp(-Q(x))dx$ on $\mathbf{R}$ if

$$F_n(Q;\mathbf{a}) = n, \qquad\qquad n = 1, 2, \ldots \tag{7}$$

making the *Freud's equations* for $\mathbf{a}$ (when Q is an even function, or else one also must consider the functionals $G_n = \int_{-\infty}^{\infty} p_n^2(x)Q'(x)d\mu(x)$ [9]). The functionals F_n are linear in Q, but nonlinear in $\mathbf{a}$, so with $Q(x) = x^4$ one recovers the example (3).

3.Asymptotic expansions.

When Q is a polynomial, (6) is explicit in a finite number of neighbours of a_n, and *asymptotic expansions* can be studied: so (4) has been completed as

$$a_n \sim \left(\frac{n}{C(\alpha)}\right)^{1/\alpha} \sum_{k=0}^{\infty} \frac{A_k}{n^{2k}} \tag{8}$$

with $A_0 = 1$, when α is an even integer ([10], see also [1]).

The **problem** now is to extend (8) when α is not an even integer, still using (6). Making the assumption that $F_n(Q;\mathbf{a})$ still depends essentially on the close neighbours of a_n (technically, that $\partial F_n(Q;\mathbf{a})/\partial a_k \to 0$ when $|n-k| \to \infty$, see further for more on $\partial F_n/\partial a_k$), we approximate $F_m(Q;\mathbf{a})$ for m near n by $F_m(Q;\mathring{\mathbf{a}}^{(n)})$, where $\mathring{\mathbf{a}}^{(n)}$ is the constant sequence $\ldots = \mathring{a}_{n-2}^{(n)} = \mathring{a}_{n-1}^{(n)} = \mathring{a}_n^{(n)} = \mathring{a}_{n+1}^{(n)} = \ldots = \tilde{a}_n$ (as we already know that $a_{n+k}/a_n \to 1$ when $n \to \infty$). The measure $\mathring{\mu}^{(n)}$ is then $d\mathring{\mu}^{(n)}(x) = (1/(\pi\tilde{a}_n))\sqrt{1-(x/(2\tilde{a}_n))^2}\, dx$ on $[-2\tilde{a}_n, 2\tilde{a}_n]$. This does not mean that $d\mathring{\mu}^{(n)}$ is close to $d\mu$, but that $F_m(Q;\mathring{\mathbf{a}}^{(n)})$ (probably) is close to $F_m(Q;\mathbf{a})$ for m close to $n\ldots$, and a lot of other trial measures would be as good. Proceeding with the computations ($\mathring{p}_m(x)$ is the Chebyshev polynomial $U_m(x/(2\tilde{a}_n))$), one finds

$$\begin{aligned} F_m(Q;\mathring{\mathbf{a}}^{(n)}) &= \tilde{a}_n \int_{-2\tilde{a}_n}^{2\tilde{a}_n} U_m\left(\frac{x}{2\tilde{a}_n}\right) U_{m-1}\left(\frac{x}{2\tilde{a}_n}\right) Q'(x) \frac{1}{\pi\tilde{a}_n}\sqrt{1-\left(\frac{x}{2\tilde{a}_n}\right)^2}\, dx \\ &= \frac{\tilde{a}_n}{\pi}\int_0^{\pi} [\cos\theta - cos(2m+1)\theta] Q'(2\tilde{a}_n\cos\theta)\, d\theta \end{aligned} \tag{9}$$

If Q is reasonably smooth, the $(2m+1)^{\text{th}}$ Fourier coefficient of $Q'(2\tilde{a}_n \cos\theta)$ may be neglected for large m and we recover the $LMRS$ approximation $\tilde{a}_n$ for a_n from (9). For instance, with $Q(x) = |x|^\alpha$:

$$\begin{aligned} F_m(|x|^\alpha; \mathring{\mathbf{a}}^{(n)}) &= \frac{\tilde{2}a_n}{\pi} \int_0^{\pi/2} [\cos\theta - cos(2m+1)\theta]\alpha(2\tilde{a}_n\cos\theta)^{\alpha-1}\, d\theta \\ &= \frac{2^\alpha \alpha \tilde{a}_n^\alpha}{\pi} \left[\frac{\Gamma((\alpha+1)/2)\Gamma(1/2)}{2\Gamma(1+\alpha/2)} - \frac{\Gamma(\alpha/2)\Gamma((\alpha+1)/2)\Gamma(1/2)}{2\Gamma(1+m+\alpha/2)\Gamma(\alpha/2\ -m)} \right] \\ &= \tilde{a}_n^\alpha C(\alpha) \left[1 - (-1)^m \sin(\pi\alpha/2) \frac{\Gamma(1+m-\alpha/2)\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi\Gamma(1+m+\alpha/2)} \right] \\ &\sim \tilde{a}_n^\alpha C(\alpha) \left[1 - (-1)^m \sin(\pi\alpha/2) \frac{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi m^\alpha} \right] \end{aligned}$$

This suggests an $(-1)^n/n^\alpha$ term in the expansion:

CONJECTURE. *The recurrence coefficients of* $\exp(-|x|^\alpha)$ *satisfy the asymptotic expansion*

$$a_n \sim \left(\frac{n}{C(\alpha)}\right)^{1/\alpha} \left(\sum_{k=0}^{\infty} \frac{A_k}{n^{i_k}} + (-1)^n \sum_{k=0}^{\infty} \frac{B_k}{n^{j_k}} \right) \tag{10}$$

with $0 = i_0 < i_1 < \ldots$, $A_0 = 1$, $\alpha = j_0 < j_1 < \ldots \quad (\alpha > 1)$.

For more accurate predictions, one relates errors on $\mathbf{a}$ to errors on $\mathbf{F}(\mathbf{a})$ ($\mathbf{F}(\mathbf{a})$ is the sequence $\{F_m(Q;\mathbf{a})\}$) by $\mathbf{F}(\mathbf{a}) - \mathbf{F}(\mathring{\mathbf{a}}^{(n)}) \sim \mathbf{J}(\mathbf{a} - \mathring{\mathbf{a}}^{(n)})$, where J is the Jacobian matrix of the partial derivatives $\partial F_m(Q;\mathbf{a})/\partial a_k$. The elements of this matrix (computed at $\mathring{\mathbf{a}}^{(n)}$) are

$$\begin{aligned} J_{m,k} &= \partial F_m(Q; \mathring{\mathbf{a}}^{(n)})/\partial \mathring{a}_k^{(n)} \\ &= 2\mathring{a}_m \int_{\mathbf{R}} \int_{\mathbf{R}} p_m(x) p_{m-1}(y) p_k(x) p_{k-1}(y) \frac{Q'(x) - Q'(y)}{x-y}\, d\mathring{\mu}^{(n)}(x)\, d\mathring{\mu}^{(n)}(y) \qquad [9] \\ &= \frac{8\tilde{a}_n}{\pi^2} \int_0^\pi \int_0^\pi \sin(m+1)\theta \sin m\psi \sin(k+1)\theta \sin k\psi \frac{Q'(2\tilde{a}_n\cos\theta) - Q'(2\tilde{a}_n\cos\psi)}{2\tilde{a}_n(\cos\theta - \cos\psi)}\, d\theta d\psi \end{aligned}$$

Here again, keeping only the lowest order Fourier coefficient:

$$J_{m,k} \sim \frac{2\tilde{a}_n}{\pi^2} \int_0^\pi \int_0^\pi \cos(m-k)\theta \cos(m-k)\psi \frac{Q'(2\tilde{a}_n\cos\theta) - Q'(2\tilde{a}_n\cos\psi)}{2\tilde{a}_n(\cos\theta - \cos\psi)}\, d\theta d\psi.$$

leaving a Toeplitz matrix of symbol $\mathring{\Phi}(\varphi) = \sum_{-\infty}^{\infty} J_{p,0} \exp(ip\varphi)$ such that

$$\frac{1}{\pi} \int_0^\pi \cos p\varphi \mathring{\Phi}(\varphi) d\varphi = \frac{2\tilde{a}_n}{\pi^2} \int_0^\pi \int_0^\pi \cos p\theta \cos p\psi \frac{Q'(2\tilde{a}_n\cos\theta) - Q'(2\tilde{a}_n\cos\psi)}{2\tilde{a}_n(\cos\theta - \cos\psi)}\, d\theta d\psi$$

$$\text{i.e.,} \qquad \mathring{\Phi}(\varphi) = \frac{2\tilde{a}_n}{\pi} \int_0^\pi \frac{Q'(2\tilde{a}_n\cos\theta) - Q'(2\tilde{a}_n\cos(\theta+\varphi))}{2\tilde{a}_n(\cos\theta - \cos(\theta+\varphi))}\, d\theta$$

$\mathbf{J}^{-1}$ is approximately the Toeplitz matrix of symbol $1/\mathring{\Phi}$, so that for $|x|^\alpha$:

$$a_n - \mathring{a}_n^{(n)} \sim \sum_{p=-\infty}^{\infty} (\mathbf{J}^{-1})_{n,n+p}(F_{n+p}(Q;\mathbf{a}) - F_{n+p}(Q;\mathring{\mathbf{a}}^{(n)}))$$

$$\sim \sum_{-\infty}^{\infty}(\mathbf{J}^{-1})_{p,0}\left(n+p-n+n(-1)^{n+p}\sin(\pi\alpha/2)\frac{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi(n+p)^\alpha}\right)$$

$$\sim (-1)^n \sin(\pi\alpha/2)\frac{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi\mathring{\Phi}(\pi)} n^{1-\alpha}$$

suggesting $B_0 = (\alpha-1)\sin(\pi\alpha/2)(\Gamma(\alpha/2))^2/(2\pi)$ in (10), using $\mathring{\Phi}(\pi) =$

$$\pi^{-1}(2\tilde{a}_n)^{\alpha-1}\int_0^{\pi/2}(\cos\theta)^{\alpha-2}d\theta = \alpha(2\tilde{a}_n)^{\alpha-1}\Gamma((\alpha-1)/2)/(\Gamma(1/2)\Gamma(\alpha/2)).$$

Now, some horribly wrong mistake must have occurred somewhere, because very high accuracy (up to 200 digits, on the IBM 3090 of the University) calculations of instances of a_n for various values of α, followed by severe extrapolation devices designed to exhibit B_0, lead to the

PROBLEM. *Show that one has* $j_0 = \alpha$ *and* $B_0 = (\alpha-1)\sin(\pi\alpha/2)(1-1/\alpha)^\alpha(\Gamma(\alpha/2))^2/(2\pi)$ *in* (10) *when* $\alpha > 1$.

Where does this $(1-1/\alpha)^\alpha$ come from ?????

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