Freud equations for Legendre polynomials on a circular arc and solution of the Grünbaum-Delsarte-Janssen-Vries problem.
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Abstract. One establishes inequalities for the coefficients of orthogonal polynomials

$$\Phi_n(z) = z^n + \xi_n z^{n-1} + \cdots + \Phi_n(0), \quad n = 0, 1, \ldots$$

which are orthogonal with respect to a constant weight on the arc of the unit circle $S = \{e^{i\theta}, \alpha\pi < \theta < 2\pi - \alpha\pi\}$, with $0 < \alpha < 1$. Recurrence relations (Freud equations), and differential relations are used. Among other results, it is shown that $\Phi_n(0) > 0, n = 1, 2, \ldots$

Keywords: unit circle orthogonal polynomials

1. Introduction and statement of results.

1.1. **Introduction.** The analysis of orthogonal polynomials on the unit circle has been limited for a long time to measures supported on the whole circle (theories of Szegő, and, later on, of Rakhmanov). Orthogonal polynomials on circular arcs were only known through special cases (Geronimus, Akhiezer). They now enter a general theory as an important subclass, as can be seen in Khrushchev's paper [20] and, of course, in B. Simon's recent work [27].

Actually, only a very special set of such orthogonal polynomials will be studied here, namely the Legendre polynomials on an arc, i.e., $\Phi_0, \Phi_1, \ldots$ are polynomials, with Φ_n of degree n, and

$$\int_{\alpha\pi}^{2\pi-\alpha\pi} \Phi_n(e^{i\theta})\overline{\Phi_m(e^{i\theta})}\, d\theta = 0$$

when $n \neq m$, and where α is given $(0 < \alpha < 1)$.

A property of these polynomials is needed in the solution of the following problem:

"3. The following Toeplitz matrix arises in several applications. Define for $i \neq j$, $A_{i,j}(\alpha) = \dfrac{\sin \pi\alpha(i-j)}{\pi(i-j)}$ and set $A_{i,i} = \alpha$. Conjecture: the matrix $M = (I - A)^{-1}$ has positive entries. A proof is known for $1/2 \leqslant \alpha < 1$. Can one extend this to $0 < \alpha < 1$? Submitted by Alberto Grünbaum, November 3, 1992. (grunbaum@math.berkeley.edu)" [18].

The question was asked by Grünbaum as a result of investigations about the limited angle tomography problem [8,17], i.e., how to reconstruct a function f of two variables, with support inside the unit disk, from the knowledge of line integrals $(P_\theta f)(t) = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)ds$ for $\theta = \theta_1, \ldots, \theta_M \in [0, \theta_{\max}]$. The approximate reconstruction formula involves functions of one variable $\alpha_k(x\cos\theta_k + y\sin\theta_k)$ conveniently expanded as series of Chebyshev polynomials U_n (for $d-$dimensional problems, the convenient polynomials are the Gegenbauer polynomials $C_n^{(d/2)}$). A least squares search of the unknown coefficients of the U_n's leads to normal equations with a matrix of elements $U_n(\cos(\theta_k - \theta_m))/U_n(1)$. For a large number of equidistant allowed directions

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$\theta_k = k\theta_{\max}/M$, one recovers a Toeplitz matrix of elements close to $\dfrac{\sin((n+1)(k-m)\theta_{\max}/M)}{(n+1)(k-m)\theta_{\max}/M}$, i.e., $1/\alpha$ times the matrix A of the problem above, when $(n+1)\theta_{\max}/M = \alpha$.

The problem also appears in [9], where the authors study the robustness of a signal recovery procedure amounting to find the polynomial $p = p_0 + \cdots + p_N z^N$ minimizing the integral of $|f(\theta) - p(e^{i\theta})|^2$ on the circular arc shown above. This elementary least-squares problem involves again the Gram matrix $I - A$ of the problem above, and the stability of the recovery procedure is related to the size of the smallest eigenvalue of the matrix. The corresponding eigenvector is shown to have elements of the same sign. The theory of this eigenvalue-eigenvector pair should be more complete if it could be shown that $(I-A)^{-1}$ has only positive elements, for any $N = 1, 2, \dots,$ and any $\alpha \in (0, 1)$. It is also reported in [9, p. 644] that Grünbaum stated this conjecture as early as 1981.

$I - A$ is the Gram matrix $[\langle z^k, z^m \rangle], k, m = 0, 1, \dots, N$ of the weight $w = 1$ on the circular arc $\alpha\pi < \theta < 2\pi - \alpha\pi$: $\langle z^k, z^m \rangle = \displaystyle\int_{\alpha\pi}^{2\pi-\alpha\pi} \exp(i(k-m)\theta)) \frac{d\theta}{2\pi}$.

As each power z^k can be expanded in the basis of the orthogonal polynomials, it follows that the Gram matrix is the product of the triangular matrices built with the coefficients of these expansions. Therefore, the inverse of the Gram matrix is the product of the triangular matrices made with the coefficients of the reverse expansions of the orthogonal polynomials in the base of monomials (from [7, lemma 8.7.1]). It follows that the inverse of the Gram matrix is positive if all the orthogonal polynomials Φ_k, $k = 0, 1, \dots$ have positive coefficients.

Remark that these coefficients are real, from the symmetry of the weight function with respect to the real axis [31].

A direct proof of positivity [24] of $(I-A)^{-1}$ when $1/2 \leqslant \alpha < 1$ is done through the writing of the orthogonal polynomial $\Phi_n(z)$ as a multiple integral of a positive weight times $(z-\exp(i\theta_1)) \cdots (z-\exp(i\theta_n))$, where $\theta_1, \dots, \theta_n$ are anywhere on the arc $(\alpha\pi, 2\pi - \alpha\pi)$ ([28, § 16.2]). From the symmetry with respect to the real axis, one makes the average of the 2^n equivalent angles θ_k and $2\pi - \theta_k$, $k = 1, \dots, n$ resulting in the integral of $(z - \cos\theta_1) \cdots (z - \cos\theta_n)$, on $\alpha\pi < \theta_k < \pi$, and all the coefficients appear to be positive, as all the cosines are negative.

By a similar argument, one also has that all the zeros of Φ_n have a real part smaller than $\cos\alpha\pi$ (Fejér, see [28, chap. 16]), so that if $\alpha \geqslant 1/2$, all the zeros of Φ_n have negative real part, so $\Phi_n(0) = (-1)^n$ times the product of all the zeros must be > 0 (conjugate pairs have no influence on the sign, and the number of real zeros is n minus an even number).

For all the entries of all the $(I - A)^{-1}$ matrices to be positive, it is necessary that all the coefficients $\Phi_n(0) > 0$, $n = 1, \dots, N$, and the condition is known to be sufficient [9, p. 645]. This will be recalled as a consequence of the recurrence relation (3).

Here are some results containing the solution of the problem:

1.2. **Theorem.** *The monic polynomials*

$$\Phi_n(z) = z^n + \xi_n z^{n-1} + \cdots + \Phi_n(0), \quad n = 0, 1, \dots$$

which are orthogonal with respect to a constant weight on the arc of the unit circle $S = \{e^{i\theta}, \alpha\pi < \theta < 2\pi - \alpha\pi\}$, *with* $0 < \alpha < 1$, *have real coefficients satisfying the following inequalities:*

(1) $0 < \Phi_n(0) < \sigma$, $n = 1, 2, \dots,$ *where* $\sigma = \sin(\pi\alpha/2)$.
(2) $n\sigma^2 < \xi_n < (n-1)\sigma^2 + \sigma$, $n = 1, 2, \dots,$
(3) $n\Phi_n(0) < (n+1)\Phi_{n+1}(0)$, $n = 1, 2, \dots,$
(4) *for any integer* $n > 0$, $\Phi_n(0)$ *is an increasing function of* α,

1.3. **Conjecture.** *Under the same conditions as above,*

$$\Phi_n(0) < \Phi_{n+1}(0)\ , \qquad n = 1, 2, \dots$$

1.4. **Method of proof of the theorem.**

The proof mimics an algorithm of numerical calculation of the sequence $\{\Phi_n(0)\}$ through a (non linear) recurrence relation. It happens that a naive calculation based on an approximate value of $\Phi_1(0)$ produces unsatisfactory values, and that such numerical instabilities in recurrence calculations can be fixed

- In section 2, a recurrence relation for the $\Phi_n(0)$'s (***Freud equations***) will be produced,
- in section 3, the set of solutions of the latter recurrence relations will be shown to be a one-parameter set of sequences $\{\boldsymbol{x} = \{x_1, x_2, \dots\}\ \}$, each solution $\boldsymbol{x}$ being completely determined by x_1.
  It will also be shown that there is at most one positive solution.
- In section 4, for each $N = 1, 2, \dots,$ one will show how to construct the unique solution $\boldsymbol{x}^{(N)}$ satisfying $0 < x_n^{(N)} < \sigma$ for $n = 1, 2, \dots, N$ and $x_{N+1}^{(N)} = \sigma$.
- Finally, in section 5, we will see that, for each $n = 1, 2, \dots,$ $x_n^{(N)}$ decreases when N increases and reaches therefore a limit x_n^* with which we build a nonnegative solution $\boldsymbol{x}^*$. This solution will finally be shown to be positive, ensuring the long sought existence of the positive solution!

1.5. **Known results.**

1.5.1. *Asymptotic results.* There are many results on asymptotic behaviour [13,14,15, etc.], where it is shown that $\Phi_n(0) \to \sigma = \sin(\alpha\pi/2)$ when $n \to \infty$, for orthogonal polynomials on the arc above, with a weight which is positive almost everywhere.

In [13, § 6], Golinskii, Nevai, and Van Assche give asymptotic expansions of $\Phi_n(0)$ for several measures on the arc S, the simplest one being $d\mu(\theta) = \sin(\theta/2)\, d\theta$. Their result in this case is $\Phi_n(0) = \sigma - \dfrac{\cos(\alpha\pi/2)\cot(\alpha\pi/2)}{8n^2} + O(1/n^3)$, very likely valid in our case too.

More subtle asymptotic estimates are also of interest in random matrix theory [1, 30].

1.5.2. *Exact connection with orthogonal polynomials on an interval.* Famous identities found by Szegő [28, § 11.5] relate orthogonal polynomials on the unit circle with respect to a weight $w(\theta)$, with $w(\theta) = w(2\pi - \theta)$, to orthogonal polynomials on the real interval $x \in [-1, 1]$ with respect to the weights $(1 - x^2)^{\pm 1/2} w(\arccos x)$, where $x = \cos\theta$. When the support of w is an arc $\alpha\pi \leqslant \theta \leqslant 2\pi - \alpha\pi$, the actual support for x if $[-1, \cos\alpha\pi]$. If we want to discuss real orthogonal polynomials on the more usual interval $y \in [-1, 1]$, one must perform a further transformation $x = y\cos^2(\alpha\pi/2) - \sin^2(\alpha\pi/2)$, resulting in the rather awkward weight $[(1+y)(1+\sin^2(\alpha\pi/2) - y\cos^2(\alpha\pi/2))]^{\pm 1/2} w(\arccos[y\cos^2(\alpha\pi/2) - \sin^2(\alpha\pi/2)])\dots$

A more symmetrical transformation by Zhedanov [31], based on formulas of Delsarte and Genin, leads to orthogonal polynomials on $x \in [-1, 1]$ with respect to the weights $(1-k^2x^2)^{\pm 1/2} w(2\arccos(kx))$, where $k = \cos(\alpha\pi/2)$.

Polynomials which are orthogonal with respect to similar weights have been reported by Chihara [5, Heine and Rees, chap. 6, § 13, (A) and (G)], but these polynomials depend on implicit parameters which may be not easier than our $\Phi_n(0)$s...

1.6. **General identities of unit circle orthogonal polynomials.**

Monic polynomials orthogonal on the unit circle with respect to any valid measure $d\mu$:

$$\Phi_n(z) = z^n + \xi_n z^{n-1} + \cdots + \Phi_n(0)\ , \langle \Phi_n, \Phi_m \rangle = \int_0^{2\pi} \Phi_n(z)\overline{\Phi_m(z)}\, d\mu(\theta) = 0 \text{ if } m \neq n,\ \ (z = e^{i\theta})$$

satisfy quite a number of remarkable identities, most of them stated by Szegő in his book [28, § 11.3-11.4]. The central one is that, with

$$\Phi_n^*(z) = \overline{\Phi_n(0)}\, z^n + \cdots + \overline{\xi_n}\, z + 1,$$

$\Phi_n^*/\|\Phi_n\|^2$ is the kernel polynomial with respect to the origin:

$$\frac{\Phi_n^*(z)}{\|\Phi_n\|^2} = K_n(z;0) = \sum_{k=0}^{n} \frac{\overline{\Phi_k(0)}}{\|\Phi_k\|^2}\, \Phi_k(z) \tag{1}$$

implying

$$\|\Phi_{n+1}\|^2 = (1 - |\Phi_{n+1}(0)|^2)\, \|\Phi_n\|^2 \tag{2}$$

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z) \tag{3}$$

$$\langle \Phi_n, z^n \rangle = \|\Phi_n\|^2\ ; \langle \Phi_n, z^{-1} \rangle = -\Phi_{n+1}(0)\, \|\Phi_n\|^2. \tag{4}$$

For the last one: $\langle \Phi_n, z^{-1} \rangle = \langle z\Phi_n, 1 \rangle = -\Phi_{n+1}(0)\langle \Phi_n^*, 1 \rangle$, and $\langle \Phi_n^*(z), P(z) \rangle = \|\Phi_n\|^2 \langle K_n, P \rangle = \|\Phi_n\|^2\, \overline{P(0)}$ if P is a polynomial of degree $\leqslant n$.

$$\Phi_{n+1}^*(z) = \frac{\|\Phi_{n+1}\|^2}{\|\Phi_n\|^2}\Phi_n^*(z) + \overline{\Phi_{n+1}(0)}\Phi_{n+1}(z) \tag{5}$$

$$\Phi_{n+1}(z) = \frac{\|\Phi_{n+1}\|^2}{\|\Phi_n\|^2} z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z) \tag{6}$$

Finally, (3) yields expressions for the coefficients of z^{n-1} and z in $\Phi_n(z)$:

$$\xi_n = \xi_{n-1} + \Phi_n(0)\overline{\Phi_{n-1}(0)} = \Phi_1(0) + \Phi_2(0)\overline{\Phi_1(0)} + \cdots + \Phi_n(0)\overline{\Phi_{n-1}(0)} \tag{7}$$

$$\Phi_n'(0) = \Phi_{n-1}(0) + \Phi_n(0)\overline{\xi_{n-1}} = (1 - |\Phi_n(0)|^2)\Phi_{n-1}(0) + \Phi_n(0)\overline{\xi_n} \tag{8}$$

2. Recurrence relations (*Freud equations*).

2.1. The Laguerre-Freud equations. In looking for special non classical orthogonal polynomials related to continued fractions satisfying differential equations, Laguerre found some families of recurrence relations for the unknown coefficients. Among the people who rediscovered some of these relations, G. Freud showed how to achieve progress in analysis by deriving from these relations a proof of inequalities and asymptotic properties, see [4, 11, 22] for more.

For orthogonal polynomials on the unit circle, the crux of the matter is that the weight function satisfies

$$dw/d\theta = Rw, \tag{9}$$

where R is a rational function of $z = \exp(i\theta)$, the same rational function iP/Q on the whole unit circle, up to a finite number of points [2]. One shall also need that $Qw = 0$ at the endpoints of the support.

2.2. **The family of Legendre measures.**

Let us consider the measure $d\mu(\theta) = w(\theta)\dfrac{d\theta}{2\pi}$, with the following weight function:

$$\begin{aligned} w(\theta) &= A\,, \qquad \alpha\pi < \theta < 2\pi - \alpha\pi,\\ &= B\,, \qquad -\alpha\pi < \theta < \alpha\pi, \end{aligned} \tag{10}$$

with A and $B \geqslant 0$, $A + B > 0$.

Our problem deals only with $B = 0$, but we will need the full family (10) in a further discussion.

From symmetry with respect to the real axis, the polynomials Φ_n have real coefficients.

Let $Q(z) = (z - e^{i\alpha\pi})(z - e^{-i\alpha\pi}) = z^2 - 2\cos(\alpha\pi)z + 1 = 2z(\cos\theta - \cos(\alpha\pi))$.

2.3. **The differential relation for the orthogonal polynomials.**

We show that $Q\Phi_n'$ is a remarkably short linear combination of some Φs and Φ^*s [2]. To this end, we look at the integral of $\dfrac{d}{dz}[z^{-1}Q(z)f(z)\Phi_n(z^{-1})]$ on the two arcs of (10) for various polynomials f. Of course, the two integrals vanish, as Q vanishes at the endpoints. So,

$$\begin{aligned} 0 &= A\int_{e^{i\alpha\pi}}^{e^{-i\alpha\pi}} d[z^{-1}Q(z)f(z)\Phi_n(z^{-1})] + B\int_{e^{-i\alpha\pi}}^{e^{i\alpha\pi}} d[z^{-1}Q(z)f(z)\Phi_n(z^{-1})]\\ &= 2\pi i\int_0^{2\pi} z\frac{d}{dz}[z^{-1}Q(z)f(z)\Phi_n(z^{-1})]w(\theta)d\theta, \end{aligned}$$

as $dz = de^{i\theta} = iz\,d\theta$.

The value is also

$$\langle z(z^{-1}Qf)', \Phi_n\rangle - \langle z^{-2}Qf, \Phi_n'\rangle = 0.$$

The second scalar product is also $\langle f, Q\Phi_n'\rangle$, as $z^{-2}Q(z) = Q(z^{-1})$, so

$$\langle f, Q\Phi_n'\rangle = \langle z(z^{-1}Qf)', \Phi_n\rangle,$$

showing already that $Q\Phi_n'$ is a polynomial of degree $n+1$ which is orthogonal to $z, \dots, z^{n-2}$. By subtracting a suitable multiple of the kernel polynomial $Q\Phi_n' - X_nK_{n-1}$ is orthogonal to all the polynomials of degree $\leqslant n-2$, where $X_n = \langle Q\Phi_n', 1\rangle = \langle z - z^{-1}, \Phi_n\rangle = \Phi_{n+1}(0)\|\Phi_n\|^2$.

$$Q\Phi_n' = X_n\|\Phi_n\|^{-2}\Phi_{n-1}^* + n\Phi_{n+1} + Y_n\Phi_n + Z_n\Phi_{n-1}, \tag{11}$$

with the value of X_n found above, even when $n = 1$, as there is no other orthogonality constraint. The coefficient of Φ_{n+1} is obvious from the leading coefficient of $Q\Phi_n'$. By looking at the coefficient of z^n in the expansion of $Q\Phi_n'$, we get

$$Y_n = (n-1)\xi_n - 2n\cos(\alpha\pi) - n\xi_{n+1} = -\xi_n - 2n\cos(\alpha\pi) - n\Phi_{n+1}(0)\Phi_n(0).$$

For Z_n,

$$\begin{aligned} Z_n\|\Phi_{n-1}\|^2 &= \langle Q\Phi_n', \Phi_{n-1}\rangle - X_n\langle K_{n-1}, \Phi_{n-1}\rangle\\ &= \langle z(z^{-1}Q\Phi_{n-1})', \Phi_n\rangle - X_n\Phi_{n-1}(0)\\ &= \langle nz^n + \cdots - \Phi_{n-1}(0)z^{-1}, \Phi_n\rangle - X_n\Phi_{n-1}(0)\\ &= n\|\Phi_n\|^2. \end{aligned}$$

$$Q\Phi_n' = (1-\Phi_n(0)^2)\Phi_{n+1}(0)\Phi_{n-1}^* + n\Phi_{n+1} - [\xi_n + 2n\cos(\alpha\pi) + n\Phi_n(0)\Phi_{n+1}(0)]\Phi_n + n(1-\Phi_n(0)^2)\Phi_{n-1}$$

or also

(12) $Q\Phi_n' = (n+1)(1-\Phi_n(0)^2)\Phi_{n+1}(0)\Phi_{n-1}^* + [nz - \xi_n - 2n\cos(\alpha\pi)]\Phi_n + n(1-\Phi_n(0)^2)\Phi_{n-1}$

which we evaluate at $z=0$:

2.4. **Recurrence relation for $\Phi_n(0)$.**

$$(n+1)\Phi_{n+1}(0) - 2\frac{\xi_n + n\cos(\alpha\pi)}{1-\Phi_n(0)^2}\Phi_n(0) + (n-1)\Phi_{n-1}(0) = 0, \tag{13}$$

for $n = 1, 2, \dots$, and where $\xi_n = \Phi_1(0) + \Phi_1(0)\Phi_2(0) + \cdots + \Phi_{n-1}(0)\Phi_n(0)$.

Which is the recurrence relation determining $\Phi_{n+1}(0)$ from $\Phi_1(0), \dots, \Phi_n(0)$, and which will be discussed in more detail in the next section.

2.5. **Differential equation for Φ_n.** Now, (12) can be transformed into a differential system for Φ_n and Φ_n^*:

(14)

$$zQ(z)\Phi_n'(z) = [nQ(z) - (\xi_n + (n+1)\Phi_n(0)\Phi_{n+1}(0))z]\Phi_n(z) + [(n+1)\Phi_{n+1}(0)z - n\Phi_n(0)]\Phi_n^*(z)$$
$$Q(z)(\Phi_n^*)'(z) = [n\Phi_n(0)z - (n+1)\Phi_{n+1}(0)]\Phi_n(z) + [\xi_n + (n+1)\Phi_n(0)\Phi_{n+1}(0)]\Phi_n^*(z)$$

Remark that, when $Q(z) = 0$,

$$\frac{\Phi_n(e^{\pm i\alpha\pi})}{\Phi_n^*(e^{\pm i\alpha\pi})} = \exp[\mp in\alpha\pi + 2i\arg\Phi_n(e^{\pm i\alpha\pi})] = \frac{(n+1)\Phi_{n+1}(0) - n\Phi_n(0)e^{\mp i\alpha\pi})}{\xi_n + (n+1)\Phi_n(0)\Phi_{n+1}(0)},$$

which makes sense if

$$|\xi_n + (n+1)\Phi_n(0)\Phi_{n+1}(0)| = |(n+1)\Phi_{n+1}(0) - n\Phi_n(0)e^{\pm i\alpha\pi})|,$$

another interesting identity about the $\Phi_n(0)$'s. By squaring[2], one has

(15) $[\xi_n + (n+1)\Phi_n(0)\Phi_{n+1}(0)]^2 = (n+1)^2\Phi_{n+1}^2(0) - 2n(n+1)\Phi_n(0)\Phi_{n+1}(0)\cos(\alpha\pi) + n^2\Phi_n^2(0).$

Also, if one writes the system (14) as $\begin{bmatrix} zQ\Phi_n' \\ Q(\Phi^*)_n' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_n \\ \Phi_n^* \end{bmatrix}$, then $AD - BC = n\xi_n Q$, one gets the scalar differential equation for Φ_n, see [2, 19].

3. Properties of the solutions of the recurrence relations.

3.1. **The set of solutions.**

We now want to investigate all the solutions of the recurrence relation

$$(n+1)x_{n+1} - 2\frac{\xi_n + n\cos(\alpha\pi)}{1-x_n^2}x_n + (n-1)x_{n-1} = 0, \tag{16}$$

for $n = 1, 2, \dots$, where $\xi_n = x_1 + x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$.

Each solution is a sequence $\{x_1, x_2, \dots\}$ completely determined by the initial value x_1 (the value $x_0 = 1$ is common to all the solutions considered here).

[2]Squaring yields a proof by induction: take the identity at $n-1$ and add $2\{\xi_n + \Phi_n(0)[(n+1)\Phi_{n+1}(0) + (n-1)\Phi_{n-1}(0)]\}\Phi_n(0)[(n+1)\Phi_{n+1}(0) - (n-1)\Phi_{n-1}(0)]$, so, (15) appears as a kind of first integral of (13). The form (15) appears essentially in Adler and van Moerbeke [1], and in Forrester and Witte [10].

The particular solution we are interested in is determined by

$$x_1 = \Phi_1(0) = -\frac{\displaystyle\int_{\alpha\pi}^{(2-\alpha)\pi} e^{\pm i\theta}\, d\theta}{\displaystyle\int_{\alpha\pi}^{(2-\alpha)\pi} d\theta} = \frac{\sin(\alpha\pi)}{(1-\alpha)\pi}.$$

But as (13) is valid for all the weights (10), we find that x_n is the related $\Phi_n(0)$, and that x_1 is the ratio of moments

$$x_1 = -\frac{A\displaystyle\int_{\alpha\pi}^{(2-\alpha)\pi} e^{\pm i\theta}\, d\theta + B\int_{-\alpha\pi}^{\alpha\pi} e^{\pm i\theta}\, d\theta}{A\displaystyle\int_{\alpha\pi}^{(2-\alpha)\pi} d\theta + B\int_{-\alpha\pi}^{\alpha\pi} d\theta} = \frac{(A-B)\sin(\alpha\pi)}{A(1-\alpha)\pi + B\alpha\pi}, \tag{17}$$

relating A/B to any x_1 (and even negative values of A/B if $x_1 \notin [-\sin(\alpha\pi)/(\alpha\pi), \sin(\alpha\pi)/((1-\alpha)\pi)]$).

3.2. **Monotonicity with respect to x_1. Proposition.** *While x_1, $x_2, \dots x_{n-1}$ are positive and less than 1, and while x_n is positive, x_n is a continuous increasing function of x_1.*

Indeed, let us write the i^{th} equation of (16) as

$$\frac{(i+1)x_{i+1}}{ix_i} = 2\frac{x_1 + x_1x_2 + \cdots + x_{i-1}x_i + i\cos(\alpha\pi)}{i(1-x_i^2)} - \cfrac{1}{\cfrac{ix_i}{(i-1)x_{i-1}}},$$

for $i = 1, 2, \dots, n-1$. As $x_1, \dots, x_n$ are positive, and $1 - x_1^2, \dots, 1 - x_{n-1}^2$ are positive too, the numerators $\xi_i + i\cos(\alpha\pi)$ are positive too up to $i = n-1$. When $i = 1$, we see that x_2/x_1, and therefore x_2, is an increasing function of x_1.

If $2x_2/x_1, \dots, ix_i/((i-1)x_{i-1})$ are continuous positive increasing functions of x_1, then so is x_{i+1}/x_i, and therefore x_{i+1}, as the two terms of the right-hand side are increasing. □

We look at the evolution of a solution with respect to $x_1 \in (0, 1)$. We guess that if x_1 is too small, some x_n will be negative, and that if x_1 is too large, some x_n will be larger than 1.

3.3. **Unicity of positive solution. Proposition.** *The recurrence* (16) *has at most one positive solution.*

Indeed, we consider four possibilities for x_1, according to the ratio A/B in (17):

(1) $x_1 = \dfrac{\sin(\alpha\pi)}{(1-\alpha)\pi}$, corresponding to $B = 0$. This is the solution we hope to show to be positive.

(2) $-\dfrac{\sin(\alpha\pi)}{\alpha\pi} < x_1 < \dfrac{\sin(\alpha\pi)}{(1-\alpha)\pi}$, corresponding to $A > 0$ and $B > 0$. We then have a Szegő weight, with $x_n \to 0$ and ξ_n remaining bounded when $n \to \infty$. For n large, and $p = 0, 1, 2, \dots, P$ fixed, we have

$$\frac{x_{n+p+1}}{x_{n+p}} \sim 2\cos(\alpha\pi) - \cfrac{1}{2\cos(\alpha\pi) - \cfrac{1}{\ddots - \cfrac{x_{n-1}}{x_n}}} = \frac{\sin((p+1)\alpha\pi + \rho_n)}{\sin(p\alpha\pi + \rho_n)},$$

so that $x_{n+p} \sim C_n \sin(\rho_n + p\alpha\pi)$, $p = 0, 1, \dots, P-1$. We now choose P so that $P\alpha$ is close to an even integer. The sines must change their signs, as the sum of these P values is close to zero (actually, is $o(C_n)$).

(3) $x_1 = -\dfrac{\sin(\alpha\pi)}{\alpha\pi}$, corresponds to $A = 0$, and has of course no chance, as x_1 is already negative!

(4) $x_1 \notin \left[-\dfrac{\sin(\alpha\pi)}{\alpha\pi}, \dfrac{\sin(\alpha\pi)}{(1-\alpha)\pi}\right]$, corresponds to a non positive weight $A/B < 0$, and we will either encounter a negative x_n, or $x_n > 1$, but then $x_{n+1} < 0$ [3].

That means that if we succeed in constructing a positive solution of (16), this solution will have to be of the type 1) above, and that will be the proof of positivity of the sought solution.

4. Construction of a positive solution for $n = 1, 2, ..., N+1$.

4.1. Iteration of positive sequences.

As it is so difficult to "push" a positive solution through an starting value x_1, we try to build a positive solution of (16) through an iterative process keeping positive sequences. A good start is to write (16) as

$$x_n = \sqrt{A_n^2(\boldsymbol{x})+1} - A_n(\boldsymbol{x}) = \frac{1}{\sqrt{A_n^2(\boldsymbol{x})+1} + A_n(\boldsymbol{x})}, \quad n = 1, 2, \dots \tag{18}$$

where

$$A_n(\boldsymbol{x}) = \frac{x_1 + x_1x_2 + \cdots + x_{n-1}x_n + n\cos(\alpha\pi)}{(n-1)x_{n-1} + (n+1)x_{n+1}}.$$

Indeed, consider (16) as an equation of degree two for x_n

$$x_n^2 + 2A_n(\boldsymbol{x})\,x_n - 1 = 0,$$

and take the unique positive root, which is (18).

Therefore, the positive solution of (16), if it exists, must satisfy (18), and if we find a (positive, of course) sequence satisfying (18), we will have found the unique positive solution of (16).

One may then consider to iterate (18), hoping to see it to converge towards the long sought positive solution.

Heavy numerical experiments (see [23, § 4.2]) suggest that convergence indeed holds, but that no easy proof is at hand. Moreover, some inequalities of Theorem 1.2 do not hold for intermediate steps of application of (18).

A modified iterative scheme will be much more satisfactory:

4.2. An iteration of finite positive sequences.Proposition.

- *For any $\alpha \in (0,1)$ and $\varepsilon \geqslant 0$, the function $\boldsymbol{F}^{(N,\varepsilon)}$ acting on a sequence $\boldsymbol{x} = \{x_n\}_1^\infty$ by*

(19)

$$F_n^{(N,\varepsilon)} = \sqrt{[A_n^{(N,\varepsilon)}\boldsymbol{x})]^2+1} - A_n^{(N,\varepsilon)}(\boldsymbol{x}) = \frac{1}{\sqrt{[A_n^{(N,\varepsilon)}(\boldsymbol{x})]^2+1} + A_n^{(N,\varepsilon)}(\boldsymbol{x})}, \quad n = 1, 2, \dots, N$$

$$= \sigma, \qquad n = N+1, N+2, \dots$$

where $\sigma = \sin\dfrac{\alpha\pi}{2}$, and

$$A_n^{(N,\varepsilon)}(\boldsymbol{x}) = \frac{N\sigma^2 + \varepsilon - x_nx_{n+1} - \cdots - x_{N-1}x_N + n\cos(\alpha\pi)}{(n-1)x_{n-1} + (n+1)x_{n+1}}, \quad n = 1, 2, \dots, N, \tag{20}$$

transforms a positive sequence into a positive sequence;
if $\boldsymbol{x} \geqslant \boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{x})$ (element-wise), then, $\boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{x}) \geqslant \boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{x}))$ when $\varepsilon \geqslant 0$.

---

[3] If x_{n-2}, x_{n-1}, and x_n are positive, with $x_{n-1} < 1$, then $\xi_{n-1} + (n-1)\cos(\alpha\pi) > x_n/x_{n-1} - x_{n-1}x_n$, using (16) with $n-1$. So, $\xi_n + n\cos(\alpha\pi) = \xi_{n-1} + (n-1)\cos(\alpha\pi) + x_{n-1}x_n + \cos(\alpha\pi) > x_n/x_{n-1} + \cos(\alpha\pi) > 0$, and $x_{n+1} < 0$

- *Iterations of* $\boldsymbol{F}^{(N,\varepsilon)}$*, starting with the constant sequence* $x_n = \sigma, n = 1, 2, \ldots,$ *converge to a positive fixed point* $\boldsymbol{x}^{(N,\varepsilon)}$ *of* $\boldsymbol{F}^{(N,\varepsilon)}$*, i.e., a positive solution of*

$$(n+1)x_{n+1} - 2\frac{N\sigma^2 + \varepsilon - x_n x_{n+1} - \cdots - x_{N-1}x_N + n\cos(\alpha\pi)}{1 - x_n^2}x_n + (n-1)x_{n-1} = 0, \tag{21}$$

for $n = 1, 2, \ldots, N$*, and* $x_n = \sigma$ *for* $n > N$*.*
- *For any* $\varepsilon \geqslant 0$*, we now consider the function*

$$f_N(\varepsilon) = N\sigma^2 + \varepsilon - x_1 - x_1x_2 - \cdots - x_{N-1}x_N$$

built with the sequence $\{x_1^{(N,\varepsilon)}, \ldots, x_N^{(N,\varepsilon)}\}$ *found above. The set of equations* (21) *can also be written as*

$$(n+1)x_{n+1} - 2\frac{f_N(\varepsilon) + \xi_n + n\cos(\alpha\pi)}{1 - x_n^2}x_n + (n-1)x_{n-1} = 0, \tag{22}$$

Then, f_N *is an increasing function,* $f_N(0) = \sigma^2 - \sigma < 0$*,* $f_N(\varepsilon) \geqslant \sigma^2 - \sigma + \varepsilon$*, so that there is a unique positive zero* ε_N *of* f_N*, and the found positive solution* $\boldsymbol{x}^{(N,\varepsilon_N)}$ *of* (21) *is then the positive solution* $\boldsymbol{x}^{(N)}$ *of the equations* (16) *for* $n = 1, 2, \ldots, N$*, and* $x_{N+1} = \sigma$*.*

Indeed, whenever $\boldsymbol{x}$ is a positive sequence, each $A_n^{(N,\varepsilon)}(\boldsymbol{x})$ is a decreasing function of the x_i's, therefore, $F_n^{(N,\varepsilon)}(\boldsymbol{x})$ is an increasing function of $\boldsymbol{x}$.

Next, the constant positive sequence $x_n = \sigma$, $n = 1, 2, \ldots$ satisfies $\boldsymbol{x} \geqslant \boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{x})$, as $A_n^{(N,\varepsilon)}(\boldsymbol{x}) = \dfrac{n\sigma^2 + \varepsilon + n\cos(\alpha\pi)}{2n\sigma} \geqslant \dfrac{\sigma^{-1} - \sigma}{2}, n = 1, 2, \ldots, N$, from (20), and $\cos(\alpha\pi) = 1 - 2\sigma^2$.

Each x_n will therefore decrease at each new iteration of $F_n^{(N,\varepsilon)}$, and will reach a nonnegative limit called $x_n^{(N,\varepsilon)}$, which satisfies (22), as stated above. Remark that this limit is not only nonnegative, but actually positive: if $x_1^{(N,\varepsilon)} = 0$, then $x_n^{(N,\varepsilon)} = 0$ for all $n > 0$; if $x_{n-1}^{(N,\varepsilon)} > 0$, and $x_n^{(N,\varepsilon)} = 0$, with $n > 0$,then $x_{n+1}^{(N,\varepsilon)} < 0$, and we could not have $x_{N+1} = \sigma$.

We also have $x_n^{(N,\varepsilon)} < \sigma$ if $\varepsilon > 0$.

Finally, we compare the values of some x_n when the iterations (19-20) are performed with two different values of ε, and find that x_n is a decreasing function of ε, whence the increasing character of the function f_N.

□

Much more general iterations with monotony properties are worked in Chapter 3 of Collatz' book [6].

5. Final limit process.

5.1. **Proposition .** *The sequence* $\boldsymbol{x}^{(N)}$ *built above as the unique positive solution of* (16) *for* $n = 1, 2, \ldots, N$ *with* $x_{N+1} = \sigma$*, decreases when* N *increases and converges to the unique positive solution* $\boldsymbol{x}$ *of* (16)*, whose existence had to be established.*

Indeed, from $x_{N+1}^{(N)} = \sigma$, and $x_{N+1}^{(N+1)} < \sigma$, $x_1^{(N+1)} < x_1^{(N)}$ must follow, from Proposition 3.2, and then $x_n^{(N+1)} < x_n^{(N)}$ for all $n \leqslant N + 1$.

Moreover, $\boldsymbol{x}$ is actually positive, and not merely nonnegative, as $x_n < \sigma$ and $\varepsilon_N > 0 \Rightarrow 0 > N\sigma^2 + \varepsilon_N - x_1^{(N)} - (N-1)\sigma^2$: $x_1 > \sigma^2$. And, as we saw above, we can not have $x_{n-1} > 0, x_n = 0$, and $x_{n+1} \geqslant 0$.

This achieves the proof of (1-2) of Theorem 1.2.

5.2. **Numerical illustration and software.**

we choose $\alpha = 1/4$, then $\sigma = \sin(\alpha\pi/2) = 0.382683...$,
We iterate $F^{(5,0.01)}$, starting with the constant sequence $x_n = \sigma$:

it.	res.	x1	x2	x3	x4	x5	x6
1	0.01306	0.38268	0.38268	0.38268	0.38268	0.38268	0.38268
2	0.01053	0.37937	0.38102	0.38157	0.38185	0.38201	0.38268
3	0.00960	0.37673	0.37939	0.38060	0.38118	0.38176	0.38268
4	0.00804	0.37436	0.37803	0.37975	0.38076	0.38157	0.38268
5	0.00679	0.37239	0.37686	0.37913	0.38041	0.38144	0.38268
6	0.00542	0.37074	0.37594	0.37860	0.38017	0.38134	0.38268
7	0.00445	0.36943	0.37517	0.37820	0.37996	0.38126	0.38268
8	0.00352	0.36837	0.37457	0.37787	0.37980	0.38120	0.38268
9	0.00285	0.36753	0.37408	0.37761	0.37968	0.38116	0.38268
10	0.00226	0.36685	0.37370	0.37740	0.37958	0.38112	0.38268

where "res" is the norm of the residue at the particular iteration step, i.e., the largest absolute value of the left-hand sides of (21), $n = 1, 2, \ldots, N$. This error norm decreases rather slowly, and we proceed up to the reception of a reasonably small value:

it.	res.	x1	x2	x3	x4	x5	x6
50	0.00000	0.36420	0.37218	0.37659	0.37918	0.38097	0.38268

one finds $f_5(0.01) = -0.18493$. We already knew that $f_5(0) = \sigma^2 - \sigma = -0.23623...$
We start the whole process again with various values of ε:

eps.	f(eps)	x1	x2	x3	x4	x5	x6
0	-0.23623	0.38268	0.38268	0.38268	0.38268	0.38268	0.38268
0.01	-0.18493	0.36420	0.37218	0.37659	0.37918	0.38097	0.38268
0.02	-0.13634	0.34700	0.36206	0.37061	0.37571	0.37927	0.38268
0.03	-0.09021	0.33097	0.35231	0.36474	0.37228	0.37758	0.38268
0.04	-0.04633	0.31600	0.34291	0.35898	0.36889	0.37591	0.38268
0.05	-0.00450	0.30200	0.33384	0.35333	0.36552	0.37424	0.38268
0.06	0.03544	0.28888	0.32509	0.34778	0.36220	0.37259	0.38268

we find $\varepsilon_5 = 0.0511$, and perform the whole thing again for several values of N:

N	eps	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
5	0.05110	0.30051	0.33286	0.35271	0.36516	0.37406	0.38268				
6	0.04124	0.30024	0.33242	0.35194	0.36370	0.37118	0.37682	0.38268			
7	0.03443	0.30015	0.33227	0.35167	0.36319	0.37019	0.37482	0.37853	0.38268		
8	0.02953	0.30012	0.33221	0.35157	0.36301	0.36984	0.37411	0.37707	0.37962	0.38268	
9	0.02585	0.30011	0.33219	0.35154	0.36295	0.36971	0.37385	0.37654	0.37852	0.38034	0.38268
10	0.02299	0.30011	0.33219	0.35152	0.36292	0.36967	0.37376	0.37634	0.37810	0.37948	0.38084

And we see that we have indeed reconstructed $x_1 = \Phi_1(0) = \dfrac{\sin(\alpha\pi)}{(1-\alpha)\pi} = 0.3001054....$

The gp-pari [3] program used here can be found at
`http://www.math.ucl.ac.be/~magnus/freud/grunbd.gp`.
A more experimental program, allowing Gegenbauer polynomials too is at
`http://www.math.ucl.ac.be/~magnus/freud/grunb2.gp`.
There is also a java program available at
`http://www.math.ucl.ac.be/~magnus/freud/grunbd.htm`.
The numerical efficiency of this demonstration is close to zero! Should somebody really need a long subsequence of the $\Phi_n(0)$'s, a Newton-Raphson procedure should be built, as in [21].

5.3. **Proof of (3) of Theorem 1.2.** We show that, if $\boldsymbol{x}$ is a positive sequence bounded by σ, and with nx_n increasing with n, then the same holds for $\boldsymbol{F}^{(N,\varepsilon)}(\boldsymbol{x})$. Indeed, by (19),

$$nF_n = \frac{1}{\dfrac{A_n}{n} + \sqrt{\left(\dfrac{A_n}{n}\right)^2 + \dfrac{1}{n^2}}}$$

is increasing if A_n/n is decreasing. Now, by (20),

$$\frac{A_n}{n} = \frac{y_n + \cos(\alpha\pi)}{(n-1)x_{n-1} + (n+1)x_{n+1}},$$

$$\text{where } y_n = \frac{N\sigma^2 + \varepsilon - x_n x_{n+1} - \cdots - x_{N-1}x_N}{n},$$

has an increasing denominator, and a decreasing numerator. Indeed,

$$y_{n+1} - y_n = \frac{(n+1)y_{n+1} - ny_n - y_{n+1}}{n} = \frac{x_n x_{n+1} - y_{n+1}}{n} < 0,$$

as $x_n < \sigma$ and $\varepsilon > 0 \Rightarrow y_n > \sigma^2$. □

6. Differential equations with respect to α.

Let Φ_n and $\tilde{\Phi}_n$ be the monic orthogonal polynomials of degree n with respect to the measures $d\mu$ and $d\tilde{\mu}$. As any polynomial of degree $n-1$, $\tilde{\Phi}_n - \Phi_n$ is represented through the kernel polynomial K_{n-1}:

$$\tilde{\Phi}_n(z) - \Phi_n(z) = \int_{|t|=1} (\tilde{\Phi}_n(t) - \Phi_n(t)) K_{n-1}(z,t)\, d\mu.$$

We may suppress in the integral Φ_n, which is orthogonal to K_{n-1}; and replace $d\mu$ by $d\mu - d\tilde{\mu}$, as $\tilde{\Phi}_n$ is orthogonal to K_{n-1} with respect to $d\tilde{\mu}$:

$$\tilde{\Phi}_n(z) = \Phi_n(z) - \int_{|t|=1} \tilde{\Phi}_n(t) K_{n-1}(z,t)\, (d\tilde{\mu} - d\mu).$$

sometimes called the Bernstein integral equation for $\tilde{\Phi}_n$ [26], also Bernstein-Korous identity by Golinskii [14, eq. (70)]. Here, $d\tilde{\mu} - d\mu$ only lives on small neighborhoods of $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$, and

$$\frac{\partial \Phi_n(z)}{\pi \partial \alpha} = (A-B)[\Phi_n(e^{i\alpha\pi})K_{n-1}(z,e^{i\alpha\pi}) + \Phi_n(e^{-i\alpha\pi})K_{n-1}(z,e^{-i\alpha\pi})] \tag{23}$$

At $z = 0$:

$$\begin{aligned} \frac{d\Phi_n(0)}{\pi d\alpha} &= (A-B)[\Phi_n(e^{i\alpha\pi})K_{n-1}(0,e^{i\alpha\pi}) + \Phi_n(e^{-i\alpha\pi})K_{n-1}(0,e^{-i\alpha\pi})] \\ &= (A-B)\|\Phi_{n-1}\|^{-2}[\Phi_n(e^{i\alpha\pi})\overline{\Phi^*_{n-1}(e^{i\alpha\pi})} + \overline{\Phi_n(e^{-i\alpha\pi})}\Phi^*_{n-1}(e^{-i\alpha\pi})] \end{aligned} \tag{24}$$

relating $\Phi_n(0)$ to values at $e^{\pm i\alpha\pi}$, which may not be easier. However,

$$\frac{d\Phi_n(0)}{\pi d\alpha} = (A-B)\frac{|\Phi_{n-1}(e^{i\alpha\pi})|^2}{\|\Phi_{n-1}\|^2}\left[\frac{\Phi_n(e^{i\alpha\pi})}{\Phi^*_{n-1}(e^{i\alpha\pi})} + \frac{\overline{\Phi_n(e^{i\alpha\pi})}}{\overline{\Phi^*_{n-1}(e^{i\alpha\pi})}}\right],$$

and we know that

$$\begin{aligned} \frac{\Phi_n(e^{i\alpha\pi})}{\Phi^*_{n-1}(e^{i\alpha\pi})} &= e^{i\alpha\pi}\frac{\Phi_{n-1}(e^{i\alpha\pi})}{\Phi^*_{n-1}(e^{i\alpha\pi})} + \Phi_n(0) \\ &= \frac{n\Phi_n(0)e^{i\alpha\pi} - (n-1)\Phi_{n-1}(0)}{\xi_n + (n-1)\Phi_{n-1}(0)\Phi_n(0)} + \Phi_n(0), \end{aligned}$$

and

$$\frac{d\Phi_n(0)}{d\alpha} = \pi(A-B)[1-\Phi_n^2(0)]\frac{|\Phi_{n-1}(e^{i\alpha\pi})|^2}{\|\Phi_{n-1}\|^2}\frac{(n+1)\Phi_{n+1}(0)-(n-1)\Phi_{n-1}(0)}{\xi_n+(n-1)\Phi_{n-1}(0)\Phi_n(0)} \tag{25}$$

which achieves the proof of (4) of Theorem 1.2. □

We certainly would like more explicit differential relations and equations (Painlevé!) with respect to α here! There are such relations in [10, 30],

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References

[1] M. Adler, P. van Moerbeke, Recursion relations for Unitary integrals, Combinatorics and the Toeplitz Lattice, *Commun. Math. Phys.* **237** (2003) 397-440; also preprint `math-ph/0201063`, 2002

[2] M. Alfaro, F. Marcellán, Recent trends in orthogonal polynomials on the unit circle. Orthogonal polynomials and their applications (Erice, 1990), 3–14, *IMACS Ann. Comput. Appl. Math.*, **9**, 1991.

[3] C. Batut, K. Belabas, D. Bernardi, H. Cohen, M. Olivier, *The PARI-GP calculator*, guides and software at `http://pari.math.u-bordeaux.fr/`

[4] S. Belmehdi, A. Ronveaux, Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials. *J. Approx. Theory* **76** (1994), no. 3, 351–368.

[5] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[6] L. Collatz, *Functional Analysis and Numerical Mathematics*, Ac. Press, 1966.

[7] P.J. Davis, *Interpolation and Approximation*, Blaisdell, Waltham, 1963 = Dover, New York, 1975.

[8] Davison, M. E.; Grünbaum, F. A., Tomographic reconstruction with arbitrary directions. *Comm. Pure Appl. Math.* **34** (1981), no. 1, 77–119.

[9] Ph. Delsarte, A.J.E.M. Janssen, L.B. Vries, Discrete prolate spheroidal wave functions and interpolation, *SIAM J. Appl. Math.* **45** (1985) 641-650.

[10] P.J. Forrester, N.S. Witte, Discrete Painlevé equations, orthogonal polynomials on the unit circle and $N-$recurrences for averages over $U(N)-$ $\mathrm{P_{VI}}$ $\tau-$functions, preprint, `http://www.arXiv.org/abs/math-ph/0308036`, 2003.

[11] G.Freud, On the coefficients in the recursion formulæ of orthogonal polynomials, *Proc. Royal Irish Acad. Sect. A* **76** (1976), 1-6.

[12] L.B. Golinskii, Reflection coefficients for the generalized Jacobi weight functions, *J. Approx. Th.* **78** (1994) 117-126.

[13] L. Golinskii, P. Nevai, W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, *J. Approx. Th.* **83** (1995) 392-422.

[14] L. Golinskii, Akhiezer's orthogonal polynomials and Bernstein-Szegő method for a circular arc, *J. Approx. Th.* **95** (1998) 229-263.

[15] L. Golinskii, P. Nevai, F. Pinter, W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle II, *J. Approx. Th.* **96** (1999) 1-33.

[16] L. Golinskii, On the scientific legacy of Ya. L. Geronimus (to the hundredth anniversary), pp. 273-281 in *Self-Similar Systems*, edited by V.B. Priezzhev and V.P. Spiridonov, Joint Institute for Nuclear Research, Dubna, 1999.

[17] Grünbaum, F. Alberto, The limited angle reconstruction problem. *Computed tomography (Cincinnati, Ohio, 1982)*, pp. 43–61, *Proc. Sympos. Appl. Math.*, **27**, Amer. Math. Soc., Providence, R.I., 1982.

[18] A. Grünbaum, Problem # 3 of *SIAM Activity Group on Orthogonal Polynomials and Special Functions Newsletter*, Summer 1993.

[19] M.E.H. Ismail, N.S. Witte, Discriminants and functional equations for polynomials orthogonal on the unit circle, *J. Approx. Th.* **110** (2001) 200-228.

[20] S.V. Khrushchev, Classification theorems for general orthogonal polynomials on the unit circle, *J. Approx. Th.* **116** (2002) 268-342.

[21] J.S.Lew, D.A.Quarles, Nonnegative solutions of a nonlinear recurrence, *J. Approx. Th.* **38** (1983), 357-379.

[22] A.P. Magnus, Freud's equations for orthogonal polynomials as discrete Painlevé equations, pp. 228-243 in *Symmetries and Integrability of Difference Equations*, Edited by Peter A. Clarkson & Frank W. Nijhoff, Cambridge U.P., Lond. Math. Soc. Lect. Note Ser. 255, 1999.

[23] A.P. Magnus, MAPA 3072A Special topics in approximation theory: Semi-classical orthogonal polynomials on the unit circle, unpublished lecture notes, Univ. of Louvain, Louvain-la-Neuve, 1999-2000, 2002-2003. Available in `http://www.math.ucl.ac.be/~magnus/num3/m3xxx99.pdf`

[24] J. Meinguet, private communication, Spring 2003.

[25] P. Nevai, Orthogonal polynomials, measures and recurrences on the unit circle, *Trans. Amer. Math. Soc.* **300** (1987), no. 1, 175–189.

[26] J. Nuttall, S.R. Singh, Orthogonal polynomials and Padé approximants associated to a sustem of arcs, *J. Approx. Theory* **21** (1977), 1-42.

[27] B. Simon, *Orthogonal polynomials on the unit circle. Part 1. Classical theory*. American Mathematical Society Colloquium Publications, **54**, Part 1. American Mathematical Society, Providence, RI, 2005; *Part 2. Spectral theory*. American Mathematical Society Colloquium Publications, **54**, Part 2. American Mathematical Society, Providence, RI, 2005.

[28] G. Szegő, *Orthogonal Polynomials*, 3^{rd} ed., Amer. Math. Soc. , Colloquium Publications, vol. **23**, Providence, 1967.

[29] J.P. Thiran, C. Detaille, Chebyshev polynomials on circular arcs in the complex plane, pp. 771-786 *in Progress in Approximation Theory* (P. Nevai & A. Pinkus, editors), Ac. Press, 1991.

[30] N. S. Witte, private communication, July 2003.

[31] A. Zhedanov, On some classes of polynomials orthogonal on arcs of the unit circle connected with symmetric orthogonal polynomials on an interval, *J. Approx. Th.* **94** (1998) 73-106.