2. Grenander,U., Szego,G., Toeplitz Forms and their Applications, Univ. of California Press, Berkeley-Los Angeles, 1958.
3. López,G., Conditions for convergence of multipoint Padé approximants for functions of Stieltjes type, Mat. Sb. 107(149), 69-83,1978; Eng. translation in Math. USSR Sb. 35, 363-376, 1979.
4. López,G., On the convergence of Pade approximants for meromorphic functions of Stieltjes type, Mat. Sb. 111(153), 308-316, 1980; Eng. translation in Math. USSR Sb. 38, 281-288, 1981.
5. López,G., On the asymptotics of orthogonal polynomials and the convergence of multipoint Pade approximants, Mat. Sb. 128(170), 216-229, 1985; to appear translated in Math. USSR Sb..
6. López,G., Asymptotics of polynomials orthogonal with respect to varying measures, (submitted to Const. Approx. Theory).
7. López,G., Unified approach to asymptotics of sequences of orthogonal polynomials on finite and infinite intervals. (submitted to Mat. Sb.).
8. Szego,G., Orthogonal Polynomials, Coll. Pub. XX , Providence, R.I..

ASSOCIATED ASKEY-WILSON POLYNOMIALS AS LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS

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ABSTRACT

One looks for [formal] orthogonal polynomials satisfying interesting differential or difference relations and equations (Laguerre-Hahn theory). The divided difference operator used here is essentially the Askey-Wilson operator

$$Df(x) = \frac{E_2 f(x) - E_1 f(x)}{E_2 x - E_1 x} = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}$$

where $y_1(x)$ and $y_2(x)$ are the two roots of $Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + f = 0$.

The related Laguerre-Hahn orthogonal polynomials are then introduced as the denominators $P_0, P_1, \ldots$ of the successive approximants Q_n/P_n of the Gauss-Heine-like continued fraction $f(x) = 1/(x-r_0-s_1/(x-r_1-s_2/\ldots))$ satisfying the Riccati equation $a(x)Df(x) = b(x)E_1f(x)E_2f(x) + c(x)Mf(x) + d(x)$ where a,b,c,d are polynomials and $Mf(x) = (E_1f(x) + E_2f(x))/2$.

In the classical case (degrees $a,b,c,d \leq 2,0,1,0$) , closed-forms for the recurrence coefficients r_n and s_n are obtained and show that we are dealing essentially with the associated Askey-Wilson polynomials.

One finds for P_n difference relations $(a_n+a)DP_n = (c_n-c)MP_n + 2s_nd_nMP_{n-1} - 2bMQ_n$ and a writing in terms of solutions of linear second order difference equations $P_n = (X_nY_{-1}-Y_nX_{-1})/(X_0Y_{-1}-Y_0X_{-1})$.

1. INTRODUCTION. DIFFERENCE OPERATORS AND EQUATIONS

Classical orthogonal polynomials are solutions of remarkable differential relations and equations. This makes them very useful as elements in the representation of solutions of problems of mathematical physics, numerical analysis, etc. (see [N1] Chap. 2, [WP] Chap. 11). Laguerre ([LA] , see also [AT] , [MC]) found a systematic way of generating orthogonal polynomials satisfying differential relations and equations. A new setting of his theory will be given here.

Classical orthogonal polynomials of a discrete variable are also used in the same fields. Now, they satisfy difference relations and equations where the fundamental difference operator is $\Delta f(x) =$

$= f(x+1)-f(x)$. It happens that the difference formulas are very similar to the differential ones ([N1] , Chap. 2, Sect. 12).

Another interesting difference operator is $\Delta_q f(x) = f(qx)-f(x)$ ([HE] p. 99 , [H2]), also used in orthogonal polynomials theory ([H1] , [H2] , [H3]).

By investigating new families of orthogonal polynomials, Wilson considered the following divided difference operator (see [AS] , Sect. 5) : put $x = a+bt^2$, then

$$\Delta_W f(x) = [f(a+b(t+1)^2)-f(a+b(t-1)^2)]/t$$

Finally, Askey and Wilson ([AS] Sect. 5) found a further extension: if $x = a+b(q^m+q^{-m})$, then

$$\Delta_{AW} f(x) = [f(a+b(q^{m+1/2}+q^{-m-1/2}))-f(a+b(q^{m-1/2}+q^{-m+1/2}))]/(q^m-q^{-m}) .$$

Each of these operators is an extension of the preceding one, which can be recovered as a special case and/or a limit case, up to a linear transformation of the variable. For instance, $\Delta f(x)$ is the limit when $q \to 1$ of Δ_q acting on X of the function $f(X-1/(q-1))$ with $X = x+1/(q-1)$... A general form of operator avoiding these troubles is given now.

We consider the most general divided difference operator of the form

$$Df(x) = \frac{E_2 f(x)-E_1 f(x)}{E_2 x-E_1 x} = \frac{f(y_2(x))-f(y_1(x))}{y_2(x)-y_1(x)} \qquad (1.1)$$

leaving a polynomial of degree $n-1$ when f is a polynomial of degree n . The two first non obvious conditions are

$$y_1(x) + y_2(x) = \text{polynomial of degree } 1$$

and $(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2$ = polynomial of degree 2

equivalent to $y_1(x)y_2(x)$ = polynomial of degree ≤ 2 .

This defines y_1 and y_2 as the two roots of an equation of the form

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + f = 0 , \qquad A \neq 0 . \qquad (1.2)$$

Then, $y^{n+1} = (y_1+y_2)y^n-y_1y_2y^{n-1}$, i.e., $Dx^{n+1} = (y_1+y_2)Dx^n - y_1y_2Dx^{n-1}$ shows that Dx^n is indeed a polynomial of degree $\leq n$ (see [MR] Section 2 for a similar derivation). Conditions ensuring Dx^n to be of exact degree $n-1$ will be given in a moment.

The most important identities involving y_1 and y_2 are

$$y_1 + y_2 = -2(Bx+d)/a \qquad (1.3a)$$

$$y_1y_2 = (Cx^2 + 2Ex + F)/A \qquad (1.3b)$$

$$(y_2-y_1)^2 = 4[(B^2-AC)x^2 + 2(BD-AE)x + D^2-AF]/A^2 \qquad (1.3c)$$

$$y_1,y_2 = -(Bx+D)/A \pm \left[(B^2-AC)\left(x + \frac{BD-AE}{B^2-AC}\right)^2 + \frac{A\theta}{B^2-AC}\right]^{1/2}$$

$$\text{if } B^2 \neq AC \qquad (1.4)$$

where

$$\theta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} .$$

In general ($B^2 \neq AC$ and $\theta \neq 0$) , we recover Δ_{AW} ; Δ_q if $B^2 \neq AC$ and $\theta = 0$; Δ_W if $B^2 = AC$ and $\theta \neq 0$; Δ if $B^2 = AC$ and $\theta = 0$ (implying $BD = AE$) . Only the differential operator d/dx must still be considered as a limit case.

The companion operator to D is

$$Mf(x) = (E_1 f(x) + E_2 f(x))/2 .$$

One has

$$D(fg) = Df\,Mg + Mf\,Dg ,$$
$$M(fg) = Mf\,Mg + (y_2-y_1)^2/4\,Df\,Dg ,$$
$$D(1/f) = -Df/(E_1 f\,E_2 f) .$$

A first-order difference equation is a link between $E_1 f(x) = f(y_1(x))$ and $E_2 f(x) = f(y_2(x))$ or , if we call $X = y_1(x)$, between $f(X)$ and $Ef(X) = f(y_2(y_{-1}(X)))$, where y_{-1} and y_{-2} are the inverse functions of y_1 and y_2 (this requires $C \neq 0$) :

$$y_{-1}(y_1(x)) \equiv x \quad , \quad y_{-2}(y_2(x)) \equiv x \quad , \quad Ef(x) = f(y_2(y_{-1}(x)))$$
$$E^{-1}f(x) = f(y_1(y_{-2}(x))) \quad , \quad E_{-1}f(x) = f(y_{-1}(x)) \quad ,$$
$$E_{-2}f(x) = f(y_{-2}(x)) ,$$
$$E = E_{-1}E_2 = (E_1)^{-1}E_2 \quad , \quad E^{-1} = E_{-2}E_1 = (E_2)^{-1}E_1$$

This introduces the adjoint operators

$$D^*f(x) = (E_{-2}f(x) - E_{-1}f(x))/(y_{-2}(x)-y_{-1}(x)) ,$$
$$M^*f(x) = (E_{-1}f(x) + E_{-2}f(x))/2$$

Remark that , as $y_1(x) + y_2(x) = -2(Bx+D)/A$,

$$Ex+x = -2(By_{-1}(x) + D)/A \quad , \quad E^{-1}x+x = -2(By_{-2}(x)+D)/A \quad . \qquad (1.5)$$

A difference equation links values on several powers $E^k(X)$, which are situated on the lattice discussed by Nikiforov and Suslov [NI], indeed, consider

$$EX + E^{-1}X = -2(B(y_{-1} + y_{-2})+2D)/A-2X = 2(2\frac{B^2}{AC} - 1)X + 4(BE-CD)/(AC)$$

and apply E^k : $E^{k+1}X + E^{k-1}X = 2(2\frac{B^2}{AC} -1)E^kX + 4(BE-CD)/(AC)$.

Let q satisfy

$$q + q^{-1} = 2(2B^2/(AC)-1) , \quad (1.6)$$

then, E^kX has the form

$$E^kX = \alpha q^k + \beta q^{-k} + \frac{CD-BE}{B^2-AC} , \quad q \neq 1 ;$$

$$E^kX = \alpha + \beta k + 2\frac{BE-CD}{AC}k^2 , \quad q = 1 \quad (1.7)$$

for a fixed X. These are indeed the forms discussed in [NI].

The relation (1.4) shows that $y_1(x)/x$ and $y_2(x)/x$ have limits $(C/A)^{1/2}q^{1/2}$ and $(C/A)q^{-1/2}$ when $x \to \infty$. Therefore, Dx^n will be of exact degree $n-1$ if $q^n \neq 1$ and $q \neq 1$; if $q = 1$ (i.e. $B^2 = AC$), (1.3a) and (1.3c) show directly that $Dx^n = n(-B/A)^{n-1}x^{n-1} + \ldots$ Remark also that if $q^N = 1$ for some integer N, each lattice $\{E^kx\}$ is reduced to a finite set, this corresponds however to useful orthogonal polynomials [AS].

Second order difference operators link $E^{-1}f(x)$, $f(x)$ and $Ef(x)$. Interesting operators with rational coefficients are (R is a rational function) :

$$D^*(R(x)Df(x)) = \{R(y_{-2}(x))\frac{f(x)-E^{-1}f(x)}{x-E^{-1}x} - R(y_{-1}(x))\frac{Ef(x)-f(x)}{Ex-x}\} / (y_{-2}(x)-y_{-1}(x)) \quad (1.8)$$

$$M^*[(y_2(x)-y_1(x))\ R(x)\ Df(x)] / (y_{-2}(x)-y_{-1}(x)) =$$

$$\frac{1}{2}[(x-E^{-1}x)\ RDf(y_{-2}(x)) + (Ex-x)RDf(y_{-1}(x))]/(y_{-2}(x)-y_{-1}(x)) =$$

$$= (x+D/A)\ D^*(R(x)Df(x)) + (B/A)\ D^*(xR(x)Df(x)) \quad (1.9)$$

2. LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS

Formal orthogonal polynomials related to the sequence of "moments" $\{\mu_n\}$ are the diagonal Padé denominators of

$$f(x) = \sum_{0}^{\infty} \mu_n x^{-n-1} \quad (2.1)$$

This means that the numerator Q_n (of degree $n-1$) and the denominator P_n (of degree n) are normally determined by

$$f(x) - Q_n(x)/P_n(x) = O(x^{-2n-1}) \qquad x \to \infty .$$

Equivalently, Q_n/P_n is the nth approximant of the Jacobi contin fraction of f

$$f(x) = 1/[x-r_o-s_1/(x-r_1-s_2/\ldots)] \qquad (\mu_o = 1) \quad (2.2)$$

and P_n and Q_n satisfy the three-term recurrence relation

$$P_{n+1}(x) = (x-r_n)P_n(x) - s_nP_{n-1}(x) \qquad P_o=1 \qquad P_{-1}=0 \quad (2.3)$$

$$Q_{n+1}(x) = (x-r_n)Q_n(x) - s_nQ_{n-1}(x) \qquad Q_1=1 \qquad Q_o = 0 \qquad s_oQ_{-1} = -1$$

Laguerre [LA] realized that families of orthogonal polynomials satisfying remarkable differential relations and equations are conveniently described through their generating functions of moment (2.1) (see also [AT] [MC]). The relevant extension follows .

Definition 2.1. Laguerre-Hahn orthogonal polynomials are diagonal Padé denominators of an expansion (2.1) satisfying a Riccati equation

$$a(x)Df(x) = b(x)E_1f(x)E_2f(x) + c(x)Mf(x) + d(x) \quad (2.4)$$

where a,b,c,d are polynomials .

This will lead to difference relations and equations in Sections 5 and 6 . Conversely, there is increasing evidence [H1] [H3] [BO] [B2] that all orthogonal polynomials satisfying non trivial difference relations and equations are those of definition 2.1. As most of this evidence has been gathered by W. Hahn (see [H1] [H3] and references therein), the corresponding polynomials are called here Laguerre-Hahn polynomials.

Other interesting forms of the Riccati equation (2.4) showing how it links E_1f and E_2f are

$$bE_1fE_2f + [(y_2-y_1)^{-1}a + c/2]E_1f - [(y_2-y_1)^{-1}a - c/2]E_2f + d = 0 \quad (2.5)$$

$$(bE_1f - a/(y_2-y_1) + c/2)\ (bE_2f + a/(y_2-y_1) + c/2) =$$

$$= -bd-a^2/(y_2-y_1)^2 + c^2/4 \quad (2.6)$$

Although measures and weights will not be studied throughly here, let us remark that if f is a Stieltjes function

$$f(x) = \int_S (x-u)^{-1} d\sigma(u) \quad , \qquad x \notin S$$

with σ increasing on its real support S , $w(x) = \sigma'(x)$ is obtained almost everywhere as limit of $-\pi^{-1}$ Im $f(x+i\varepsilon)$ when $\varepsilon \to 0$ $(\varepsilon > 0)$. It is then easy to show that, if $b \equiv 0$,

$$aDw = cMw \ , \qquad (2.7)$$

equivalently $(a-(y_2-y_1)c/2)E_2 w = (a+(y_2-y_1)c/2)E_1 w$.

If there is a masspoint of weight $\mu(X)$ at some $X = E_1 x$ (pole of f with residue $\mu(X)$) , (2.5) tells that , still if $b = 0$, there is another one at $EX = E_2 x$ with a weight

$$\mu(EX) = \mu(E_2 x) = \frac{a+(y_2-y_1)c/2}{a-(y_2-y_1)c/2} \frac{dy_2}{dy_1} \mu(E_1 x) \ .$$

This shows that there are masspoint on lattices $\{E^k X\}$ interrupted at zeros of $a \pm (y_2-y_1)c/2$ [if $b \equiv 0$] . If $b \neq 0$, it will be shown in section 6 how f can be represented as a ratio of solutions of second order difference equations. The measure can then be expressed in terms of these solutions , as in sect. 5 of [BE] . A very large set of measures, still far from being completely explored , are concerned .

3. FUNDAMENTAL RECURRENCE RELATIONS

The importance of the Riccati form (2.4) is that the same form holds for the equation satisfied by f_1 $(f(x) = 1/(x-r_o-s_1 f_1))$. Iteration of this remark ([BA] p. 163) will generate a sequence of Riccati equations whose coefficients will eventually enter difference relations and equations.

<u>Theorem 3.1</u>. Let the Jacobi continued fraction (2.2) satisfy the Riccati equation (2.4) where a,b,c and d are polynomials of degrees $\leq m+2$, m , $m+1$ and m . Then, $f_n(x) = 1/(x-r_n-s_{n+1}/(x-r_{n+1} - \dots))$ satisfies

$$a_n(x)Df_n(x) = b_n(x)E_1 f_n(x) E_2 f_n(x) + c_n(x) M f_n(x) + d_n(x) \qquad (3.1)$$

with polynomials a_n b_n c_n d_n of degrees still bounded by $m+2$, m , $m+1$ and m .

Moreover,

$$a_{n+1}(x) = a_n(x) - (y_2-y_1)^2 d_n(x)/2 \qquad (3.2a)$$

$$b_{n+1}(x) = s_{n+1} d_n(x) \qquad (3.2b)$$

$$c_{n+1}(x) = -c_n(x) - 2(Mx-r_n)d_n(x) \qquad (3.2c)$$

$$d_{n+1}(x) = [a_n(x) + b_n(x) + (Mx-r_n)c_n(x) + (y_1-r_n)(y_2-r_n)d_n(x)]/s_{n+1} \qquad (3.2d)$$

$(a_o \equiv a \ , \ b_o \equiv b \ , \ c_o \equiv c \ , \ d_o \equiv d)$.

Indeed , put $f_n(x) = 1/(x-r_n-s_{n+1}f_{n+1}(x))$ in (3.1) and (3.2) follows easily by an induction starting with (2.4) at $n = 0$ $(f_o \equiv f)$. Only the degree of d_{n+1} must still be discussed . As $f_n(x) = 1/(x-r_n) + O(x^{-3})$ for large x , (3.1) yields $-a_n D(x-r_n)^{-1} + b_n E_1(x-r_n)^{-1} E_2(x-r_n)^{-1} + c_n M(x-r_n)^{-1} + d_n = O(x^{m-2})$, or $a_n + b_n + (Mx-r_n)c_n + (y_1-r_n)(y_2-r_n)d_n = O(x^m)$ and this means degree $d_{n+1} \leq m$. #

<u>Remark 3.2</u>. The polynomial

$$\chi(x) = (a_n(x))^2 + (y_2(x)-y_1(x))^2 [b_n(x)d_n(x) - (c_n(x))^2/4] \qquad (3.3)$$

of degree $2m+4$ is independent of n ;

$$\Sigma_n(x) = b_n(x)d_n(x) - (c_n(x))^2/4 =$$
$$= b_o(x)d_o(x) - (c_o(x))^2/4 + (1/2)(a_o(x)+a_n(x)) \sum_{o}^{n-1} d_k(x) \qquad (3.4)$$

Remark also that χ appears at the right-hand side of (2.6) .

We proceed now with a first exploration of the polynomials a_n , b_n , c_n and d_n :

<u>Lemma 3.3</u>.

The dominant coefficients of $a_n(x) = a_{n,o}x^{m+2} + \dots$, $c_n(x) = c_{n,o}x^{m+1} + \dots$ and $d_n(x) = d_{n,o}x^m + \dots$ have the form

$$a_{n,o} = -(K/2)\left(q^{-n-\mu} + q^{n+\mu}\right) ,$$

$$c_{n,o} = K(A/C)^{1/2}\left(q^{-n-\mu} - q^{n+\mu}\right) / \left(q^{-1/2} - q^{1/2}\right) ,$$

$$d_{n,o} = (KA/C)\left(q^{-n-\mu-1/2} - q^{n+\mu+1/2}\right) / \left(q^{-1/2} - q^{1/2}\right) , \quad \text{if} \quad q \neq 1 \ ;$$

$$a_{n,o} = -K \ , \quad c_{n,o} = 2K(A/C)^{1/2}(n+\mu) \ ,$$

$$d_{n,o} = (2KA/C)(n+1/2+\mu) \qquad \text{if} \quad q = 1 \ .$$

Indeed , from (3.2a..d) , (1.3c) and (1.3a) , one has

$$a_{n+1,o} = a_{n,o} - 2A^{-2}(B^2 - AC)\,d_{n,o}\ , \qquad c_{n+1,o} = -c_{n,o} + (2B/A)\,d_{n,o}$$

and $a_{n,o} - (B/A)c_{n,o} + (C/A)d_{n,o} = 0$.

After elimination of $a_{n,o}$ and $a_{n+1,o}$:

$$\begin{bmatrix} c_{n+1,o} \\ d_{n+1,o} \end{bmatrix} = \begin{bmatrix} -1 & 2B/A \\ -2B/C & 4B^2/(AC)-1 \end{bmatrix} \begin{bmatrix} c_{n,o} \\ d_{n,o} \end{bmatrix}$$

This matrix has eigenvalues q and q^{-1} (from (1.6)) . The square root of this matrix with eigenvalues $q^{1/2}$ and $q^{-1/2}$ is $\begin{bmatrix} 0 & (C/A)^{1/2} \\ -(A/C)^{1/2} & 2B/(AC)^{1/2} \end{bmatrix}$, so that $c_{n+1/2,o} = (C/A)^{1/2} d_{n,o}$,

$d_{n+1/2,o} = (q^{1/2} + q^{-1/2})\,d_{n,o} - d_{n-1/2,0}$. This is solved in terms of powers of q and q^{-1} if $q \neq 1$, of linear functions of n if $q = 1$. The constants K and μ of the theorem are derived from the initial values $c_{o,o}$ and $d_{o,o}$. #

There is no simple rule for the other coefficients of the polynomials, they are related by equations where the further unknowns r_n and s_n are also involved. If the d_n's are supposed to be known, it is possible to eliminate the a's , b's and c's and to end with equations for r_n and s_n . When $m = 0$ (classical case), d_n is a constant known by lemma 3.3 and formulas in closed form can be found. The polynomials studied by Askey and Wilson [AN] [AS] [PE] [WI] will be recovered.

4. RELATIONS WITH ASKEY-WILSON POLYNOMIALS

Theorem 4.1. If the degrees of a,b,c,d are less or equal than $2,0,1,0$, the orthogonal polynomials of definition 2.1 satisfy the recurrence relation (2.3) where

$$s_n = S((n+\mu)^2)/[d_{n-1}(d_{n-1/2})^2 d_n]$$

and where r_n can be written in four different ways as

$$r_n = y_1(x_i) - R_i(n+\mu)/(d_n d_{n-1/2}) - R_i(-n-\mu-1)/(d_n d_{n+1/2})$$

$$= y_2(x_i) - R_i(-n-\mu)/(d_n d_{n-1/2}) - R_i(n+\mu+1)\ (d_n d_{n+1/2})$$

with $R_i(x)R_i(-x) \equiv S(x^2)$, $i=1,\dots,4$. If $q \neq 1$, $S(x^2)$ is a polynomial of degree four in $q^x + q^{-x}$, each $R_i(x)$ is a Laurent polynomial of degree two in q^x (i.e, a linear combination of q^{-2x}, q^{-x} , 1 , q^x and q^{2x}), and $d_n = \text{const.}(q^{n+\mu+1/2} - q^{-n-\mu-1/2})$; if $q = 1$, S and each R_i are polynomials of degree four and $d_n = \text{const.}\ (n+\mu+1/2)$.

Indeed, from theorem 3.1, r_n and s_n can be obtained if we find completely the expressions of the polynomials a_n , b_n , c_n and d_n . As $m = 0$, d_n is a constant already known (lemma 3.3). From (3.2a) , $a_n(x) = a_o(x) - (1/2)(y_2 - y_1)^2 \sum_o^{n-1} d_k$. Let us introduce D_n such that $D_{n+1} - D_n = d_n$, a function of n that will be very useful. From lemma 3.3, $D_n = (KA/C)\ (q^{-1/2} - q^{1/2})^{-2}(q^{-n-\mu} + q^{n+\mu})$ + constant if $q \neq 1$, $D_n = (KA/C)(n+\mu)^2$ + const. if $q = 1$.

Considered as a function of n , $a_n(x)$ is therefore a polynomia[l] of first degree in D_n .

In the polynomial $c_n(x) = c_{n,o}x + c_{n,1}$, only $c_{n,1}$ remains to be determined. This is done by looking at the coefficient of x in (3.4) exhibiting $c_{n,o}c_{n,1}$ as a quadratic polynomial in D_n .

This shows that $c_{n,o}c_n(x)$ as a function of n is a polynomial of second degree in D_n (in (3.3) , the coefficient of x^4 in χ shows that $(c_{n,o})^2$ is quadratic in D_n) .

We have now all the material needed for the recurrence coefficients : from (3.2c) , $r_n = Mx + (c_n(x) + c_{n+1}(x))/(2d_n)$ for any choice of x ; in (3.3) , everything is known but $b_n = s_n d^2_{n-1}$ (from (3.2b)) , so that $s_n = [(y_2(x) - y_1(x))^{-2}(\chi(x) - (a_n(x))^2) + (c_n(x))^2/4]/(d_{n-1}d_n)$ for any choice of x . By multiplying the numerator and the denominator by $(c_{n,o})^2$, s_n appears indeed as a quartic polynomial in D_n divided by $d_{n-1}(c_{n,o})^2 d_n$ and the first part of the theorem follows from lemma 3.3 ($c_{n,o} = (C/A)^{1/2} d_{n-1/2,o}$). There is a connection between r_n and s_n which appears when one chooses x as one of the four roots of $\chi(x) = 0$ (whence the four possibilities announced in the theorem). Then

$$\Big([c_n(x) + 2(y_2(x) - y_1(x))^{-1}a_n(x)]/(2d_n)\Big)\Big([c_n(x) - 2(y_2(x) - y_1(x))^{-1}a_n(x)]/(2d_{n-1})\Big)$$

and

$$\Big([c_n(x) - 2(y_2(x) - y_1(x))^{-1}a_n(x)]/(2d_n)\Big)\Big([c_n(x) + 2(y_2(x) - y_1(x))^{-1}a_n(x)]/(2d_{n-1})\Big)$$

are two different factorizations of s_n . After multiplication of the numerators and the denominators by $(1/2)d_{n-1/2}$, each of the numerators appears as an even function of $n+\mu$ plus or minus an odd function of $n+\mu$: $-R_i(n+\mu)$ and $-R_i(-n-\mu)$. Finally, the sum of the factors of the first factorization (where n is replaced by $n+1$ in the second factor) gives (use (3.2a) and (3.2.c)) $r_n - Mx + (y_2(x)-y_1(x))/2 = r_n - y_1(x)$. The second factorization gives $r_n - y_2(x)$. #

Problem 4.2. Find a geometric description of the roots of $\chi(x) = 0$.

Theorem 4.1. Tells that we are dealing with a family of orthogonal polynomials depending on 6 essential parameters (q , μ , and the 4 zeros of S) and 2 inessential parameters wich are the dominant coefficient of S (dilation of the variable) and one of the $y_i(x_j)$'s (translation). Apparently, the conic (1.2) defining y_1 and y_2 requires 5 parameters and a,b,c,d of degrees 2,0,1,0 ask for another 7 of whom we subtract 1 for homogeneity and 1 for the constraint $\mu_o = 1$, leaving 10 parameters. Several operators E_1 and E_2 will therefore lead to the same family of orthogonal polynomials. The two remaining degrees of freedom can be used to make $E_1 x \equiv x$ (possible only if $\theta = 0$ (see (1.4))) or to have a convenient symmetry $A = C$ and $D = E$ implying $E_{-1} = E_2 = E^{1/2}$, $E_{-2} = E_1 = E^{-1/2}$ (Askey and Wilson's choice : [AS] Section 5).

Theorem 4.3. Under the conditions of Theorem 4.1 and if $b = 0$, the orthogonal polynomials of definition 2.1 are the Askey-Wilson polynomials [AN] [AS] [WI] . If $b \neq 0$, they are the associated Askey-Wilson polynomials.

Indeed, if we look at different recurrence relation writing for the Askey-Wilson polynomials, such as [AN] p. 56 or [AS] p. 5 , we obtain the forms of Theorem 4.1 (our s_n is the $A_{n-1}c_n$ of [AN] and [AS]). However, there is a constraint $S(\mu^2) = 0$ so that $s_o = 0$ in [AN] and [AS]. This happens indeed if $b=0$. If $b\neq 0$, let ν be such that $R_i(\nu)=0$ and let us write $r_n = r_{n+\nu-\mu}$, $s_n = s_{n+\nu-\mu}$. Then r_n and s_n are the recurrence coefficients of a valid set of Askey-Wilson polynomials with ν insted of μ . This shows that we are dealing with the associated Askey-Wilson polynomials.

Wilson [WI] and Askey-Wilson [AN] [AS] polynomials have been introduced through contiguity relations for ${}_4F_3$ and ${}_4\varphi_3$ hypergeometric functions. These hypergeometric expansions can also be

recovered here:

Theorem 4.4. With r_n and s_n given by Theorem 4.1, the recurrenc relation

$$Z_{n+1} = (x-r_n)Z_n - s_n Z_{n-1} \qquad (4.1)$$

has special solutions of the form

$$Z_n = \prod^{n} \frac{R_i(\mp(m+\mu))}{d_{m-1}d_{m-1/2}} \sum_{m=0}^{\infty} \rho^{-m} \prod_{k=1}^{m} \frac{\delta(n+\mu\mp\nu_1-k+1)}{\delta(k)\delta(k\pm\nu_1\pm\nu_2)} \frac{\delta(n+\mu\pm\nu_1+k)}{\delta(k\pm\nu_1\pm\nu_3)} \frac{(x-E^{\mp(k-1)}y_j(x_i))}{\delta(k\pm\nu_1\pm\nu_4)} \qquad (4.2)$$

where the upper signs correspond to $j=1$ and the lower sings to $j=2$ and where $\delta(n) = d_{n/2-\mu-1/2} = (KA/C)(q^{-n/2}-q^{n/2})/(q^{-1/2}-q^{1/2})$ if $q \neq 1$; $\delta(n) = KAn/C$ if $q = 1$.

Indeed, for one of the values of $i = 1,\dots,4$ allowed by Theorem 4.1, (4.1) becomes

$$Z_{n+1} = \left(x-y_j(x_i) + \frac{R_i(\pm(n+\mu))}{d_n d_{n-1/2}} + \frac{R_i(\mp(n+\mu+1))}{d_n d_{n+1/2}}\right)Z_n - \frac{R_i(\pm(n+\mu))R_i(\mp(n+\mu))}{d_{n-1}(d_{n-1/2})^2 d_n} Z_{n-1} .$$

If $x = y_j(x_i)$, (4.1) has the simple particular solution $\prod^{n} \frac{R_i(\mp(m+\mu))}{d_{m-1}d_{m-1/2}}$. Writing $Z_n = \prod^{n} \frac{R_i(\mp(m+\mu))}{d_{m-1}d_{m-1/2}} \tilde{Z}_n$, the recurrence relation becomes

$$\frac{R_i(\mp(n+\mu+1))}{d_n d_{n+1/2}})(\tilde{Z}_{n+1}-\tilde{Z}_n) = (x-y_j(x_i))\tilde{Z}_n + \frac{R_i(\pm(n+\mu))}{d_n d_{n-1/2}}(\tilde{Z}_n-\tilde{Z}_{n-1}) \qquad (4.3)$$

Consider $\tilde{Z}_n$ as a function of $n+\mu+1/2$. Then this recurrence accepts solutions which are even functions of $n+\mu+1/2$ (d_n is an odd function of $n+\mu+1/2$) . Wether there are interesting forms of $\tilde{Z}_n$ as and odd function of $n+\mu+1/2$ is not investigated here. Furthermore, from Theorem 4.1, $R_i(t)$ can be factorized as $R_i(t) = \rho\delta(t-\nu_1)\delta(t-\nu_2)\,\delta(t-\nu_3)\delta(t-\nu_4)$. So if $\pm(n+\mu) = \nu_1$, i.e. $n = -\mu \pm \nu_1$, there is a simple relation between $\tilde{Z}_n$ and $\tilde{Z}_{n+1}$. This suggests for $\tilde{Z}_n$ an expression containing $\delta(n+\mu\mp\nu_1)$, $\delta(n+\mu\mp\nu_1)\delta(n+\mu\mp\nu_1-1),\dots$ and as it must be even in $n+\mu+1/2$, the following is proposed:

$$\tilde{Z}_n = \sum_{m=o}^{\infty} \xi_m \prod_{k=1}^{m} \delta(n+\mu\mp\nu_1-k+1)\delta(n+\mu\pm\nu_1+k) . \qquad (4.4)$$

Expanding $\tilde{Z}_n - \tilde{Z}_{n-1}$, one encounters $\delta(n+\mu\mp\nu_1)\delta(n+\mu\pm\nu_1+m)$ -

$\delta(n+\mu\mp\nu_1-m)\delta(n+\mu\pm\nu_1)$ which is neatly factorized as $\delta(m)\delta(2n+2\mu) = \delta(m)d_{n-1/2}$. After a little work on (4.3) , one must satisfy the identity

$$\sum_{m=0}^{\infty}\xi_m\prod_{k=1}^{m-1}\delta(n+\mu\mp\nu_1-k+1)\delta(n+\mu\pm\nu_1+k)(X_{n,m}-(x-y_j(x_i))d_n\delta(n+\mu\mp\nu_1-m+1)\delta(n+\mu\pm\nu_1+m)) = 0,$$

where $X_{n,m} = \rho\delta(m)[\delta(n+\mu\pm\nu_1+m)\prod_2^4\delta(n+\mu+1\pm\nu_k) - \delta(n+\mu\mp\nu_1-m+1)\prod_2^4\delta(n+\mu\mp\nu_k)$ is an odd function of $n+\mu+1/2$ and can be factorized as $d_n[\alpha_m+\beta_m\delta(n+\mu\mp\nu_1-m+1)\delta(n+\mu\pm\nu_1+m)$. Then $\alpha_m\xi_m + (\beta_{m-1}-x+y_j(x_i))\xi_{m-1} = 0$ follows , so that (4.4) becomes

$$\tilde{Z}_n = \sum_{m=0}^{\infty}\prod_{k=1}^{m}(\alpha_k)^{-1}\delta(n+\mu\mp\nu_1-k+1)\delta(n+\mu\pm\nu_1+k)(x-y_j(x_i)-\beta_{k-1}) , \quad (4.5)$$

α_m is easily found by taking $n = -\mu+m-1\pm\nu_1$ as $\rho\delta(m)\prod_2^4\delta(m\pm\nu_1\pm\nu_k)$; from the highest powers of q^n and q^{-n} in $X_{n,m}$ and $\beta_o = 0$, one finds $\beta_m = \rho\delta(m)\delta(m+1\pm\sum_1^4\nu_k)$. So β_m is a combination of q^m, q^{-m} and a constant. The coefficients of q^m and q^{-m} are the same as in $[C(q^{-1/2}-q^{1/2})/KA]^2R_i(\mp(m/2+1/4))$, from the factorization of R_i . In order to see the link between β_m and $E^{\mp m}y_j(x_i)$, one returns to the definition of R_i in theorem 4.1 in terms of $a_n(x_i)$ and $c_n(x_i)$ to get finally $\beta_m = (-1/2)(C/A)^{1/2}$

$$[x_i + \frac{BD-AE}{B^2-AC}](q^{-m-1/2}+q^{m+1/2}) \mp (1/2)(y_2-y_1)\frac{q^{-m-1/2}-q^{m+1/2}}{q^{-1/2}-q^{1/2}} +$$

+ constant , where the constant is such that $\beta_o = 0$. It then happens that $y_j(x_i)+\beta_m$ is of the form (1.7) . Checking that $2y_j(x_i)+\beta_o+\beta_{-1} = -2(Bx_i+D)/A$ and comparing with (1.5) yields indeed $y_j(x_i)+\beta_m = E^{\mp m}y_j(x_i)$ and (4.2) follows . #

5. DIFFERENCE RELATIONS

We return to the general theory of sections 2 and 3 in order to show how the polynomials of definition 2.1 satisfy difference relations involving the polynomials a_n , b_n , c_n and d_n of section 3. Before that, the following functions will be found useful:

$$\tilde{a}_n(x) = -s_1\ldots s_{n-1}\,(a_n(x)/(y_2-y_1)+c_n(x)/2)$$
$$\tilde{c}_n(x) = s_1\ldots s_{n-1}\,(a_n(x)/(y_2-y_1)-c_n(x)/2)$$
$$\tilde{b}_n(x) = s_1\ldots s_{n-1}\,b_n(x)/s_n$$
$$\tilde{d}_n(x) = s_1\ldots s_n\,d_n(x)$$

Remark that the Riccati forms (2.5) and (2.6) (with indexes n)

can be written

$$(s_n)^2\tilde{b}_nE_1f_nE_2f_n - s_n\tilde{a}_nE_1f_n - s_n\tilde{c}_nE_2f_n + \tilde{d}_n = 0 \quad (5.1)$$

and

$$(s_n\tilde{b}_nE_1f_n - \tilde{c}_n)(s_n\tilde{b}_nE_2f_n - \tilde{a}_n) = -(s_1\ldots s_{n-1})^2\chi/(y_2-y_1)^2 \quad (5.2)$$

<u>Lemma 5.1</u>. Let X_n and Y_n be two independent solutions of $Z_{n+1} = (x-r_n)Z_n - s_nZ_{n-1}$, then there are functions $\alpha,\beta,\gamma,\delta$ of x , independent of n , such that

$$\tilde{a}_n = \alpha(E_1X_{n-1})(E_2X_n)+\beta(E_1Y_{n-1})(E_2X_n)+\gamma(E_1X_{n-1})(E_2Y_n)+\delta(E_1Y_{n-1})(E_2Y_n)$$
$$\tilde{c}_n = \alpha(E_1X_n)(E_2X_{n-1})+\beta(E_1Y_n)(E_2X_{n-1})+\gamma(E_1X_n)(E_2Y_{n-1})+\delta(E_1Y_n)(E_2Y_{n-1})$$
$$\tilde{b}_n = \alpha(E_1X_{n-1})(E_2X_{n-1})+\beta(E_1Y_{n-1})(E_2X_{n-1})+\gamma(E_1X_{n-1})(E_2Y_{n-1})+\delta(E_1Y_{n-1})(E_2Y_{n-1})$$
$$\tilde{d}_n = \alpha(E_1X_n)(E_2X_n)+\beta(E_1Y_n)(E_2X_n)+\gamma(E_1X_n)(E_2Y_n)+\delta(E_1Y_n)(E_2Y_n)$$

$$\tilde{a}_nE_1X_n - \tilde{d}_nE_1X_{n-1} = (\beta E_2X_n + \delta E_2Y_n)\,E_1\Gamma_n \quad (5.3a)$$
$$\tilde{c}_nE_2X_n - \tilde{d}_nE_2X_{n-1} = (\gamma E_1X_n + \delta E_1Y_n)\,E_2\Gamma_n \quad (5.3b)$$

$$\tilde{a}_nE_1Y_n - \tilde{d}_nE_1Y_{n-1} = -(\gamma E_2Y_n + \alpha E_2X_n)\,E_1\Gamma_n \quad (5.4a)$$
$$\tilde{c}_nE_2Y_n - \tilde{d}_nE_2Y_{n-1} = -(\beta E_1Y_n + \alpha E_1X_n)\,E_2\Gamma_n \quad (5.4b)$$

where Γ_n is the Casorati determinant $X_nY_{n-1}-X_{n-1}Y_n$ satisfying $\Gamma_{n+1} = s_n\Gamma_n$.

$$\tilde{a}_n\tilde{c}_n-\tilde{b}_n\tilde{d}_n = (s_1\ldots s_{n-1})^2(-(a_n)^2/(y_2-y_1)^2+(c_n)^2/4-b_nd_n)$$
$$= -(s_1\ldots s_{n-1})^2\chi/(y_2-y_1)^2 = (\beta\gamma-\alpha\delta)E_1\Gamma_n\,E_2\Gamma_n . \quad (5.5)$$

Indeed, from (3.2) , the recurrence relations for $\tilde{a}_n,\ldots,\tilde{d}_n$ can be written

$$\begin{bmatrix}\tilde{a}_{n+1}\\ \tilde{c}_{n+1}\\ \tilde{b}_{n+1}\\ \tilde{d}_{n+1}\end{bmatrix} = \begin{bmatrix}0 & -s_n & 0 & y_2-r_n\\ -s_n & 0 & 0 & y_1-r_n\\ 0 & 0 & 0 & 1\\ -s_n(y_2-r_n) & -s_n(y_1-r_n) & (s_n)^2 & (y_1-r_n)(y_2-r_n)\end{bmatrix}\begin{bmatrix}\tilde{a}_n\\ \tilde{c}_n\\ \tilde{b}_n\\ \tilde{d}_n\end{bmatrix}$$

Now, if $\Xi_{n+1} = (y_2-r_n)\Xi_n - s_n\Xi_{n-1}$ and $\gamma_{n+1} = (y_1-r_n)\gamma_n - s_n\gamma_{n-1}$, that is exactly the recurrence relation for $[\Xi_n\gamma_{n-1}, \Xi_{n-1}\gamma_n, \Xi_{n-1}\gamma_{n-1}, \Xi_n\gamma_n]^T$. A basis of solutions of this recurrence is obtained by the four choices $\Xi_n = E_2X_n$ or E_2Y_n , $\gamma_n = E_1X_n$ or E_1Y_n (tensorial product). (5.3) and (5.4) , which are the actual difference relations, follow easily.

<u>Theorem 5.2</u>. The monic orthogonal polynomials defined in Section 2

satisfy the difference relation

$$(a_n+a)DP_n = (c_n-c)MP_n + 2s_nd_nMP_{n-1}-2bMQ_n \qquad (5.6)$$

Indeed, with $X_n = P_n$ and $Y_n = Q_n$, α , β , γ and δ of lemma 5.1 are readily found to be (take $n = 0$ and use $P_{-1} = 0$, $P_o = 1$, $Q_{-1} = -1/s_o$, $Q_o = 0$) , $\alpha = \tilde{d}_o = d$, $\beta = -s_o\tilde{a}_o = a/(y_2-y_1) + c/2$, $\gamma = -s_o\tilde{c}_o = -a/(y_2-y_1)+c/2$, $\delta = (s_o)^2\tilde{b}_o = b$. Now, adding (5.3a) and (5.3b) :

$$s_1\cdots s_{n-1}\ [a_n(E_2P_n-E_1P_n)/(y_2-y_1)-c_n(E_1P_n+E_2P_n)/2 -$$
$$- s_nd_n(E_1P_{n-1}+E_2P_{n-1})] = [a(E_2P_n-E_1P_n)/(y_2-y_1) +$$
$$+ c(E_1P_n+E_2P_n)/2 + b(E_1Q_n+E_2Q_n)]\Gamma_n$$

and this is exactly (5.6) after using $\Gamma_n = s_o\cdots s_{n-1}\Gamma_o = -s_1\cdots s_{n-1}$. #

6. DIFFERENCE EQUATIONS

Theorem 6.1. If $b \equiv 0$ in (2.4) , the orthogonal polynomials defined in section 2 satisfy the second-order difference equation with rational coefficients

$$D^*\left(\frac{a}{d_n}DP_n\right) - (y_{-2}-y_{-1})^{-1}M^*\left((y_2-y_1)\frac{c}{d_n}DP_n\right) = \left[(y_{-2}-y_{-1})^{-1}M^*\frac{2(a-a_n)}{(y_2-y_1)d_n} + D^*\frac{c_n-c}{2d_n}\right]P_n \qquad (6.1)$$

Other interesting forms are

$$D^*\left(\frac{\tilde{w}}{d_n}DP_n\right) = \left[(v_{-2}-y_{-1})^{-1}M^*\left((y_2-y_1)\frac{\sum_0^{n-1} d_k}{d_n}\right) + D^*\frac{c_n-c}{2d_n}\right]wP_n \qquad (6.2)$$

with w form (2.7) and $\tilde{w} = (a+(y_2-y_1)c/2)E_1w = (a-(y_2-y_1)c/2)E_2w$,

$$D^*\left(\frac{\chi^{1/2}}{d_n}D(w^{1/2}P_n)\right) = \left[D^*\frac{c_n}{2d_n} + 2(y_{-2}-y_{-1})^{-1}M^*\left(\frac{\chi^{1/2}-a_n}{(y_2-y_1)d_n}\right)\right]w^{1/2}P_n \qquad (6.3)$$

Indeed, as $b \equiv 0$, choosing $X_n = X_oP_n$, $Y_n = -s_oY_{-1}Q_n$ in lemma 5.1 , one finds $\delta \equiv 0$. Therefore (5.3) involves only X_n and X_{n-1} , with $\beta = \tilde{a}_o/(E_1Y_{-1}E_2X_o)$, $\gamma = \tilde{c}_o/(E_1X_oE_2Y_{-1})$.

Elimination of X_{n-1} yields

$$E_{-1}(\beta/\tilde{d}_n)\Gamma_nEX_n-E_{-2}(\gamma/\tilde{d}_n)\Gamma_nE^{-1}X_n =$$
$$= [E_{-1}(\tilde{a}_n/\tilde{d}_n)-E_{-2}(\tilde{c}_n/\tilde{d}_n)]X_n$$

which can be rewritten using (1.8) and (1.9) as

$$D^*\left(\frac{\beta-\gamma}{2d_n}(y_2-y_1)DX_n\right) - (y_{-2}-y_{-1})^{-1}M^*\left(\frac{\beta+\gamma}{d_n}(y_2-y_1)DX_n\right) =$$
$$\left(\frac{2}{s_o\Gamma_o(y_{-2}-y_{-1})}M^*\frac{a_n}{(y_2-y_1)d_n} - \frac{1}{2s_o\Gamma_o}D^*\frac{c_n}{d_n} - \frac{1}{2}D^*\frac{\gamma+\beta}{d_n} - \frac{1}{y_{-2}-y_{-1}}M^*\frac{\gamma-\beta}{d_n}\right)X_n$$

With $X_o = Y_{-1} = 1$, β and $\gamma = [-c_o/2\mp a_o/(y_2-y_1)\ /s_o$, $\Gamma_o = 1$ and this gives (6.1) which is the extension of Laguerre differential equation [AT] [H1] [LA] [MA] . For the two other forms, $\beta = -\gamma$, possible if $\tilde{a}_oE_1(X_o/Y_{-1}) = -\tilde{c}_oE_2(X_o/Y_{-1})$: $X_o/Y_{-1} = w$. With $Y_{-1} = 1/w$, $X_o = 1$, one has (6.2) , which is the form used by Askey and Wilson ([AS] Section 5). Remark also that, in the classical case (degrees a,b,c,d = 2,0,1,0) , the right-handsides of (6.1) and (6.2) have the form λ_nP_n and λ_nwP_n : d_n is independent on x , c_p is degreee 1 and use (1.9) [and (3.2a)]. With $Y_{-1} = w^{-1/2}$, $X_o = w^{1/2}$, and using (5.5) , one obtains (6.3) .

#

Theorem 6.2. The orthogonal polynomials of definition 2.1 satisfy a linear fourth-order difference equation whose coefficients depend on a,b,c,d and a_n , b_n , c_n and d_n .

Indeed, we consider again lemma 5.1 with $X_n = P_n$ and $Y_n = Q_n$ and α , β , γ and δ determined from $\tilde{a}_o$, $\tilde{b}_o$, $\tilde{c}_o$ and $\tilde{d}_o$ i.e. from a , b , c and d , but now $\delta \not\equiv 0$ in general. Elimination of $X_{n-1} = P_{n-1}$ and $Y_{n-1} = Q_{n-1}$ from (5.3) and (5.4) yields two equations involving P_n , EP_n , $E^{-1}P_n$, Q_n , EQ_n and $E^{-1}Q_n$. Applying the operators E and E^{-1} to the two equivalent equations involving only Q_n and $E^{-1}Q_n$ for the first, and Q_n and EQ_n for the second, we get two new equations. Final elimination of Q_n , EQ_n and $E^{-1}Q_n$ from the four resulting equations leaves a single relation involving $E^{-2}P_n$, $E^{-1}P_n$, P_n , EP_n and E^2P_n , which is the fourth-order difference equation.

The complete writing of this equation would be very tedious. It is much more interesting to see how its solutions are related to solutions of second order difference equations ([H1] eq. 17) :

Theorem 6.3. The orthogonal polynomials of definition (2.1) can be written

$$P_n = (X_nY_{-1}-Y_nX_{-1})/(X_oY_{-1}-Y_oX_{-1}) , \qquad (6.4)$$

where the functions of x and n X_n and Y_n satisfy simultaneously the three-terms recurrence relation

$$Z_{n+1} = (x-r_n)Z_n - s_n Z_{n-1} , \qquad (6.5)$$

the second order difference equation

$$D^*\left(\frac{\chi^{1/2}}{d_n} DZ_n\right) = \left[D^*\frac{c_n}{2d_n} + 2(y_{-2}-y_{-1})^{-1} M^*\left(\frac{\chi^{1/2}-a_n}{(y_2-y_1)d_n}\right)\right] Z_n , \qquad (6.6)$$

and the difference relation

$$(\chi^{1/2}+a_n)DZ_n = c_n MZ_n + 2s_n d_n MZ_{n-1} \qquad (6.7)$$

Indeed, let us start with X_n and Y_n , two independent solutions of (6.5). Then, (6.4) is obviously a solution which is a polynomial of degree n in x , and this is P_n . We are still free to choose the four functions of x X_o , X_{-1} , Y_o and Y_{-1} . Here is a way to have $\alpha = \delta = 0$ in Lemma 5.1 , which will lead to considerable simplification : it is based on the fact that if $Z_o/(s_o Z_{-1})$ is a solution of the Riccati equation (2.4) , and if the recurrence relation (6.5) holds , then $Z_n/(s_n Z_{n-1})$ is a solution of the Riccati equation (3.1) . This is a consequence of the construction of (3.1) . So let us choose a solution g of (2.4) and fix the ratio $X_o/(s_o X_{-1}) = g$ (of course, g could be f itself, and this will indicate a representation of f as a ratio of solutions of second-order differences equations), and let $g_n = X_n/(s_n X_{n-1})$. We multiply now the four lines of lemma 5.1 by $-s_n E_1 g_n$, $-s_n E_2 g_n$, $(s_n)^2 E_1 g_n E_2 g_n$, 1 and we make the sum . The left-hand side vanishes, as we have reconstructed the Riccati equation of g_n in the form (5.1) . In the right-hand side, we find various products with the vanishing factor $s_n g_n X_{n-1} - X_n$ and the single product left $\delta E_1(s_n g_n Y_{n-1}-Y_n)E_2(s_n g_n Y_{n-1}-Y_n)$. Therefore, $\delta = 0$. Fixing $Y_o/(s_o Y_{-1}) = h$, another solution of (2.4) , we have also $\alpha = 0$.

Now, eliminating X_{n-1} and Y_{n-1} from (5.3) and (5.4) :

$$\Gamma_n E_{-1}(\beta/\tilde{d}_n)\, EX_n - \Gamma_n E_{-2}(\gamma/\tilde{d}_n)\, E^{-1}X_n = \left[E_{-1}(\tilde{a}_n/\tilde{d}_n)-E_{-2}(\tilde{c}_n/\tilde{d}_n)\right]X_n$$
$$\qquad (6.8)$$
$$-\Gamma_n E_{-1}(\gamma/\tilde{d}_n)\, EY_n + \Gamma_n E_{-2}(\beta/\tilde{d}_n)\, E^{-1}Y_n = \left[E_{-1}(\tilde{a}_n/\tilde{d}_n)-E_{-2}(\tilde{c}_n/\tilde{d}_n)\right]Y_n$$

with $\beta\gamma = -(\tilde{a}_o\tilde{c}_o-\tilde{b}_o\tilde{d}_o)/(E_1\Gamma_o E_2\Gamma_o) = -\chi/[s_o^2(y_2-y_1)^2 E_1\Gamma_o E_2\Gamma_o]$. These equations are the some if $\gamma = -\beta$. With the choice $\Gamma_o = 1$, we proceed as in the proof of Theorem 6.1 and we get (6.6), which has indeed the same form as (6.3) .

Now, let us define

$$W_n = E_{-1}\frac{\chi^{1/2}}{(y_2-y_1)d_n}\, EZ_n - E_{-1}\frac{1}{d_n}\left(\frac{a_n}{y_2-y_1} + \frac{c_n}{2}\right) Z_n$$
$$= -E_{-2}\frac{\chi^{1/2}}{(y_2-y_1)d_n}\, E^{-1}Z_n + E_{-2}\frac{1}{d_n}\left(\frac{a_n}{y_2-y_1} - \frac{c_n}{2}\right) Z_n , \qquad (6.9)$$

the second equality coming from (6.8) . Applying E and E^{-1} to (6.9) and eliminating Z_n , EZ_n and $E^{-1}Z_n$, it appears that W_n satisfies (6.8) with $n-1$ instead of n . Writing $W_n = s_n Z_{n-1}$, (6.9) is a difference relation which can be put in the form (6.7) , and (6.5) can be checked using (3.2). #

In the classical case, it would be interesting to get expressions of P_n and $Q_n = (X_n Y_o - Y_n X_o)/(X_1 Y_o - Y_1 X_o)$ in terms of hypergeometric expansions, as in [HE] p. 281 and 285.

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REFERENCES

[AN] G.E. ANDREWS, R. ASKEY : Classical orthogonal polynomials, pp. 36-62 in "Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984" , Lecture Notes Math. 1171 (C. BREZINSKI et al. editors), Springer, Berlin 1985.

[AS] R. ASKEY , J. WILSON : Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Memoirs AMS vol. 54 nr . 319 , AMS, Providence, 1985.

[AT] F.V. ATKINSON, W.N. EVERITT : Orthogonal polynomials which satisfy second order differential equations, pp. 173-181 in "E.B. Christoffel", (P.L. BUTZER and F. FEHER, editors), Birkhäuser, Basel 1981.

[BA] G.A. BAKER, Jr , P. GRAVES-MORRIS Padé Approximants II, Addison Wesley, Reading 1981.

[BE] V. BELEVITCH : The Gauss hypergeometric ratio as a positive real function, SIAM J. Math. An. 13 (1982) 1024-1040.

[BO] S.S. BONAN, P. NEVAI : Orthogonal polynomials and their derivatives I, J. Approx. Theory 40 (1984) 134-147.

[B2] S.S. BONAN, D.S. LUBINSKY, P. NEVAI : Orthogonal polynomials and their derivatives II, Preprint.

[H1] W. HAHN : On differential equations for orthogonal polynomials, Funk. Ekv. 21 (1978) 1-9.

[H2] W. HAHN :Lineare geometrische Differenzengleichungen, Bericht Nr. 169 (1981), Math.-Stat. Sektion Forschungszentrum Graz.

[H3] W. HAHN : Über Orthogonalpolynome, die linearen Functionalgleichungen genügen, pp. 16-35 in "Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984", Lecture Notes Math.

1171 (C. BREZINSKI et al., editors), Springer, Berlin 1985.

[HE] E. HEINE : Handbuch der Kugelfunctionen I, 2nd edition, G. Reimer, Berlin 1978.

[LA] E. LAGUERRE : Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels, J. de Math. 1 (1885) 135-165 - Oeuvres II 685-711, Chelsea 1972.

[MA] A.P. MAGNUS : Riccati acceleration of Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials, pp. 213-230 in "Padé Approximation and its Applications, Bad Honnef 1983" , Lecture Notes Math. 1071 (H. WERNER and H.J. BUNGER, editors), Springer Berlin 1984.

[MR] P. MARONI : Introduction à l'étude dés δ-polynômes orthogonaux semiclassiques,

[MC] J. McCABE : Some remarks on a result of Laguerre concerning continued fraction solutions of first order linear differential equations, pp. 373-379 in "Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984", Lecture Notes Math. 1171 (C. BREZINSKI et all., editors), Springer, Berlin 1985.

[N1] A. NIKIFOROV, V. OUVAROV : Fonctions spéciales de la physique mathématique, Mir, Moscou 1983.

[NI] A.F. NIKIFOROV, S.K. SUSLOV : Classical orthogonal polynomials of a discrete variable on nonuniform lattices, Letters Math. Phys. 11 (1986) 27-34.

[PE] M. PERLSTADT : A property of orthogonal polynomial families with polynomial duals, SIAM J. Math. An. 15 (1984), 1043-1054.

[WI] J.A. WILSON Three-term contiguous relations and some new orthogonal polynomials, pp. 227-232 in "Padé and Rational Approximation" (E.B. SAFF and R.S. VARGA, editors), Ac. Press, N. Y. 1977.

[WP] J. WIMP : Computations with Recurrence Relations, Pitman, Boston 1984.

LE CALCUL DES FORMES LINEAIRES ET LES POLYNÔMES ORTHOGONAUX SEMI-CLASSIQUES

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INTRODUCTION

Dans l'étude des suites de polynômes, la forme linéaire par rapport à laquelle une suite est orthogonale, joue un rôle essentiel et il est utile de déterminer ses propriétés indépendamment de toute représentation. Par example, la donnée d'une forme linéaire est équivalente a la donnée de ses moments ; très souvent, la suite des moments vérifie une relation de récurrence. A l'aide d'un cadre algébrique convenable cela se traduit par une équation vérifiée par la forme : c'est une propriété intrinsèque.

L'objet de cet article est double: formaliser un certain nombre d'opérations habituelles sur les formes linéaires et appliquer le formalisme obtenu aux polynômes orthogonaux semi-classiques.

Dans le §1, on introduit le cadre topologique adéquat [7] de manière à définir systématiquement, par transposition, des opérations sur les formes linéaires à partir d'opérations sur les polynômes. De ce point de vue, l'espace vectoriel des séries formelles apparait comme un miroir de l'espace vectoriel des formes linéaires. On introduit un produit multiplicatif de deux formes linéaires, qui intervient naturellement dans la détermination de la forme canonique de la suite associée a une suite orthogonale.

Dans le §2 , on introduit la transformée de Fourier d'une forme linéaire et le produit de convolution de deux formes linéaires [2] , [8].

On établit en passant un résultat équivalent au théorème de Borel sur les séries formelles ;une forme linéaire peut toujours être représentée par une ultradistribution à décroissance rapide.

Enfin dans le §3 , on indique la définition et les diverses caractérisations des suites orthogonales semi-classiques [3] . En particulier, on montre l'identité de l'ensemble des suites semi-classique et de l'ensemble des suites dont la fonction de Stieltjes formelle