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# Three easy pieces, incomplete and unfinished:

**beautiful Freud equations, "1/9", and elliptic lattices.**


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The present file is http://perso.uclouvain.be/alphonse.magnus/Threepieces2012s.pdf

To A.H., A.R., B.D., C.B., D.D, D.L., G.B., H.S., J.G., J.L., J.M., J.N., L.L., Ll.N.T., L.W., M.G., M.G., M.E.H.I., P.N., R.A., R.B., R.C., R.V., S.A., t.S.g.[1], E.T., V.P., W.G., W.H., W.V.A.

*DR KOPAK: Estamos vivos?*
*BRICK BRADFORD: Sim* [2]

[1] the Spanish gang

[2] from a Brazilian edition of W.Ritt & C.Gray's "Adrift in an Atom" ( 1937 1938)

## 1. The quest for beautiful Freud equations.

In the mid 1970's, Bernard Danloy made the following comment of Stroud & Secrest's book on numerical integration:
*Let $P_n$ and $\Pi_n$ be the monic orthogonal polynomial of degree $n$, and the monic kernel polynomial at the origin of degree $n$, with respect to the same measure $d\mu$ on $(0,\infty)$, so that one has the general recurrence relations $P_{n+1}(x) = x\Pi_n(x) - u_n P_n(x)$, and $\Pi_{n+1}(x) = P_{n+1}(x) - v_n\Pi_n(x)$, with $u_n = \|\Pi_n\|^2/\|P_n\|^2, v_n = \|P_{n+1}\|^2/\|\Pi_n\|^2$. When $d\mu(x) = w(x)dx = \exp(-x^2)dx,\ 0 < x < \infty$, one has the beautiful relations*

$$2u_n(v_{n-1} + u_n + v_n) = n + 1,$$

$$2v_n(u_n + v_n + u_{n+1}) = n + 1.$$

indeed, $0 = \displaystyle\int_0^\infty \underbrace{[\Pi_n(x)P_n(x)xw(x)]'}_{\underbrace{x\Pi_n'(x)P_n(x)}_{nP_n^2(x)+\cdots} w(x)dx+\cdots+\Pi_n(x)P_n(x)w(x)-2\Pi_n(x)\underbrace{xP_n(x)}_{\cdots+(v_{n-1}+u_n+u_n)\Pi_n(x)+\cdots}xw(x)} dx$

$$0 = \int_0^\infty \underbrace{[P_{n+1}(x)\Pi_n(x)xw(x)]'}_{\underbrace{P_{n+1}'(x)\Pi_n(x)}_{(n+1)\Pi_n^2(x)+\cdots} xw(x)dx+\cdots-2P_{n+1}(x)\underbrace{x^2\Pi_n(x)}_{\cdots+(u_n+v_n+u_{n+1})P_{n+1}(x)+\cdots}w(x)} dx.$$

B. DANLOY Construction of gaussian quadrature formulas for $\int_0^\infty e^{-x^2} f(x)dx$.

NFWO-FNRS Meeting Leuven 20 Nov. 1975 [unpublished]. Numerical construction

Jean Meinguet recognized interesting instances of the $LR$ algorithm (confirmed later by experts), I intended to establish a convergent numerical method of computation (I did). We considered to present these results to the numerical analysis world. And the numerical analysis world rejected this as meaningless junk (it still does).

Then I received a flood of (p)reprints from Columbus, OH 43210, and we, er, I learned everything on Shohat, Freud, etc. The equations are:
*if* $w(t) = \exp(-P(t))$ *on* $\mathbb{R}$,*where* $P$ *is a polynomial, then*

$$\left.\begin{aligned} (P'(J))_{n,n} &= 0,\ n = 0, 1, \dots \\ a_n (P'(J))_{n,n-1} &= n, \quad n = 1, 2, \dots \end{aligned}\right\}$$

$J$ *being the Jacobi matrix* $\begin{bmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$. Indeed,

$$0 = \int_{\mathbb{R}} \underbrace{[p_n(t)p_{n-i}(t)w(t)]'}_{(p_n'(t)=(n/a_n)p_{n-1}(t)+\cdots)p_{n-i}(t)w(t)+\cdots-p_n(t)p_{n-i}(t)P'(t))w(t)} dt$$

see the “Bar-le-Duc” stuff in `http://perso.uclouvain.be/alphonse.magnus/freud/freud.`

This is what could be called the beautiful Freud equations.

*God created the beautiful Freud equations*

Recently, M.Ismail & pupils started a quest for discrete Freud eq.

Let $\{w_k = w(y_k)\}$ be a sequence of weights satisfying the difference equation $(\mathscr{D}w)(x_k) := \dfrac{w(y_{k+1}) - w(y_k)}{y_{k+1} - y_k} = p(x_k)(w_k + w_{k+1})$ on the lattices $\{x_k\}$ and $\{y_k\}$. Then, if $w(y_{k_1}) = w(y_{k_2}) = 0$,

$$0 = \sum_{k_1}^{k_2-1} [\mathscr{D}(p_n p_{n-i} w q)(x_k)]\Delta y_k = \sum_{k_1}^{k_2-1} [p_n p_{n-i}(y_{k+1}) \underbrace{w_{k+1}}_{\frac{1+p(x_k)\Delta y_k}{1-p(x_k)\Delta y_k} w_k} q_{k+1} - p_n p_{n-i}(y_k) w_k q_k]$$

And now?

For other weights, it is quite easy to produce ugly equations, see for instance

$$2(n+\alpha+\beta+\gamma+2)\tilde{a}_n^2(\tilde{a}_{n-1}^2+\tilde{a}_n^2+\tilde{a}_{n+1}^2)-2[\alpha\tilde{x}_0^2+(n+\beta+1)(\tilde{x}_0^2+1)+\gamma]\tilde{a}_n^2+$$

$$+2(2\tilde{a}_n^2-\tilde{x}_0^2-1)\sum_{j=1}^{n-1}\tilde{a}_j^2+n\tilde{x}_0^2-2\tilde{a}_n^2\tilde{a}_{n-1}^2+2\sum_{j=1}^{n-1}(\tilde{a}_j^4+2\tilde{a}_j^2\tilde{a}_{j-1}^2)+(2\beta+1)\tilde{x}_0^2\,(1-(-1)^n)/2=0,$$

$$n=1,2,\dots$$

such contraptions result from the identity $0=\int_S[ap_np_{n-i}w]'dx$, when the measure is $d\mu(x)=w(x)dx+$ masses at the zeros of the polynomial $a$, and $aw'=bw$.

Or this one on the unit circle:

$$w(\theta)=\begin{cases} r_1e^{-\gamma\theta}|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}\ , & \theta_1<\theta<\theta_2 \\ r_2e^{-\gamma\theta}|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}\ , & \theta\notin[\theta_1,\theta_2] \end{cases}$$

on the two arcs of endpoints $\exp(i\theta_1)$ and $\exp(i\theta_2)$.

$x_n := \Phi_n(0)$:

$$(n+1+\alpha+\beta+i\gamma)x_{n+1} = \frac{e^{i(\theta_1+\theta_2)}\overline{\xi_{n-1,n}} + \xi_{n-1,n} + n(e^{i\theta_1}+e^{i\theta_2})}{1-|x_n|^2}\, x_n$$

$-(n-1+\alpha+\beta-i\gamma)e^{i(\theta_1+\theta_2)}x_{n-1}$, where $\xi_{n-1,n} = x_1\overline{x_0} + x_2\overline{x_1} + \cdots + x_n\overline{x_{n-1}}$, $n = 1, 2, \ldots$ We need $x_1$. For a general ratio $r_2/r_1$, one has $\mathrm{Im}\,[\exp(-i(\theta_1 + \theta_2)/2)\ x_1] = \dfrac{\beta-\alpha}{\alpha+\beta+1}\sin((\theta_2-\theta_1)/2)$, (here, $\gamma = 0$), and we try various real parts.

One obviously expects $\Phi_n(0)$ to behave like a combination of $e^{in\theta_1}$ and $e^{in\theta_2}$ with slowly varying coefficients.

we know that $x_n \to 0$, and even that $\overline{\xi_{n-1,n}} \to_{n\to\infty} \lambda_1$, from Szegő-Geronimus theory, where $\log w = \underbrace{\cdots + \lambda_{-2}z^{-2} + \lambda_{-1}z^{-1} + \lambda_0}_{\log q(z)} + \underbrace{\lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots}_{\log r(z)}$,

Let $i\delta_j = \dfrac{\lambda_1}{e^{-i\theta_k} - e^{-i\theta_j}} + \dfrac{\overline{\lambda_1}}{e^{i\theta_j} - e^{i\theta_k}} - (\alpha+\beta)\dfrac{e^{i\theta_j} + e^{i\theta_k}}{e^{i\theta_j} - e^{i\theta_k}}$, $j = 1, 2$, and we suspect

$$x_n = \Phi_n(0) = \frac{A_1 n^{i\delta_1}}{n} e^{in\theta_1} + \frac{A_2 n^{i\delta_2}}{n} e^{in\theta_2} + o(1/n)$$

without full proof, alas. But we may recover $A_1$ and $A_2$ from an enormous number of numerical runs, checked with P.Nevai's Christoffel reconstruction procedure.

$A_1 = (\alpha - \rho i)e^{i\beta(\pi-\theta_2+\theta_1)+i\psi_1}$ , $A_2 = (\beta + \rho i)e^{-i\alpha(\pi-\theta_2+\theta_1)+i\psi_2}$ , with $\rho = \dfrac{\log(r_1/r_2)}{2\pi}$.

The absolute values of $A_1$ and $A_2$ are probably correct, as they agree with what is needed in the Hartwig-Fisher formula.Known cases: when $\theta_2 - \theta_1 = \pi$, and when $w(\theta)/[|\sin(\theta-\theta_1)/2|^{2\alpha}|\sin(\theta-\theta_2)/2|^{2\beta}]$ is the same constant on the whole circle, one must have $A_1 = \alpha$, $A_2 = \beta$, and $\delta = 0$ (Badkov, Golinskii, etc.).

The phases $\psi_1$ and $\psi_2$ are conjectured here, only for $\alpha = \beta = 0$ (sorry)

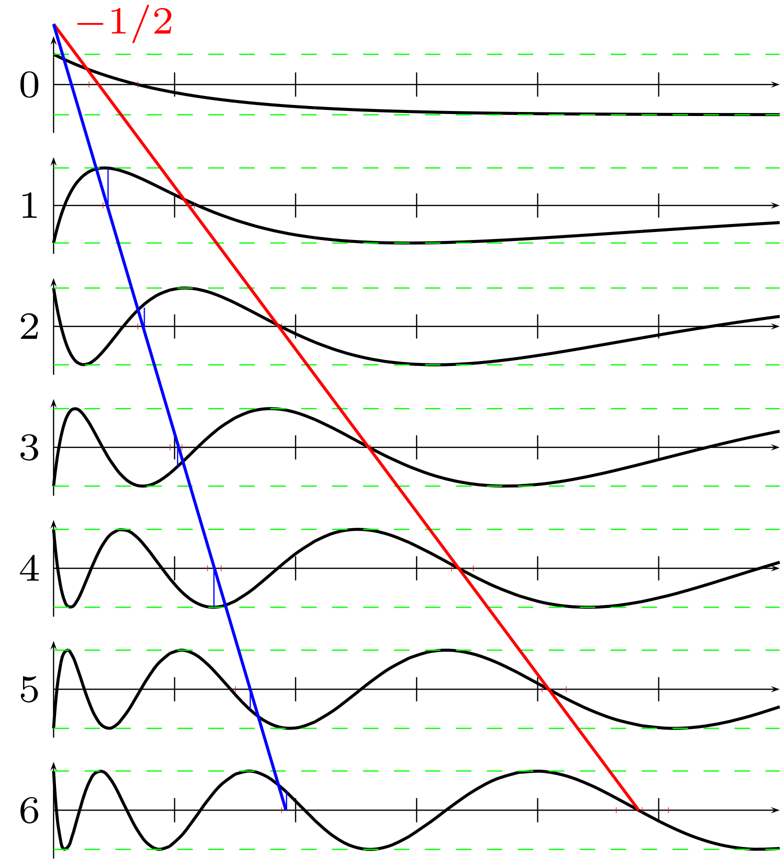
$$\psi_1 = -\psi_2 = -2\rho \log \sin((\theta_2 - \theta_1)/2) + 2 \arg \Gamma(1 + i\rho) - 2\rho \log 2$$

see `http://perso.uclouvain.be/alphonse.magnus/num3/m3xxx99.pdf`

## 2. "1/9" ou comment s'en débarrasser.

Let $e^{-t} - \dfrac{P_n(t)}{Q_n(t)} = (-1)^n E_n \cos((2n+1)\theta_n(t)),\ \ 0 \leqslant t < +\infty.$

and we look where $\theta_n(t) = \pi/2$ and $\pi/3$,
$\theta_n((n+1/2)u) \to_{n\to\infty} \theta(u)$.
$\left(e^{-z} - \dfrac{P_n(z)}{Q_n(z)}\right)^{1/n} \approx \exp(2\mathcal{V}(z/n) - 2c)$, inside a contour, where the complex potential $\mathcal{V}(z) = \mathcal{V}_i(z) - \mathcal{V}p(z) = \lim n^{-1}[\sum[\log(z - \text{interp.}) - \log(z - \text{poles})]]$ satisfies
1. its derivative $\mathcal{V}'$ takes opposite pure imaginary values on the two sides of the positive real axis $= E$, where $\mathcal{V}(u) = \pm i\theta(u)$.
2. $\mathcal{V}' - f'/(2nf) = \mathcal{V}' - 1/2$ takes opposite values on the two sides of the scaled arc of poles $F$, and vanishes at the endpoints $a$ and $b$.
3. for $\mathcal{V}$ itself, $\mathcal{V}(+\infty+0i) - \mathcal{V}(+\infty-0i) = 2\pi i$.




One has $\mathcal{V}''(z) = \dfrac{1}{2\sqrt{z^3(z/a-1)(z/b-1)}}$,
(Gonchar and Rakhmanov).

$$(72) \quad 1 - 9x - 25x^3 + 49x^6 + 81x^{10} + \ldots + (2n+1)^2(-x)^{\frac{n(n+1)}{2}} \ldots = 0.$$

Cette équation a une et une seule racine $x$, comprise entre zéro et l'unité, ce qui sera prouvé en toute rigueur au Chapitre XIII, et la forme (72) permet de la calculer avec six décimales exactes au moyen des quatre premiers termes seulement; on trouve

$$(73) \qquad x = 0{,}107653\ldots.$$

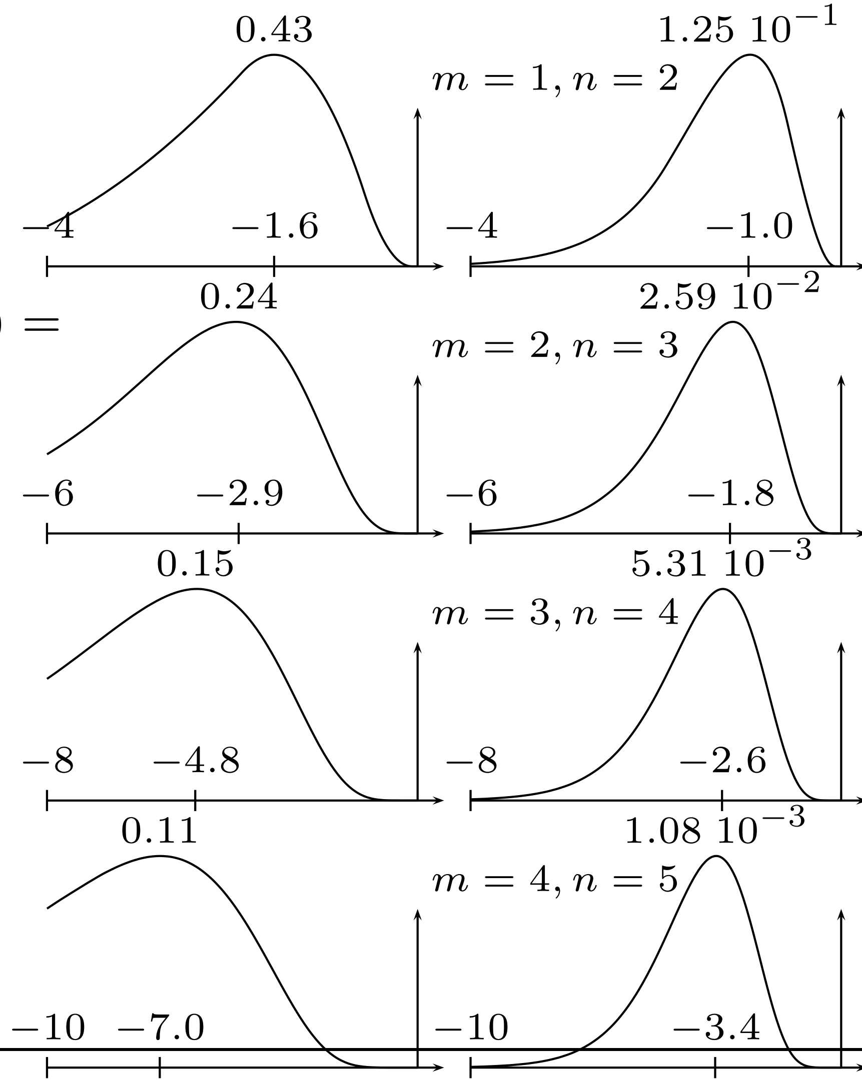
$\mathcal{V}'_{n,i}(z) = z^{-1} + \mathcal{V}'_n(z) - n(\mathcal{V}'_n(z))^2$, $\mathcal{V}'_{n,p}(z) = \mathcal{V}_{n,i}(z) - \mathcal{V}_n(z) = z^{-1} - n(\mathcal{V}'_n(z))^2$. Denominator $= \prod(1 - z/\text{poles}) \sim \exp(n(\mathcal{V}_{n,p}(z) - \mathcal{V}_{n,p}(0)) = \exp(-2z\cos\theta/\sqrt{a_1b_1} + z^2/(6na_1b_1) + \cdots)$, has a fixed limit when $n \to \infty$. Moreover, $\exp(-2z\cos\theta/\sqrt{a_1b_1}) = \exp(-0.71203\ldots z)$ fits with tables.

$q(x)e^x - p(x) = [x_0, \ldots, x_{m+n}, x]_{q(x)\exp(x)}(x - x_0)\cdots(x - x_{m+n})$. The product of the $x - x_i$'s behaves like $\exp(n\mathcal{V}_n(x))$, and the divided difference will be explored right now.

Denominator $q$ is the orthogonal polynomial of degree $n$ with respect to the scalar product $\langle f,g\rangle_n = [x_0,\dots,x_{m+n}]_{f(x)g(x)\exp(x)} = \sum_{j=0}^{m+n} \frac{f(x_j)g(x_j)\exp(x_j)}{\prod_{m\neq j}(x_j-x_m)} = \frac{1}{2\pi i}\int_{C_n} \frac{f(t)g(t)\exp(t)\,dt}{(t-x_0)\cdots(t-x_{m+n})}$

B-spline formula $\langle f,g\rangle_n = \int_{x_{m+n}}^{x_0} B(x)\frac{d^n}{dx^n}[f(x)g(x)e^x]\,dx$ ,

**The shape of things to come.** Here are some instances of $(m+n)!$ times $B(x)$ and $B(x)e^x$ on the $x_i$'s of best approximants, $m=n-1$. And CF [Gutknecht & Trefethen]?




It seems that the scaled $B(x)e^x$ tends towards a fixed function.

***Application to best approximation to*** $\exp(-(nA+B)z)$ on $[0,c]$

$$\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\,\mathcal{V}'(z) = \frac{A}{2\pi i}\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\,\frac{dt}{z-t}$$

$$\Rightarrow \mathcal{V}''(z) = \frac{\text{const. } z + \text{ const.}}{\sqrt{z^3(z-c)^3(z-a)(z-b)}}$$ (Herbert and the Russians).

$a, b = \dfrac{c(1+\alpha)(1\pm i\tan\xi)}{2(1\pm i\alpha\tan\xi)}$ found from two equations $Ac = \dfrac{\pi^2\alpha}{(1-\alpha^2)\mathsf{E}(\mathsf{K}-\mathsf{E})}$ and $\displaystyle\int_0^\pi \frac{\sqrt{1+\tan^2\xi\cos^2\theta}}{(1+i\alpha\tan\xi\cos\theta)^2}d\theta = 0$, (integral of ***third*** kind), with unknowns $\alpha$ and $k=\sin\xi$.

The error decreases essentially (root asymptotics) like $\rho^n$, with $\log\dfrac{1}{\rho} = \pi\dfrac{\alpha(\mathsf{K}-\mathsf{E})(\mathsf{K}'-\mathsf{E}')-\mathsf{E}\mathsf{E}'}{(\alpha-1)\mathsf{E}(\mathsf{K}-\mathsf{E})}$

***Strong asymptotics*** from Aptekarev (2001): $E_n \sim 2\rho^n\rho_B$, where $2\log\rho_B =$ Re $\{(\mathcal{V}_{B,+} + (z) + \mathcal{V}_{B,-}(z))_E - [\mathcal{V}_{B,+}(z) + \mathcal{V}_{B,-}(z) + 2Bz]_F\}$, $\mathcal{V}'_B = \widetilde{\mathcal{V}}'$ being analytic outside $E \bigcup F$, taking opposite values on the two sides of $E = [0, c]$, $\mathcal{V}'_B(z) + B$ taking opposite values on the two sides of $F$, or any arc of endpoints $a$ and $b$, and corresponding to a positive unit charge on $F$, and a negative unit charge on $E$, and finally $\mathcal{V}'_B(z) =$ const. $z^{-2} + \cdots$ when $z \to \infty$.

The problem is solved by $\mathcal{V}_{A/2} = \mathcal{V}$ if $B = A/2$. And if $B = 0$? Then, $\mathcal{V}'_0$ is the simple algebraic function $\mathcal{V}'_0(z) = \dfrac{\text{constant}}{\sqrt{z(z-c)(z-a)(z-b)}}$ associated to the potential of a plain (and plane) condenser $(E, \widetilde{F})$, although we do not need to know what $\widetilde{F}$ is. The capacity is $2\mathsf{K}/(\pi\mathsf{K}')$, and $\rho_0 = \exp\left(-\dfrac{\pi}{2}\dfrac{\mathsf{K}'}{\mathsf{K}}\right)$. And for any $B$, $\mathcal{V}_B = \dfrac{2B}{A}\mathcal{V} + \left(1 - \dfrac{2B}{A}\right)\mathcal{V}_0$ does the trick, see Meinguet [2000] for such relations. So, $\rho_B = \rho^{B/A}\rho_0^{(1-2B/A)}$ . and we just have to get $\rho_0 = \exp(-1/C)$, where $C$ is the plain condenser capacity of $(E, \widetilde{F})$.

Now, we look at some error norms $E_n = \left\| e^{-nz} - \dfrac{p_n(z)}{q_n(z)} \right\|_\infty$ on $[0, c]$, and the products $\rho^{-n}E_n/2$ which should tend towards $\rho_0$:

| $n$ | $c =$ | $1$ | $c =$ | $5$ | $c =$ | $\infty$ |
|---|---|---|---|---|---|---|
| | $E_n$ | $\rho^{-n}E_n/2$ | $E_n$ | $\rho^{-n}E_n/2$ | $E_n$ | $\rho^{-n}E_n/2$ |
| 1 | $1.58E-3$ | $0.04509$ | $3.13E-2$ | $0.1946$ | $6.68E-2$ | $0.3104$ |
| 2 | $3.19E-5$ | $0.05206$ | $2.76E-3$ | $0.2140$ | $7.35E-3$ | $0.3175$ |
| 4 | $1.06E-8$ | $0.05667$ | $1.88E-5$ | $0.2248$ | $8.65E-5$ | $0.3221$ |
| 5 | $1.9E-10$ | $0.05771$ | $1.52E-6$ | $0.2270$ | $9.34E-6$ | $0.3232$ |
| lim | | $0.06241$ | | $0.236$ | | $0.328$ |
| $e^{-\frac{\pi K'}{2K}}$ | | $0.06240$ | | $0.2362$ | | $0.328$ |

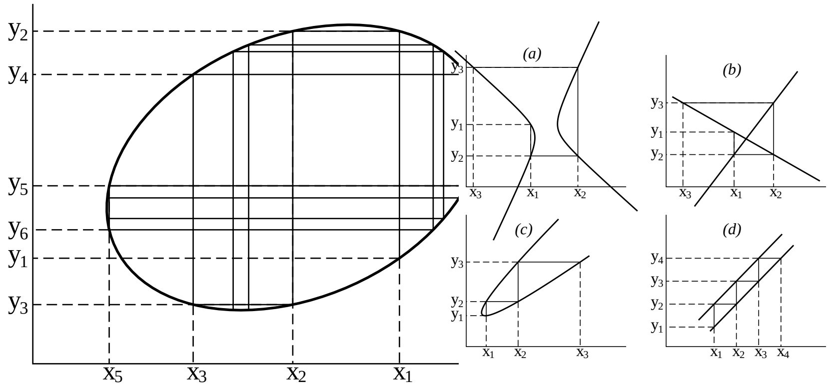
The last rows are: the limit when $n \to \infty$ estimated through a simple step of Thiele interpolatory continued fraction, i.e., $\lambda$ from $\lambda + \mu/(n+\nu)$ interpolation three values, $=$ first nontrivial step of $\rho-$algorithm [Brezinski];
and the formula $\exp(-\pi \mathsf{K}'/(2\mathsf{K}))$.

For more, see `http://perso.uclouvain.be/alphonse.magnus/num3/m3xxx00.pdf`

Was elliptic functions such a dead subject (before Brent, Salamin, and the Borweins)? Not so in Leuven-Louvain: Georges Lemaître, of big bang fame, taught analytical mechanics with elliptic functions examples (to often bewildered students), and Vitold Belevitch used them in filtering problems (see papers by J. Todd) in MBLE research lab. J.Meinguet was in both lines, so was J.P.Thiran (Namur). See papers and books by M. d'Udekem-Gevers (Namur). And now, cryptography (J.J. Quisquater).

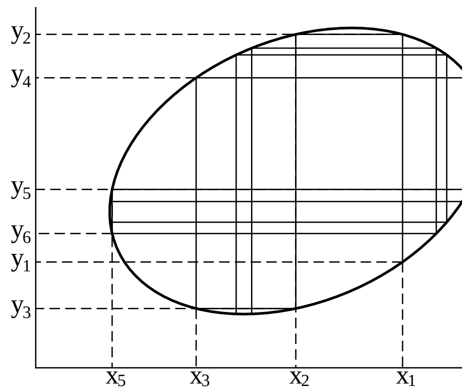
## 3. These &@* elliptic lattices.

I met the Devil in Bar-le-Duc (1984). He told me this:




**Manifesto.** *(Bi)orthogonal polynomials and rational functions satisfy remarkably simple (?) differential or difference equations and relations when the orthogonality form satisfies itself a differential or difference equation of first order, a Riccati equation* $a\mathscr{D}f = bf^+f^- + c(f^+ + f^-) + d$, with *polynomial coefficients. The statement is valid for any difference operator of the form* $(\mathscr{D}f)(x) = \dfrac{f^+(x) - f^-(x)}{x^+ - x^-} = \dfrac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}$, *where* $\varphi$ *and* $\psi$ *are the two* $y-$*roots of* $F(x,y) = 0$ *of second degree in* $y$*. The* **adjoint** *operator* $(\mathscr{D}^\dagger g)(y) = \dfrac{g(\psi^{-1}(y)) - g(\varphi^{-1}(y))}{\psi^{-1}(y) - \varphi^{-1}(y)}$ *must be in the same class* $\Rightarrow F$ **biquadratic**.

(The Devil did not tell everything). In a nutshell: $f$ is approximated by $q_n/p_n$ with an error of order $2n$ (Padé setting), or vanishing at $2n$ points (interpolation setting) $f = \dfrac{q_n}{p_n} + \varepsilon_{2n}, 0 = a\mathscr{D}\dfrac{q_n}{p_n} - b\dfrac{q_n^+ q_n^-}{p_n^+ p_n^-} - c\left(\dfrac{q_n^+}{p_n^+} + \dfrac{q_n^-}{p_n^-}\right) - d+$ something small, say $\eta_{2n}$, so $a\dfrac{q_n^+ p_n^- - q_n^- p_n^+}{x^+ - x^-} - bq_n^+ q_n^- - c[q_n^+ p_n^- + q_n^- p_n^+] - dp_n^+ p_n^- = -\eta_{2n} p_n^+ p_n^-$ which is a rational function of bounded degree, whence a lot of relations, together with Casorati-Wronski relations, etc. Higher order Riccati-like equations match other orthogonality-approximation schemes: multiple (or $d-$) orthogonality.

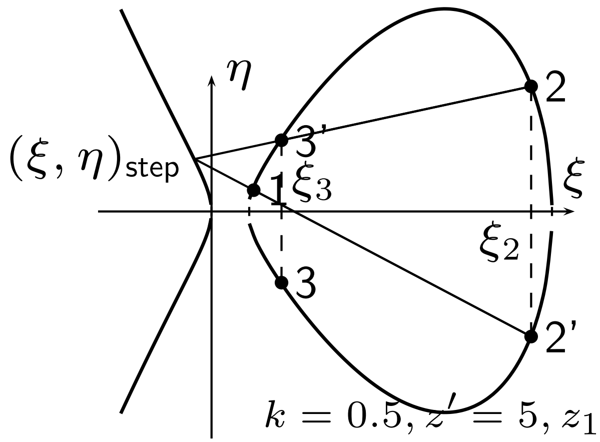



**Why elliptic?** Successive points on the curve $F(x,y) = X_2(x)y^2 + X_1(x)y + X_0(x) = 0$ are $(x_n, y_n)$, then $(x_n, y_{n+1})$ where $y_{n+1}$ is the other $y-$root of $F(x_n, y) = 0$, and $(x_{n+1}, y_{n+1})$ where $x_{n+1}$ is the other $x-$root of $F(x, y_{n+1}) = 0$. Relation with the more familiar picture of points in arithmetic progression on a cubic (***elliptic curve***): with $P := X_1^2 - 4X_0X_2$, of zeros $z_1, \ldots, z_4$, so that $y = \dfrac{-X_1 \pm \sqrt{P}}{2X_2}$, let the birational transformation

$$\xi = \frac{1}{x - z_1}, \qquad x = z_1 + \frac{1}{\xi},$$

$$\eta = \frac{X_1(x) + 2yX_2(x)}{(x - z_1)^2}, \quad y = \frac{-X_1(z_1 + 1/\xi) + \eta/\xi^2}{2X_2(z_1 + 1/\xi)},$$

(Appell & Goursat), then $\eta^2 = \xi^4 P(z_1 + 1/\xi)$ which is the cubic polynomial $P_3(\xi) = P'(z_1)\xi^3 + (P''(z_1)/2)\xi^2 + (P'''(z_1)/6)\xi + P''''/24$.




The line $\eta = \eta_n - \dfrac{\eta_{n+1} + \eta_n}{\xi_{n+1} - \xi_n}(\xi - \xi_n)$ joining $(\xi_n, \eta_n)$ to $(\xi_{n+1}, -\eta_{n+1})$ meets the cubic curve at $(\xi, \eta)_{\text{step}}$ independent of $n$. see `http://perso.uclouvain.be/alphonse.magnus/num3/m32006.pdf`

What are the classical recurrence coefficients?

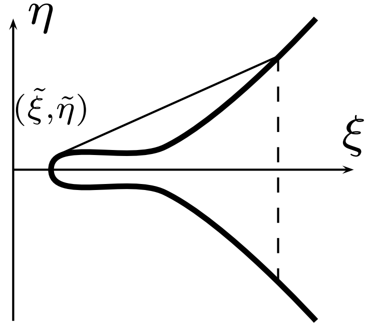
$$\left.\begin{matrix} d/dx \\ \Delta \end{matrix}\right\} \text{rat. f. of degree } 4 \text{ in } n, \quad \left.\begin{matrix} D_q \\ \text{ANSUW} \end{matrix}\right\} \text{rat. f. of degree } 4 \text{ in } q^n, \quad \left.\begin{matrix} \text{Ellipt.} \\ \phantom{x} \end{matrix}\right\} \text{and now?}$$

Discussing recurrence relations is equivalent to discussing the continued fraction $f_n(x) = \dfrac{x - y_{2n}}{\alpha_n + \beta_n x - (x - y_{2n+1}) f_{n+1}(x)}$, where $\alpha_n + \beta_n x$ interpolates $(x - y_{2n})/f_n(x)$ at $y_{2n+1}$ and $y_{2n+2}$. If $a_n \mathscr{D} f_n = b_n f^+ f_n^- + c_n(f_n^+ + f_n^-) + d_n$, $f_n(y_{2n+1}) = d_n(x_{2n})/[a_n(x_{2n})/(y_{2n+1} - y_{2n}) - c_n(x_{2n})]$, etc.

**Manifesto (continued).** *When the degrees of $a, b, c, d$ are $\leqslant 3$, **classical** setting holds: explicit formulas for recurrence relations (done by V. Spiridonov and A. Zhedanov in 2000 with theta functions), elliptic hypergeometric expansions (?), elliptic Rodrigues formulas (??), difference relations and equations.*

Partial check: at each of the 4 zeros $z_1, \ldots, z_4$, one has $a_n^2(z_i) = C_n \dfrac{z_i - x_{2n-1}}{z_i - x_{-1}} a_0^2(z_i)$ (Luminy 2007 - JCAM 2009). As a polynomial of $3^{\text{rd}}$ degree is completely determined by 4 values, we see why explicit formulas are possible. But how can $z_i - x_{2n-1}$ be a square?

In the simpler trigonometric case, $1-\sin\theta = 1-\cos(\theta-\pi/2) = 2\sin^2(\theta/2-\pi/4)$. Also, the three differences $e_i - \wp(x)$ of the Weierstrass elliptic function are squares of remarkable functions (see the *Painlevé Handbook* of Conte & Musette).




Here: we relate $(x,y)$ on the biquadratic curve to $(\tilde x,\tilde y)$ on the same curve, such that $(\xi,\eta)$ is the double of $(\tilde\xi,\tilde\eta)$ according to the addition rule of elliptic curves: the tangent at $(\tilde\xi,\tilde\eta)$ must meet the cubic curve $\eta^2 = P_3(\xi)$ at $(\xi,-\eta)$. Equality of slopes: $\dfrac{-\eta-\tilde\eta}{\xi-\tilde\xi} = \dfrac{P_3'(\tilde\xi)}{2\tilde\eta}$ : $4\eta^2\tilde\eta^2 = 4P_3(\xi)P_3(\tilde\xi) = [2P_3(\tilde\xi)+(\xi-\tilde\xi)P_3'(\tilde\xi)]^2$

$\Rightarrow \xi = \tilde\xi + \dfrac{(P_3'(\tilde\xi))^2 - 2P_3''(\tilde\xi)P_3(\tilde\xi)}{2P_3'''P_3(\tilde\xi)/3} = -2\tilde\xi + \Sigma + \dfrac{3(P_3'(\tilde\xi))^2}{2P_3'''P_3(\tilde\xi)}$, where $\dfrac{P'''}{6}(\tilde\xi^3 - \Sigma\tilde\xi^2 + \Pi\xi - Q) = P_3(\tilde\xi) = \tilde\xi^4 P(z_1 + 1/\tilde\xi)$. Note that $\Sigma = \rho_2+\rho_3+\rho_4$, where $\tilde\xi = \rho_i = (z_i - z_1)^{-1}, i = 2,3,4$ are the three roots of $P_3(\tilde\xi) = \tilde\xi^4 P(z_1 + 1/\tilde\xi) = 0$. Then, $\xi = \dfrac{\tilde\xi^4 - 2\Pi\tilde\xi^2 + 8Q\tilde\xi + \Pi^2 - 4Q\Sigma}{4(\tilde\xi^3 - \Sigma\tilde\xi^2 + \Pi\xi - Q)}$,

$$\xi - \rho_i = \frac{(\tilde\xi^2 - 2\rho_i\tilde\xi + \Pi - 2Q/\rho_i)^2}{4(\tilde\xi^3 - \Sigma\tilde\xi^2 + \Pi\xi - Q) = 4P_3(\tilde\xi)/(P_3'''/6) = 4\tilde\eta^2/(P_3'''/6)},$$

$$\frac{z_i - x_{2n-1}}{z_1 - x_{2n-1}} = \left[\frac{1}{\rho_i} - \frac{1}{\xi_{2n-1}}\right]\xi_{2n-1} = \frac{(\tilde\xi_n^2 - 2\rho_i\xi_n + \Pi - 2Q/\rho_i)^2}{4\rho_i\tilde\eta_n^2/(P'''/6)},.\ \text{YES!}$$

**Manifesto (end).** *And when the degrees $> 3$, one gets nonlinear equations for the recurrence coefficients, probably discrete Painlevé equations.* Or Freud equations (probably the ugly kind).


Estamos vivos?
Sim... Vou ver como está Beryl.