Diffusion of euro currency.

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CONTENTS

Abstract: European currency has been distributed in the various european countries in different, economically equivalent, forms.

1. **Introduction**

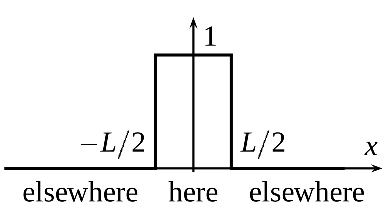
People of Euroland now use coins with a distinctive feature of the country where they come from. It may be the head of a king, queen, or grand duke, or a famous building, or the face of a great human being of the past (Dante, Mozart, Cervantes), or some other symbol, see **http://www.geocities.com/eurocoin2003/revers/**

One takes a big interest in understanding how fast foreign coins travel.

2. **Diffusion equation for euro coins**

2.1. A very simple model. People a_1, a_2,..., a_N regularly spread on the real line look how the proportion of their own currency decreases with time.

Let $y_i(t)$ be the proportion of local currency owned by a_i at time t. Let t_0 be the average time needed for a full exchange of the coins of a_i with the ones of its two neighbours, so



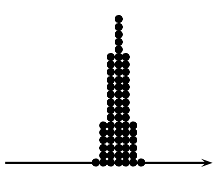
$$y_i(t+t_0) = \frac{y_{i-1}(t)+y_{i+1}(t)}{2} \quad (1)$$

starting with $y_i(0) = 1$ on an interval of length L and $y_i(0) = 0$ elsewhere (il ne sagit que de la proportion de monnaie du pays considéré, la quantité d'argent détenue par les gens ne change pas[1].

[1] Mais alors, à quoi sert l'argent???

Should the country of interest be very narrow, several applications of the rule (1) would rapidly show the typical shape of Galton's nailed board

Google Search
Searched the web for galton nail board. Results 1 - 10 of about 233. Search took 0.10 seconds.
Galton board and the binomial distribution ... simulates Galton's Board, in which balls are dropped through a triangular array of nails. This device is also called a quincunx. Every time a ball hits a nail ...
www.stattucino.com/berrie/dsl/Galton.html - 8k - Cached - Similar pages
www.stat.ucla.edu/˜dinov/courses_students.dir/Applets.dir/QuincunxApplet.html - 3k - Cached - Similar pages
QUINCUNX ... Galton named this board "the quincunx", by a peculiar Latin derivation. ... finally, use board 1/2" x 1" x 18", driven in, to prop up top of the nail-board. ...
members.fortunecity.com/jonhays/quincunx.htm - 6k - Cached - Similar pages
Untitled ... Galton is also responsible for the pinball machine! ... finally, use board 1/2" x 1" x 18", driven in, to prop up top of the nail-board. ...
members.fortunecity.com/jonhays/galton.htm - 7k - Cached - Similar pages
Quincunx or Galtons Board ... Introduction: The Galtons Board resembles a Pascals triangle. ... A 5 row Quincunx has 1 nail on the top and 5 in the bottom. ...
math.allconet.org/synergy/sami/quincunx.htm - 14k - Cached - Similar pages
Lab III ... to simulate the particle movement in Galton's Board, in which balls are dropped through a triangular array of nails. Every time a ball hits a nail it has a ...
www.cz3.nus.edu.sg/˜chenk/gem2503/lab3.htm - 5k - Cached - Similar pages
[PDF]APL IN COMPUTER - ASSISTED INSTRUCTION: SIMULATION OF STOCHASTIC ... File Format: PDF/Adobe Acrobat ... demonstrate how much we took advantage of the interactivity of APL, the Galton board program will ... 0 ˜ p ˜ I). The event is a ball's deflection at a nail to the ...
delivery.acm.org/10.1145/810000/803708/p447-wittig.pdf?key1=803708&key2=0258764401&coll=portal&am... Similar pages
[PDF]arXiv:cond-mat/0101281 v1 18 Jan 2001 File Format: PDF/Adobe Acrobat - View as HTML ... This is analogous to the statistics of a falling ball on a Galton's nail-board where also a 3 Page 4. single trajectory is not touching all nails but is random.
arxiv.org/pdf/cond-mat/0101281 - Similar pages
©2003 Google
http://archives.math.utk.edu/software/msdos/probability/jkgalton/.html



associated to entries of the Pascal's triangle:

1
1/2 1/2
1/4 1/2 1/4
1/8 3/8 3/8 1/8
1/16 1/4 3/8 1/4 1/16

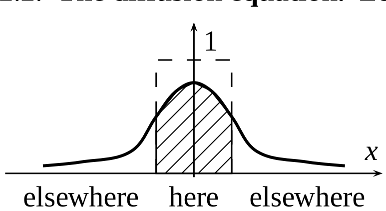
The continuous limit is better understood by writing (1) as

$$y_i(t+t_0) - y_i(t) = \frac{y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)}{2}$$

$$\text{or} \quad \frac{y_i(t+t_0) - y_i(t)}{t_0} = \sigma\frac{y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)}{d_0^2}$$

si d_0 est la distance moyenne entre deux acteurs, et $\sigma := \dfrac{d_0^2}{2t_0}$.

2.2. **The diffusion equation.** Les quotients ainsi mis en évidence sont proches de dérivées:



$$\frac{\partial y(x,t)}{\partial t} = \sigma\frac{\partial^2 y(x,t)}{\partial x^2} \tag{2}$$

Poisson:

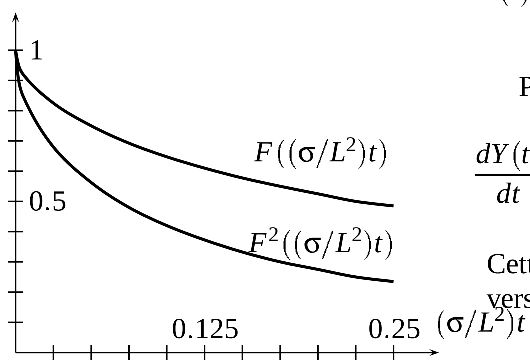
$$y(x,t) = \int_{-L/2}^{L/2} \sqrt{\frac{t_0}{2d_0^2\pi t}} \exp\left(-\frac{t_0(x-x_0)^2}{2d_0 t}\right)\, dx_0 = \frac{\operatorname{erf}\left(\sqrt{\frac{t_0}{2td_0^2}}\left(\frac{L}{2}-x\right)\right) + \operatorname{erf}\left(\sqrt{\frac{t_0}{2td_0^2}}\left(\frac{L}{2}+x\right)\right)}{2}$$

où $\mathrm{erf}(X) := \frac{2}{\sqrt{\pi}}\int_0^X e^{-u^2}\,du$. Remarquons que

$$\frac{\partial y(x,t)}{\partial x} = \sqrt{\frac{t_0}{2d_0^2\pi t}}\left[\exp\left(-\frac{t_0(L/2+x)^2}{2td_0^2}\right) - \exp\left(-\frac{t_0(L/2-x)^2}{2td_0^2}\right)\right]$$

Au temps $t > 0$, la distribution uniforme confinée au pays d'origine s'est affaisée et a quelque peu envahi les pays voisins. La proportion encore disponible dans le pays d'origine (c'est la valeur la plus facile à mesurer) est donnée par la partie hachurée

$$Y(t) = \frac{1}{L}\int_{-L/2}^{L/2} y(x,t)\,dx$$



Par (2), on a aussi

$$\frac{dY(t)}{dt} = 2\frac{d_0^2}{2Lt_0}\frac{\partial y(L/2,t)}{\partial x} = \sqrt{\frac{d_0^2}{2L^2t_0\pi t}}\left(\exp\left(-\frac{t_0L^2}{2td_0^2}\right) - 1\right)$$

Cette moyenne nationale décroît avec t selon une fonction universelle F de $(\sigma/L^2)t$.

$$Y(t) = F((\sigma/L^2)t); \qquad F(X) = \sqrt{\frac{4X}{\pi}}\left(\exp\left(-\frac{1}{4X}\right) - 1\right) + \mathrm{erf}\left(\sqrt{\frac{1}{4X}}\right).$$

$(\sigma/L^2)t$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
$F((\sigma/L^2)t)$	0.822	0.748	0.692	0.647	0.610	0.578	0.550	0.526	0.505	0.486
$F^2((\sigma/L^2)t)$	0.676	0.560	0.479	0.419	0.372	0.334	0.303	0.277	0.255	0.236

(Au fait, en appliquant directement (1), on obtient $Y = 1 - \frac{3}{2}\frac{5}{4}\cdots\frac{2k-1}{2k-2}\frac{d_0}{L}$ après $2k$ pas de temps, si $d_0 << L$ Cela donne $\approx 1 - \sqrt{\frac{4k}{\pi}}\frac{d_0}{L} = 1 - \sqrt{\frac{2t}{t_0\pi}}\frac{d_0}{L} = 1 - \sqrt{\frac{4\sigma t}{L^2\pi}}$ pour $(\sigma/L^2)t$ petit.

For a ***two-dimensional*** problem, the equation is $\partial u(x,y,t)/\partial t = (\sigma/2)[\partial^2 u(x,y,t)/\partial x^2 + \partial^2 u(x,y,t)/\partial y^2]$. Pour un **rectangle**, on obtient $Y_1(t)Y_2(t) = F((\sigma/L_1^2)t)F((\sigma/L_2^2)t)$, avec les longueurs L_1 et L_2 des deux côtés. Avec un rectangle, on peut déjà respecter le rapport superficie/longueur de frontière d'un pays (N.B.: frontière commune avec l'Euroland!). Et tenir compte du nombre moyen de connexions (monétaires) entre les agents et leurs voisins.

Bon, avec $d_0 \approx 1$ km (distance moyenne au supermarché ou à la pompe la plus proche[2]), $L \approx$ une centaine de km, et $t_0 \approx 1$ mois, on devrait être tout au plus à un millième dans la figure ci-dessus.

[2] oui, mais si on paie par carte?

date	BEL.	NED.	LUX
1 fév. 2002	0.888	0.910	0.750
1 mar. 2002	0.812	0.869	0.810
1 avr. 2002	0.814	0.822	0.660
1 mai 2002	0.788	0.826	0.660
1 juin 2002	0.766	0.780	0.510
1 juil. 2002	0.802	0.789	0.570
1 aoû. 2002	0.734	0.749	0.590
1 sep. 2002	0.722	0.755	0.510
1 Oct. 2002	0.753	0.759	
1 Nov. 2002	0.751	0.757	
1 Dec. 2003	0.743	0.772	
1 Jan. 2003	0.737	0.770	
1 Feb. 2003	0.748	0.757	

On voit que la théorie précède l'observation. Eh bien, observons. Une énorme documentation portant sur des mesures effectuées par des centaines de gens aux Pays-Bas et en Belgique (du nord...) est rassemblée à **http://www.wiskgenoot.nl/eurodiffusie**)

On y voit[3], parmi d'autres informations disponibles (voir leur section "download"), la répartition des monnaies en Belgique, aux Pays-Bas et au Luxembourg[4] en fonction du temps.

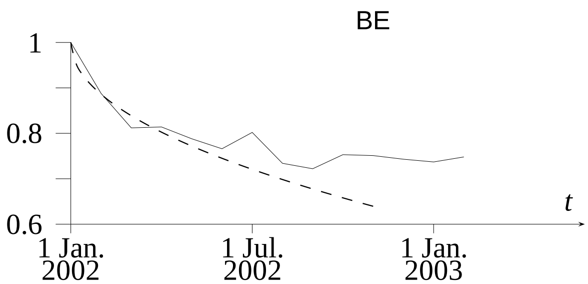
On en estime très approximativement le σ/L^2 de la Belgique et des Pays-Bas à ≈ 0.0022 et $0.0018\,\text{mois}^{-1}$, et à $\approx 0.01\,\text{mois}^{-1}$ pour le Luxembourg.

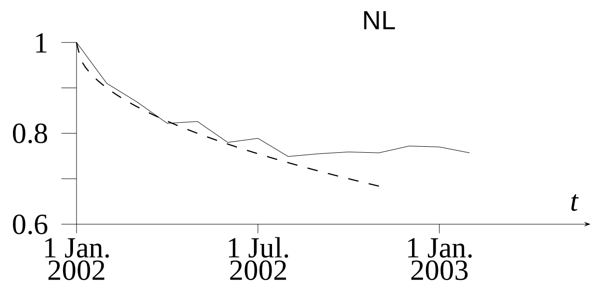
```
c: eurofit .m       1636  19.02.103  12:02   \calc\matlab4

%  eurofit.m       local euros in Belgium and Netherlands
%
%  data from     http://www.wiskgenoot.nl/eurodiffusie
%
%
%    U(t) = proportion of local euros at time t, averaged on a
%           whole country.
%
x=0:13;     % 1 Jan 2002  ,  1 Feb 2002 , ... , 1 Feb 2003.
%BE
y=[1 0.888  0.812   0.814   0.788  0.766  0.802  0.734  0.722 0.753  0.751 0.743 0.737 0.748];
plot(x,y);
title(' Local euros in Belgium');
hold on;
%ys=(1-y).^2;
%[p,S]=polyfit(x,ys,3);
%p,
%xp=0:0.1:10;
%p(4)=0;
%[yp,delta]=polyval(p,xp,S);
%plot(xp,1-sqrt(0.00000000001+yp),'r');
%
%    model solution is: U(t)=mean value on a square of the solution
%      of the diffusion equation
%
%         du(x,y,t)/dt = sigma Laplacian u(x,y,t)
%
%   with initial u= 1 on the square   , and u=0 elsewhere.
%
%  Solution: U(t)= square of one-dimensional solution
%                                             (see below)
xe=0.2:0.2:13;
sig=0.0022;s4=1/(4*sig);ye=sqrt(4*sig*xe/pi).*(exp(-s4./xe)-1)+ ...
          erf(sqrt(s4./xe));    ye=ye.*ye;
   xe(1:5),ye(1:5),
 for ix=12:12:96,  xx=ix;
   yy=sqrt(4*sig*xx/pi)*(exp(-s4/xx)-1)+ erf(sqrt(s4/xx));yy=yy^2;[xx,yy],
```

[3]New site is now **`http://www.eurodiffusie.nl`**

[4]For Luxemburg, data furnished by family of T. Hitzky.

```
 end;
plot([0,xe],[1,ye],'g');
print -dps eurofb.ps
pause;
hold off;
%NL
y=[1 0.910  0.869  0.822 0.826 0.780 0.789 0.749 0.755 0.759 0.757 0.772 0.770 0.757];
plot(x,y);
title(' Local euros in Netherlands');
hold on;
sig=0.0018;s4=1/(4*sig);ye=sqrt(4*sig*xe/pi).*(exp(-s4./xe)-1)+ ...
          erf(sqrt(s4./xe));    ye=ye.*ye;
plot([0,xe],[1,ye],'g');
print -dps eurofn.ps
%LUX
%y=[1 0.750  0.810  0.660  0.660  0.510  0.570  0.590  0.510 ];
```





3. **Effect of adding fresh coins.**

After about 6 months, the decreasing trend seems to have died.

We consider that during each exchange period t_0, the national bank makes available a fraction c of fresh national coins. As the number of various coins in each pocket is still supposed constant, the same fraction c of owned coins is removed, so that (1) is now

$$y_i(t+t_0) = (1-c)\frac{y_{i-1}(t)+y_{i+1}(t)}{2} + c, \qquad -L/2 \leqslant x \leqslant L/2, \tag{3}$$

whereas fresh coins brought abroad are of course of THEIR own kind, and our own brand decreases even faster:

$$y_i(t+t_0) = (1-c)\frac{y_{i-1}(t)+y_{i+1}(t)}{2}, \qquad |x| > L/2, \tag{4}$$

In the continuous limit:

$$\frac{\partial y(x,t)}{\partial t} = \sigma\frac{\partial^2 y(x,t)}{\partial x^2} - \gamma y(x,t) + \gamma\chi_{(-L/2,L/2)}(x), \tag{5}$$

with $\gamma = c/t_0$, and where $\chi_{(a,b)}(x) = 1$ when $a < x < b$ and vanishes elsewhere.

Limit solution when $t \to \infty$:

$$\begin{aligned} y(x,\infty) &= 1 - \exp\left(-\sqrt{\frac{\gamma}{\sigma}}\frac{L}{2}\right)\cosh\left(\sqrt{\frac{\gamma}{\sigma}}\,x\right), \qquad -L/2 \leqslant x \leqslant L/2, \\ &= \sinh\left(\sqrt{\frac{\gamma}{\sigma}}\frac{L}{2}\right)\exp\left(-\sqrt{\frac{\gamma}{\sigma}}\,|x|\right), \qquad |x| \geqslant L/2. \end{aligned}$$

Average on $(-L/2, L/2)$:

$$\frac{1}{L}\int_{-L/2}^{L/2} y(x,\infty)\,dx = 1 - \frac{1-\exp\left(-\sqrt{\frac{\gamma}{\sigma}}\,L\right)}{\sqrt{\frac{\gamma}{\sigma}}\,L} \tag{6}$$

In http://www.senat.fr/rap/l01-087-344/l01-087-34429.html, it is stated that the French bank intended to release about 6.5 10^9 coins on January 2002, followed by a regular flow of about 3 10^8 coins each month. This means γ about 0.05 month^{-1}. Assuming the same value here, and with σ/L^2 near to 0.002, we have $\gamma L^2/\sigma \approx 25$. The value of γ may decrease with time...

It figures: if $G(t) = \frac{1}{L}\int_{-L/2}^{L/2} y(x,\infty)\,dx$, we must consider G^2 for a two-dimensional problem, and here are some limit values, from (6):

$(\gamma L^2/\sigma)$	1	2	3	4	5	10	15	20	25	30	50
$G(\infty)$	0.368	0.465	0.525	0.568	0.601	0.697	0.747	0.779	0.801	0.818	0.859
$G^2(\infty)$	0.135	0.216	0.275	0.322	0.361	0.486	0.558	0.607	0.642	0.669	0.737

Simple discretization of (5):

$$y_i(t+\Delta t) = y_i(t) + \sigma\Delta t\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \gamma\frac{y_{i-1}+y_{i+1}}{2} + \gamma\Delta t\chi_i$$

```
 10   '  eurodi2.ub
 20   N=500:point 2:emaword 2:dim ema(0;2*N,2)
 30   ' screen 21:color:console 0,*,0
 50   Lh=100
 60   for I=N-Lh to N+Lh:ema(0;I,0)=1:next I
 68   ' Gls = gamma L^2/sigma   ,  Delta t=t0 h^2/d0^2 = h^2/(2 sigma)
 70   Gls=25.0:Gdt=Gls/(8*Lh^2):Limp=1-(1-exp(-sqrt(Gls)))/sqrt(Gls)
 72   print "gamma L^2/sigma= ";Gls;"  ,  limit=";Limp;" square= ";Limp^2
100   for It=1 to 50000:Som=0:for I=N-Lh to N+Lh:Som=Som+ema(0;I,0):next I
105   ' if It@25=0 then print It*Gdt/Gls;" ";ema(0;N,0);" ";0.5*Som/Lh
106   if It@25=0 then print "(";10*It*Gdt/Gls;", ";5*(Som/Lh)^2;")%"
107   ' if It@25=0 then for I=1 to 2*N-1:line (I,420-400*ema(0;I,0))-(I+1,420-400*ema(0;I+1,0)):next I:line (I
110    for I=1 to N-Lh:ema(0;I,1)=0.5*(1-Gdt)*(ema(0;I-1,0)+ema(0;I+1,0)):next I
120    for I=N-Lh to N+Lh:ema(0;I,1)=0.5*(1-Gdt)*(ema(0;I-1,0)+ema(0;I+1,0))+Gdt:next I
130    for I=N+Lh to 2*N-1:ema(0;I,1)=0.5*(1-Gdt)*(ema(0;I-1,0)+ema(0;I+1,0)):next I
150    for I=0 to 2*N:ema(0;I,0)=ema(0;I,1):next I
175   next It
```