Formulas for recurrence coefficients of orthogonal polynomials related to Lorentzian-like weights.

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Abstract. One considers the recurrence relation of orthogonal polynomials related to weights $|t|^A(1+t^{2r}/c^{2r})^{-B}$ on the whole real line, for various integer exponents $2r$.

1. Introduction.

We consider a set of polynomials p_k of degree k for each $k = 0, 1, \ldots$ and all of which are mutually orthogonal over the weight function w:

$$\int_a^b w(x)p_k(x)p_m(x)\mathrm{d}x = 0 \qquad \text{if } k \neq m. \tag{1}$$

Monic orthogonal polynomials can be generated from the general recurrence relation

$$\begin{aligned} p_{-1}(x) &= 0 \\ p_0(x) &= 1 && (2) \\ p_{k+1}(x) &= (x-\alpha_k)p_k(x) - \beta_k p_{k-1}(x) \qquad k = 0, 1, ... && (3) \end{aligned}$$

where α_k and β_k are inductively given by:

$$\alpha_k = \frac{\int_a^b x p_k^2(x)w(x)\mathrm{d}x}{\int_a^b p_k^2(x)w(x)\mathrm{d}x} \qquad k = 0, 1, ... \tag{4}$$

$$\beta_k = \frac{\int_a^b p_k^2(x)w(x)\mathrm{d}x}{\int_a^b p_{k-1}^2(x)w(x)\mathrm{d}x} \qquad k = 1, 2, ... \tag{5}$$

A Lorentz weight

$$w(t) = |t|^A(1+t^2/C)^{-B} \tag{6}$$

has simple known recurrence coefficients (see further on) if the weight is considered on the whole real line. Remark also that, if $B = C$ is large, we are close to the Hermite weight $|t|^A e^{-t^2}$.

However, many applications ask for Gaussian integration formulas associated to particular weight functions only on the positive half of the real line. A typical example is the Maxwell distribution $x^2e^{-x^2}$ needed on $x > 0$. Some authors [15,16,18] work with 'kappa' distributions $x^2(1+x^2/\kappa)^{-\kappa-1}$ which look like the Maxwell's distribution for small x, but decay much slower for large x.

2. Recurrence coefficients.

We proceed with general identities which will be needed here, and which can be found in any general textbook on orthogonal polynomials, such as [5, 9, 21], also books on formal orthogonal polynomials [4]

We only consider even weight functions on symmetric intervals $(-a, a)$, so that $\alpha_k = 0$. Then, $p_{2k}(\sqrt{x})$ are the orthogonal polynomials related to $w(\sqrt{x})/\sqrt{x}$ on $(0, a^2)$ [5,].

So, the weight functions $x^2e^{-x^2}$ and $x^2(1+x^2/\kappa)^{-\kappa-1}$ on the positive half-line are related to $|t|^5e^{-t^4}$ and $|t|^2(1+t^4/\kappa)^{-\kappa-1}$ on $t \in \mathbb{R}$.

We consider here the Lorentz-like weight

$$w(t) = |t|^A(1+t^{2r}/c^{2r})^{-B} \tag{7}$$

on the whole real line $-\infty < t < \infty$.

With these definitions, the first coefficients can be calculated analytically: the moments of (7) are

$$
\begin{aligned}
m_{2n} &= \int_{-\infty}^{\infty} |t|^{A+2n}(1+t^{2r}/c^{2r})^{-B}\mathrm{d}t \\
&= \frac{c^{A+2n+1}}{r}\int_0^1 u^{B-1-(A+2n+1)/(2r)}(1-u)^{(A+2n+1)/(2r)-1}\mathrm{d}u \\
&= \frac{c^{A+2n+1}}{r}\frac{\Gamma\left(B-\dfrac{A+2n+1}{2r}\right)\Gamma\left(\dfrac{A+2n+1}{2r}\right)}{\Gamma(B)},
\end{aligned}
$$

using $u = 1/[1+t^{2r}/c^{2r}]$. Of course, these moments are finite only while $n < (2rB - A - 1)/2$.

Then,

$$
\beta_1 = \frac{m_2}{m_0} = c^2\frac{\Gamma\left(B-\dfrac{A+3}{2r}\right)\Gamma\left(\dfrac{A+3}{2r}\right)}{\Gamma\left(B-\dfrac{A+1}{2r}\right)\Gamma\left(\dfrac{A+1}{2r}\right)}, \beta_2 = \frac{m_4}{m_2} - \frac{m_2}{m_0},
$$

etc. Further recurrence coefficients may be computed through the qd scheme (rhombus rules, see [10, p. 527], see also Brezinski [4, p. 166] for a relation with the $\varepsilon-$algorithm), as shown in the following small matlab/octave program [17]:

```
%  lor2006.m:   moments and recurrence coeff for
%
%     |t|^A (1+(t/c)^(2r))^(-B)  on  -infty, +infty
%
clear;rABc=input(' enter  r A B c  between [ ] ');
r=rABc(1);A=rABc(2);B=rABc(3);c=rABc(4);
nmx=floor(r*B-(A+1)/2);if nmx>=r*B-(A+1)/2, nmx=nmx-1;end;n=0:nmx;
% moments
mom=beta(B-(A+2*n+1)/(2*r),(A+2*n+1)/(2*r));
% qd
e=zeros(1,nmx);
q=c^2*mom(2:nmx+1)./mom(1:nmx);[1,q(1:min(nmx,5))],
for k=2:2:nmx,
   eaux=q(2:nmx-k+2)-q(1:nmx-k+1)+e(2:nmx-k+2);[k,eaux(1:min(nmx-k+1,5))],
 if k<nmx,q=(eaux(2:nmx-k+1)./eaux(1:nmx-k)).*q(2:nmx-k+1);
      [k+1,q(1:min(nmx-k,5))], end;
 e(1:nmx-k+1)=eaux(1:nmx-k+1);
end;
```

The first column of the output is made of the recurrence coefficients $\beta_1, \beta_2, \ldots$ related to the weight $w(t)$; the second, third,etc. columns are related to the weigths $t^2w(t), t^4w(t)$,etc.

Elegant as it may be, this algorithm is unsatisfactory in finite precision.

We will often need expansions of products $t^s p_k(t)$ in the basis $\{p_0, p_1, \ldots\}$. We only have to iterate (3) in the form $tp_k(t) = p_{k+1}(t) + \beta_k p_{k-1}(t)$:

$$
t\mathbf{p}(t) = \boldsymbol{M}\,\mathbf{p}(t)\ :\qquad t\begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ \beta_1 & 0 & 1 & & \\ & \beta_2 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ \vdots \end{bmatrix} \tag{8}
$$

then, $t^s\mathbf{p}(t) = \boldsymbol{M}^s\mathbf{p}(t)$, so that $\boldsymbol{M}^s_{i,j}$ is the coefficient of $p_j(t)$ in the expansion of $t^s p_i(t)$ (the indexes start at 0).

First instances:

$$
\begin{array}{cccccccc}
 & p_{i-3}(t) & p_{i-2}(t) & p_{i-1}(t) & p_i(t) & p_{i+1}(t) & p_{i+2}(t) & p_{i+3}(t) \\
tp_i(t) & & & \beta_i & 0 & 1 & & \\
t^2p_i(t) & & \beta_i\beta_{i-1} & 0 & \beta_i+\beta_{i+1} & 0 & 1 & \\
t^3p_i(t) & \beta_i\beta_{i-1}\beta_{i-2} & 0 & \beta_i(\beta_{i-1}+\beta_i+\beta_{i+1}) & 0 & \beta_i+\beta_{i+1}+\beta_{i+2} & 0 & 1
\end{array}
\tag{9}
$$

For t^4, $t^4p_i(t) = \beta_i\beta_{i-1}\beta_{i-2}\beta_{i-3}p_{i-4}(t) + \beta_i\beta_{i-1}(\beta_{i-2}+\beta_{i-1}+\beta_i+\beta_{i+1})p_{i-2}(t) + +(\beta_i\beta_{i-1}+\beta_i^2+2\beta_i\beta_{i+1}+\beta_{i+1}^2+\beta_{i+1}\beta_{i+2})p_i(t) + (\beta_i+\beta_{i+1}+\beta_{i+2}+\beta_{i+3})p_{i+2}(t) + p_{i+4}(t)$

3. Differential relations.

3.1. Orthogonal polynomials satisfying differential relations and equations.

We now come to a special class of weight functions allowing remarkable relations for the recurrence coefficients:

Lemma.*If the logarithmic derivative w'/w is a rational function, the recurrence coefficients satisfy exactly computable equations*

$$F_k(\alpha_0,\beta_1,\ldots,\alpha_{k+d-1},\beta_{k+d}) = 0, \qquad G_k(\alpha_0,\beta_1,\ldots,\alpha_{k+d-1},\beta_{k+d},\alpha_{k+d}) = 0,$$

for $k = 1,2,\ldots$, where d depends on the degree of the rational function w'/w. Moreover, the orthogonal polynomials satisfy differential relations and equations of the form

$$Pp_k' = Q_kp_k + R_kp_{k-1}, \qquad PR_kp_k'' + U_kp_k' + V_kp_k = 0,$$

where P, Q_k, R_k, U_k, and V_k are polynomials of fixed degree.

This statement has been discovered and rediscovered in various forms, see [14, 20]. Most authors are interested in the differential formulas for p_k, but the computationnaly interesting items are the F_k and G_k's.

3.2. The making of the equations.

We give here only a part of the proof for an even weight w on $|u| < a$, then $G_k = 0$. Suppose we have $w'(t)/w(t) = q(t)/p(t)$, where we also manage to have $\lim p(t)w(t) = 0$ when $t \to \pm a$. Then, by integration by parts,

$$
\begin{aligned}
\int_{-a}^{a} p(t)p_{k-1}(t)\,p_k'(t)\,w(t)\,\mathrm{d}t &= -\int_{-a}^{a} p_k(t)\,[p(t)p_{k-1}(t)w(t)]'\,\mathrm{d}t \\
&= -\int_{-a}^{a} p_k(t)\,p'(t)p_{k-1}(t)w(t)\,\mathrm{d}t \\
&\quad -\int_{-a}^{a} p_k(t)\,p(t)p_{k-1}'(t)w(t)\,\mathrm{d}t \\
&\quad -\int_{-a}^{a} p_k(t)\,p_{k-1}(t)\,p(t)w'(t)\,\mathrm{d}t
\end{aligned}
\tag{10}
$$

The first and the third of these three latter integrals involve the product of p_k and the polynomial of fixed degree $p'+q$ (after replacement of pw' by qw). By the rules of (9), the product is a linear combination of, say, p_{k+d}, $p_{k+d-1},\ldots$ with coefficients $[(p'+q)(\boldsymbol{M})]_{k,j}, j = k+d, k+d-1,\ldots$ which are simple polynomials in $\beta_{k+d},\ldots,\beta_{k-d}$. Then, by orthogonality of the p_ks with respect to w, the value of the two integrals comes out as the coefficient of p_{k-1} times $\|p_{k-1}\|^2$.

The left-hand side and the second of the three latter integrals are estimated in a similar way, after having written the derivative of a p polynomial in its own basis:

$$p_k'(t) = kp_{k-1}(t) + \delta_kp_{k-3}(t) + \epsilon_kp_{k-5}(t) + \cdots, \tag{11}$$

so that the left-hand side of (10) is

$$k[p(\boldsymbol{M})]_{k-1,k-1}\|p_{k-1}\|^2 + \delta_k[p(\boldsymbol{M})]_{k-3,k-1}\|p_{k-1}\|^2 + \epsilon_k[p(\boldsymbol{M})]_{k-5,k-1}\|p_{k-1}\|^2 + \cdots$$

and the right-hand side is

$$-(k-1)[p(\boldsymbol{M})]_{k-2,k}\|p_k\|^2 - \delta_{k-1}[p(\boldsymbol{M})]_{k-4,k}\|p_k\|^2 - \epsilon_{k-1}[p(\boldsymbol{M})]_{k-6,k}\|p_k\|^2 + \cdots - [(p'+q)(\boldsymbol{M})]_{k,k-1}\|p_{k-1}\|^2,$$

and the final equation is found after dividing by $\|p_{k-1}\|^2$, reminding that $\|p_k\|^2/\|p_{k-1}\|^2 = \beta_k$,

$$\begin{aligned} F_k(\beta_1,\dots,\beta_{k+d}) = -[(p'+q)(\boldsymbol{M})]_{k,k-1} - k[p(\boldsymbol{M})]_{k-1,k-1} - \delta_k[p(\boldsymbol{M})]_{k-3,k-1} - \epsilon_k[p(\boldsymbol{M})]_{k-5,k-1} - \cdots \\ - \beta_k\{(k-1)[p(\boldsymbol{M})]_{k-2,k} + \delta_{k-1}[p(\boldsymbol{M})]_{k-4,k} + \cdots\} = 0, \end{aligned} \tag{12}$$

for $k = 1, 2, \dots$ One needs a number of terms of the expansion in $\delta, \epsilon, \dots$ which depends on the width of the bandmatrix $p(\boldsymbol{M})$.

The coefficients δ_k, $\epsilon_k, \dots$ in (11) are polynomials in the β's too [2]: from $\dfrac{p_k(t)}{p_{k-1}(t)} = t - \dfrac{\beta_{k-1}}{t - \dfrac{\beta_{k-2}}{t - \cdots}} = t - \beta_{k-1}t^{-1} - \beta_{k-1}\beta_{k-2}t^{-3} + O(t^{-5})$,

$$\frac{p_k'(t)}{p_k(t)} - \frac{p_{k-1}'(t)}{p_{k-1}(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{p_k(t)}{p_{k-1}(t)}\right) = t^{-1} + 2\beta_{k-1}t^{-3} + [2\beta_{k-1}^2 + 4\beta_{k-1}\beta_{k-2}]t^{-5} + O(t^{-7})$$

whence, by summing,

$$\frac{p_k'(t)}{p_k(t)} = k\frac{p_{k-1}}{p_k} + \delta_k\frac{p_{k-3}}{p_k} + \epsilon_k\frac{p_{k-5}}{p_k} + O(t^{-7}) = kt^{-1} + 2\sum_1^{k-1}\beta_i t^{-3} + \sum_1^{k-1}[2\beta_i^2 + 4\beta_i\beta_{i-1}]t^{-5} + O(t^{-7}) \tag{13}$$

Comparing the coefficients of t^{-3} and t^{-5} in the expansions:

$$\delta_k = 2\sum_1^{k-1}\beta_i - k\beta_{k-1} \qquad \epsilon_k = 2\sum_1^{k-1}[\beta_i^2 + 2\beta_i\beta_{i-1}] - k[\beta_{k-1}^2 + \beta_{k-1}\beta_{k-2}] - \delta_k[\beta_{k-1} + \beta_{k-2} + \beta_{k-3}] \tag{14}$$

3.3. **Exercise 1: Laguerre and Hermite polynomials.**

$$w(t) = |t|^A \exp(-t^2) \text{ on } (-\infty,\infty): \qquad \frac{w'(t)}{w(t)} = \frac{A}{t} - 2t,$$

corresponds to the Laguerre weight $x^{(A-1)/2}\exp(-x)$ on $(0,\infty$.

The sensible choice seems to be $p(u) = t^2$ (p must be an even polynomial), and $q(u) = At - 2t^3$. As $\boldsymbol{M}^2$ is a five-diagonal matrix, the nonzero terms of (12) are

$$-(A+2)[\boldsymbol{M}]_{k,k-1} + 2[\boldsymbol{M}^3]_{k,k-1} - k[\boldsymbol{M}^2]_{k-1,k-1} - \delta_k[\boldsymbol{M}^2]_{k-3,k-1} - (k-1)\beta_k[\boldsymbol{M}^2]_{k-2,k} = 0,$$

or

$$-(2A+3)\beta_k + 2\beta_k(\beta_{k-1}+\beta_k+\beta_{k+1}) - k(\beta_{k-1}+\beta_k) - \delta_k - (k-1)\beta_k = -(2A+3)\beta_k + 2\beta_k(\beta_{k-1}+\beta_k+\beta_{k+1}) - (2k-1)\beta_k - 2\sum^{k-1}\beta_i = 0$$

We keep the degree of p as low as possible, so to avoid big bandmatrices in (12): with $p = 1$, q is not a polynomial, but a polynomial divided by t, and (12) has a strange term with $\boldsymbol{M}^{-1}$:

$$-A[\boldsymbol{M}^{-1}]_{k,k-1} + 2[\boldsymbol{M}]_{k,k-1} - k = 0.$$

But $[f(\boldsymbol{M})]_{i,j}$ is a shorthand for the integral of $f(u)p_i(u)p_j(u)w(u)$ divided by $\|p_j\|^2$. There is nothing wrong with of $u^{-1}p_k(u)p_{k-1}(u)$, as p_kp_{k-1} is an *odd* polynomial! The result is 0 if k is even, and 1 if k is odd. One then gets immediately

$$\beta_k = \frac{k + A[1-(-1)^k]/2}{2}.$$

3.4. **Exercise 2: Maxwellian weight.**

$$x^{(A-1)/2}\exp(-x^2)\,\mathrm{d}x = 2w(t)\,\mathrm{d}t = 2|t|^A\exp(-t^4)\,\mathrm{d}t: \qquad \frac{w'(t)}{w(t)} = \frac{A}{t} - 4t^3 \tag{15}$$

on the whole real line for t.

$$F_k = -A[\boldsymbol{M}^{-1}]_{k,k-1} + 4[\boldsymbol{M}^3]_{k,k-1} - k = 0.$$

the sought relation is

$$F_k = -A[1-(-1)^k]/2 + 4\beta_k(\beta_{k+1} + \beta_k + \beta_{k-1}) - k = 0, \qquad k = 1, 2, \dots \tag{16}$$

established by various authors through history [3, 8, 12, 14, 20, ,...]!

This remarkably simple relation seems to allow the computation of any sequence $\{\beta_1, \dots, \beta_N\}$ from the knowledge of the single β_1! However, the obvious repetition of $\beta_{i+1} = [i + A(1-(1)^i)/2 - \beta_i - \beta_{i-1}]/(4B\beta_i)$ soon turns into a numerical nightmare. Any numerical error in β_1 is strongly amplified in the subsequent β_i's. This is a consequence of unicity of positive solution [22].

Instead of considering (16) as an initial value problem, we have to consider it as a nonlinear boundary value problem for $\beta_1, \dots,$, given $\beta_0 = 0$, and knowing that $\beta_i > 0$ for $i = 1, 2, \dots$. A numerically valuable use of (16) consists in correcting a whole positive sequence $\beta_{1,\text{old}}, \dots,$ by seeing each instance of (16) as an algebraic equation for β_i:

$$\beta_{i,\text{new}} = -\frac{\beta_{i+1,\text{old}} + \beta_{i-1,\text{old}}}{2} + \sqrt{\left(\frac{\beta_{i+1,\text{old}} + \beta_{i-1,\text{old}}}{2}\right)^2 + \frac{i + (2A+1)(1-(-1)^i)/2}{4B}}\,, \quad i = 1, 2, \dots \tag{17}$$

which sends positive sequences on positive sequences, may be shown to be contractive, and has interesting by-products, such as to allow a formal proof of the asymptotic behaviour

$$\beta_i = \sqrt{\frac{i}{12B}} + o(i^{1/2}) \tag{18}$$

when $i \to \infty$, [8, 14]. One sees then how to build a satisfactory finite sequence $\beta_1, \dots, \beta_N$ by putting the boundary value $\beta_{N+1} = \sqrt{(N+1)/(12B)}$ for a large N. Asymptotic behaviour is also used by Kolb [11], and by Clarke & Shizgal [6].

Much more efficient Newton-Raphson iteration: see [12]

4. Lorentzian weight.

4.1. General power r.

Now, from (7), one has

$$\frac{w'(t}{w(u)} = \frac{A}{t} - \frac{2Brt^{2r-1}}{c^{2r} + t^{2r}} \tag{19}$$

on the whole real line for t.

So,

$$p(t) = c^{2r} + t^{2r}, \qquad q(t) = \frac{Ac^{2r}}{t} + (A - 2Br)t^{2r-1}, \qquad p'(t) + q(t) = \frac{Ac^{2r}}{t} + (A - 2(B-1)r)t^{2r-1}.$$

There is nothing wrong in considering the Lorentzian weight $w(t) = |t|^A\,(1 + t^{2r}/c^{2r})^{-B}$, $A > -1, B > 0$, on $(-\infty, \infty)$, as long as the integrals in (**??**) only involve functions decreasing faster than $|t|^{-1}$ when $|t| \to \infty$. So, β_k still exists if $\|p_k\| < \infty$, i.e., $2k + A - 2Br < -1$, or

$$k < Br - (A+1)/2. \tag{20}$$

Equation (12) is now

$$\begin{aligned} F_k = &-Ac^{2r}[1-(-1)^k]/2 + (2(B-1)r - A)[\boldsymbol{M}^{2r-1}]_{k,k-1} - kc^{2r} - k[\boldsymbol{M}^{2r}]_{k-1,k-1} - \delta_k[\boldsymbol{M}^{2r}]_{k-3,k-1} - \epsilon_k[\boldsymbol{M}^{2r}]_{k-5,k-1} - \cdots \\ &- \beta_k\{(k-1)[\boldsymbol{M}^{2r}]_{k-2,k} + \delta_{k-1}[\boldsymbol{M}^{2r}]_{k-4,k} + \cdots\} = 0, \end{aligned} \tag{21}$$

which is practically untractable, unless if $r = 1$ or $r = 2$, this latter one being our example of interest anyhow. The first case is taken as exercise:

4.2. Exercise $r = 1$: Romanovski, Lesky.

$$\begin{aligned} F_k(\beta_1, \dots, \beta_k) &= -Ac^2[1-(-1)^k]/2 + (2B - 2 - A)[\boldsymbol{M}]_{k,k-1} - kc^2 - k[\boldsymbol{M}^2]_{k-1,k-1} - \delta_k[\boldsymbol{M}^2]_{k-3,k-1} - \beta_k(k-1)[\boldsymbol{M}^2]_{k-2,k} \\ &= -Ac^2[1-(-1)^k]/2 + (2B - A - 1 - 2k)\beta_k - kc^2 - 2\sum_1^{k-1}\beta_i = 0, \end{aligned}$$

which receives the explicit solution

$$\beta_k = c^2 \frac{(k + A[1-(-1)^k]/2)(2B-k-A[1-(-1)^k]/2)}{(2B-A-2k-1)(2B-A-2k+1)}, \beta_1+\beta_2+\cdots+\beta_k = c^2 \frac{(k + A[1-(-1)^k]/2)(A+k+[1-(-1)^k]/2)}{2B-A-2k-1}, \tag{22}$$

a special case of pseudo-Jacobi polynomials [13].

4.3. **Lorentz case,** $r = 2$**.**

Now,

$$F_k(\beta_1,\ldots,\beta_k,\beta_{k+1}) = 2(2\beta - A - 2 - k)\beta_k\beta_{k+1} + \rho_k = 0, \tag{23}$$

where ρ_k is the sum of terms in F_k not involving β_{k+1}:

$$\begin{aligned}\rho_k &= -(A+1/2)[1-(-1)^k] + (4\beta - 2A - 2)\beta_k(\beta_{k-1}+\beta_k)\\ &- k[1+\beta_{k-1}\beta_{k-2}+\beta_{k-1}^2+2\beta_{k-1}\beta_k+\beta_k^2] - \delta_k(\beta_{k-3}+\beta_{k-2}+\beta_{k-1}+\beta_k) - \epsilon_k\\ &- (k-1)\beta_k(\beta_{k-2}+\beta_{k-1}+\beta_k) - \delta_{k-1}\beta_k\\ &= -(A+1/2)[1-(-1)^k] + 2(2\beta - A - 2 - k)\beta_k(\beta_{k-1}+\beta_k)\\ &- k - 4\beta_k\sum^{k-2}\beta_i - 2\beta_k\beta_{k-1} - 2\sum^{k-1}[\beta_i^2+2\beta_i\beta_{i-1}],\end{aligned} \tag{24}$$

which we may as well compute, while $k < 2B - A - 2$, directly for β_{k+1}, although in high precision:

$$\beta_{k+1} = -\frac{\rho_k}{4B - A - 2k - 3)\beta_k},$$

as in the following matlab/octave program [17]:

```
%
% lor2005.m  Lorentz
%  w(x) = x^(A-1/2)  (1+x^2/c)^(-B)  on (0,infty)
%
%     monic pol. :  Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \beta_n Q_{n-1}(x)
%
% intermediate orthog. pol. R_n w.r.t.
%  |t|^A (1+t^4/c)^(-B)   on (-\infty, \infty)
%
%      R_{2n}(t)=Q_n(t^2)     let R_{n+1}(t)=tR_n(y)-\gamma_n R_{n-1}(t)
%
%                             then \alpha_n = \gamma_{2n}+\gamma_{2n+1}
%                                   \beta_n = \gamma_{2n} \gamma_{2n-1}
%
%
clear;ABc=input(' enter  A B c  between [ ] ');
A=ABc(1);B=ABc(2);c=ABc(3);
nmx=floor(2*B-(A+1)/2);if nmx>=2*B-(A+1)/2, nmx=nmx-1;end;
 norm0=c^((A+1)/4)* gamma(B-(A+1)/4)*gamma((A+1)/4) /(2*gamma(B));
 norm1=c^((A+3)/4)* gamma(B-(A+3)/4)*gamma((A+3)/4) /(2*gamma(B));
   d=norm1/norm0; d1=0;
   ioddn=1;
   [1,d], % gamma1
   sum1=0;sum2=0;
   for n=1:nmx-1,
     %  relation gam(1) ... gam(n+1)
     coefn=2*B-(A+3)/2;
     coefdp1=2*(coefn-n)*d ;
     rho=  -A*c*ioddn+2*(coefn-n)*(d+d1)*d -n*c  -2*d*sum1-2*sum2;
```

```
    if abs(coefdp1)<0.00001 , fprintf(1," ! \n ");coefdp1=d ;end;
    dp1=-rho/coefdp1;
    norm1=norm1*dp1;
    sum1=sum1+d+d1 ; sum2=sum2+d*(d+2*d1);
    [n+1,dp1,sum1,sum2],
    ioddn=1-ioddn;
    d1=d ;d=dp1 ;
  end;
```

but numerical instability soon settles in if β is large.

Sensible way is again to compute the β_n's from the two boundary values $\beta_0 = 0$, and some β_N with N not far from $rB - A$, should such a value be available...

This can be done for special values of A and B. Here is a formula when $A = 0$:

4.4. **Particular values.**

Proposition. *When r and B are positive integers, and when $A = 0$,*

$$\beta_{rB-1} = c^2 \left[\frac{\sin(\pi/(2r))}{3\sin(3\pi/(2r))} + \frac{B^2}{3\sin^2(\pi/(2r))} \right] \tag{25}$$

Indeed, we have to consider orthogonal polynomials with respect to $w(t) = (1 + t^{2r}/c^{2r})^{-B}$ on $(-\infty, \infty)$. The two last ones are

$$p_{rB-1}(t) = \frac{R(t)^B - (-1)^{rB} R(-t)^B}{2iBc/\sin(\pi/(2r))}, p_{rB}(t) = \frac{R(t)^B + (-1)^{rB} R(-t)^B}{2},$$

where $R(t)R(-t) = (-1)^r (t^{2r} + c^{2r})$ is the factorization of $(-1)^r (t^{2r} + c^{2r})$, where R is a monic polynomial of degree r with zeros of negative imaginary part:

$$\begin{aligned} R(t) &= \prod_{k=1}^{r} [t + ic\exp(i\pi(2k - r - 1)/(2r))] \\ &= t^r + \frac{ict^{r-1}}{\sin(\pi/(2r))} + \frac{c^2 t^{r-2}}{2} \left[\delta_{r,1} - \frac{1}{\sin^2(\pi/(2r))} \right] + ic^3 t^{r-3} \left[\frac{\delta_{r,1}}{2\sin(\pi/(2r))} + \frac{1}{3\sin(3\pi/(2r))} - \frac{1}{6\sin^3(\pi/(2r))} \right] + \cdots \\ R(t)^B &= t^{rB} + \frac{icBt^{rB-1}}{\sin(\pi/(2r))} + \frac{c^2 t^{rB-2}}{2} \left[B\delta_{r,1} - \frac{B^2}{\sin^2(\pi/(2r))} \right] + ic^3 t^{rB-3} \left[\frac{B^2\delta_{r,1}}{2\sin(\pi/(2r))} + \frac{B}{3\sin(3\pi/(2r))} - \frac{B^3}{6\sin^3(\pi/(2r))} \right] + \cdots \end{aligned} \tag{26}$$

Remark that the polynomial p_{rB} has a meaning despite its infinite norm, as we only have to check orthogonality with powers of same evenness up to t^{rB-2}. To prove this orthogonality, as $(-1)^r (c^{2r} + t^{2r})^B$ is the product of $R(t)^B$ and $R(-t)^B$, we must check that the integrals of $t^k / R(\pm t)^B$ on $(-\infty, \infty)$ all vanish for $k = 0, 1, \ldots, rB - 2$. And that appears by making a contour integral by adding a big semicircle in the upper or the lower half complex plane, so as to avoid the poles (which have imaginary part of the same sign).

We then find $\beta_{rB-1}, \beta_{rB-2}, \ldots$ from the continued fraction expansion about ∞

$$\frac{p_{rB}(t)}{p_{rB-1}(t)} = \frac{t^{rB} + \dfrac{c^2 t^{rB-2}}{2} \left[B\delta_{r,1} - \dfrac{B^2}{\sin^2(\pi/(2r))} \right] + \cdots}{t^{rB-1} + c^2 t^{r-3} \left[\dfrac{B\delta_{r,1}}{2} + \dfrac{\sin(\pi/(2r))}{3\sin(3\pi/(2r))} - \dfrac{B^2}{6\sin^2(\pi/(2r))} \right] + \cdots} = t - \cfrac{\beta_{rB-1}}{t - \cfrac{\beta_{rB-2}}{t - \cdots}}$$

whence (25) follows. □

Remark also that $\beta_1 + \cdots + \beta_{rB-1} = \dfrac{c^2}{2} \left[-B\delta_{r,1} + \dfrac{B^2}{\sin^2(\pi/(2r))} \right]$, opposite to the coefficient of t^{rB-2} in $p_{rB}(t)$.

The other β's are more complicated. To consider only β_{rB-2}, $oneshouldexpand \beta_{rB-1} p_{rB-2}(t) = tp_{rB-1}(t) - p_{rB}(t)$, showing that $p_{rB-2}(t)$ is a constant times a sum or a difference of $(t - \varphi)R(t)^B$ and $(t + \varphi)R(-t)^B$, so as to ensure the vanishing of the t^{rB+1}, t^{rB}, and t^{rB-1} terms. This leads to $\varphi = icB/\sin(\pi/(2r))$.

Actually, simpler formulas for β_n are found in the forbidden region $n > rB - 1$... Now, $p_{rB+1}(t)$ is again a constant times a sum or a difference of $(t - \psi)R(t)^B$ and $(t + \psi)R(-t)^B$, but where ψ is such that the result is orthogonal to t^{rB-1}, t^{rB-3}...

and lower powers of the same evenness. To this end, one extends the scalar product of $f(t)$ and $R(\pm t)^B$ to a contour integral of $f(t)/R(\mp t)^B$ avoiding the zeros of $R(\mp t)$. The value of the integral reduces then to $2\pi i$ times the residue at ∞, i.e., the coefficient of t^{-1} in the Laurent expansion about ∞. With $t^{rB-1}(t\pm\psi)/R(\pm t)^B$, the orthogonality condition is again $\psi = icB/\sin(\pi/(2r))$! The coefficient of t^{rB-1} in $p_{rB+1}(t)$ is then the coefficient of t^{rB-2} in $p_{rB}(t)$ minus ψ^2, whence finally $\beta_{rB} = -c^2B^2/\sin^2(\pi/(2r))$.

Larger even integer A could also be studied through orthogonal polynomials with respect to $w_A(t) = t^A(1+t^{2r}/c^{2r})^{-B} = t^A w_0(t)$ on $(-\infty,\infty)$. The orthogonal polynomials p_n with respect to this weight is a kernel polynomial built with orthogonal polynomials relative to the weight $w_{A-2}(t) = w_A(t)/t^2$. The formula relating the two families of orthogonal polynomials is

$$p_n(t))_{t^2w(t)} = \frac{(p_{n+2}(t))_{w(t)} - c_n(p_n(t))_{w(t)}}{t^2},$$

where c_n is such that the numerator is a multiple of t^2.

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