Elliptic difference equations.
The elliptic logarithm.

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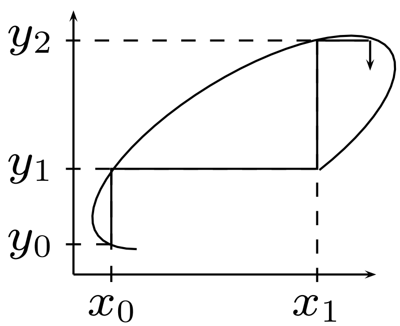
Nacht und Stürme werden Licht *Choral Fantasy, Op. 80*

Difference equations and lattices.

Difference equations in history involve $(f(x), f(x+h))$, or $(f(x), f(x-h))$, or $f(x-h/2), f(x+h/2))$, or $f(x), f(qx))$, or $(f(q^{-1/2}x), f(q^{1/2}x))$, and more recently, values of f at $r(x) \pm \sqrt{s(x)}$, where r and s are rational functions.

Where will it end?

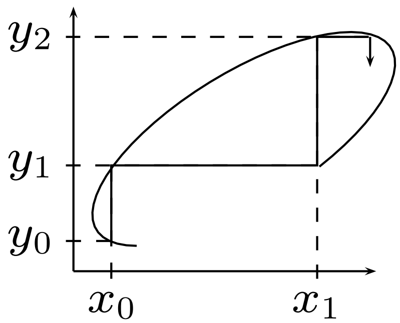
$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{1}$$



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x,y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{2a}$$

where X_0, X_1, and X_2 are rational functions. φ and $\psi = \dfrac{-X_1 \pm \sqrt{P}}{2X_2}$, where $P = X_1^2 - 4X_0X_2$.

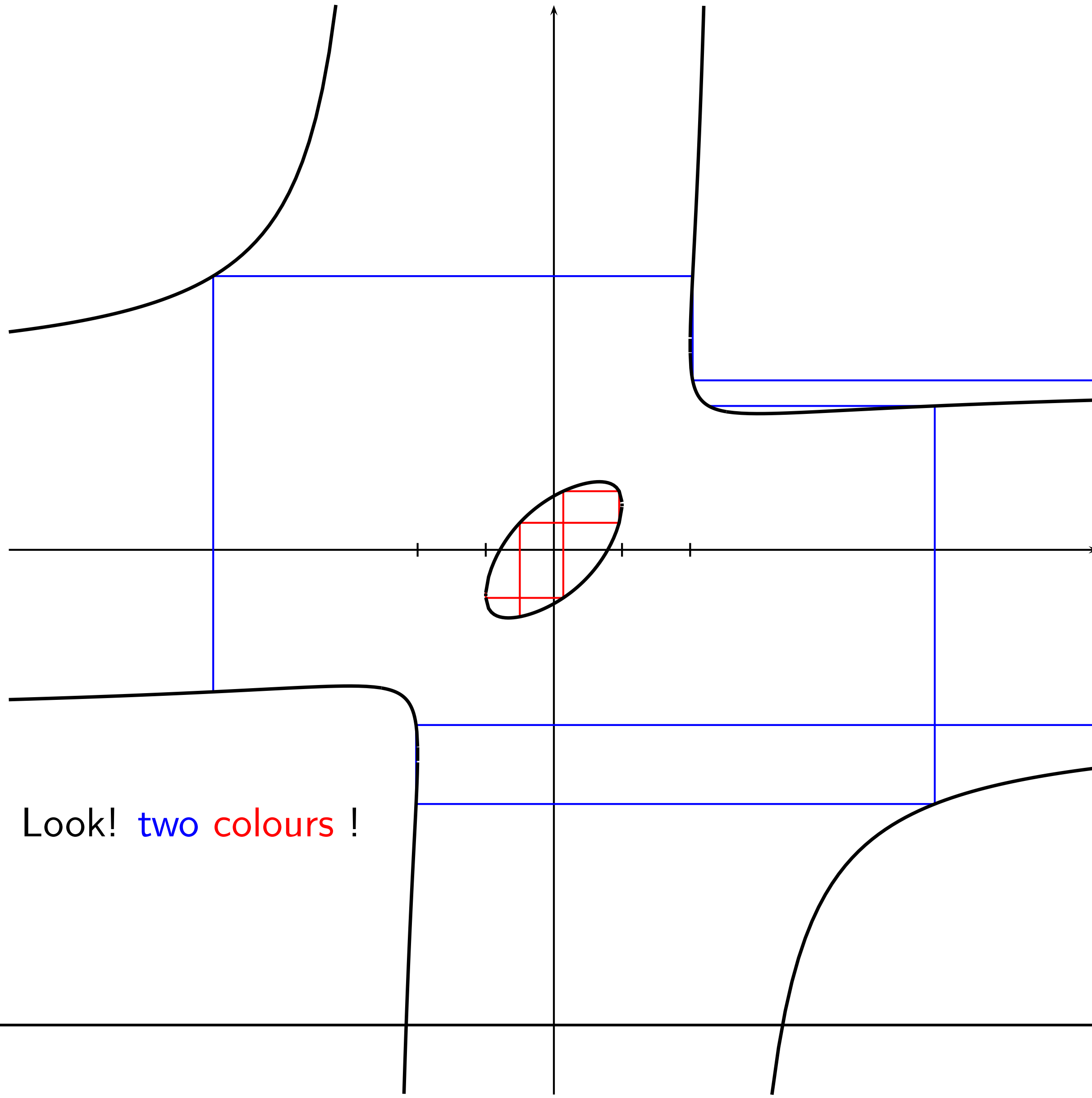


The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (2a) at x_0. We need x_1 such that y_1 is one of the two roots of (2a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

$$F(x, y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{2b}$$

Both forms (2a) and (2b) hold simultaneously when F is ***biquadratic***:

$$F(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2} c_{i,j} x^i y^j. \tag{3}$$



Look! two colours !

Why elliptic?

Equivalent definitions:

1. $\{x_n\}$ *is elliptic if* $E(x_n, x_{n+1}) = 0$*, where* E *is a* **symmetric** *biquadratic*,

2. $\{x_n\}$ *in the continued fraction arrangement* $f_n(x) = \dfrac{S_n(x) - \sqrt{P(x)}}{(x - z_0)\zeta_n(x - x_n)}$

$$= \frac{\alpha_n(x - z_0)}{1 - \beta_n(x - z_0) - (x - z_0)f_{n+1}(x)} = \cfrac{\alpha_n(x - z_0)}{1 - \beta_n(x - z_0) - \cfrac{\alpha_{n+1}(x - z_0)^2}{1 - \beta_{n+1}(x - z_0) - \ddots}}$$

is an instance of ***elliptic lattice***. Here, $E(z_0, z_0) = 0$: let the Taylor expansion about z_0 be $\sqrt{P(z)} = \gamma + \delta(x - z_0) + \cdots$ which is matched by the two first Taylor coefficients of $S_n(z)$, so that only one unknown remains in S_n: $S_n(x) = \gamma + \delta(x - z_0) + \xi_n(x - z_0)^2$. $S_n^2(x) - P(x) =$ const. $(x - z_0)^2(x - x_n)(x - x_{n+1})$. The elimination of $\xi_n = y_n$ yields a symmetric algebraic relation between x_n and x_{n+1}, whis a polynomial in $2z_0 - x_n - x_{n+1}$ and $(z_0 - x_n)(z_0 - x_{n+1})$ without constant term.

3. Jacobi and Abel related the continued fraction above to a closed formula for x_n through the Jacobi inversion problem,

$$\frac{1}{2\pi i}\int_C \frac{t^k}{\sqrt{P(t)}}\log\left(\frac{\widetilde{A}_{n-1}(t)+B_{n-1}(t)\sqrt{P(t)}}{\widetilde{A}_{n-1}(t)-B_{n-1}(t)\sqrt{P(t)}}\right)dt=0,$$

$$0=-2n\int_{z_0}^{z_1}\frac{t^k}{\sqrt{P(t)}}\,dt+\sum_{\text{zeros of }Z_0,Z_n}\pm\int_{z_1}^{\text{zero}}\frac{t^k}{\sqrt{P(t)}}\,dt+\sum_j N_j\int_{z_j}^{z_{j+1}}\frac{t^k}{\sqrt{P(t)}}\,dt, \tag{4}$$

$k=0,\ldots,m-2$, which brings us to

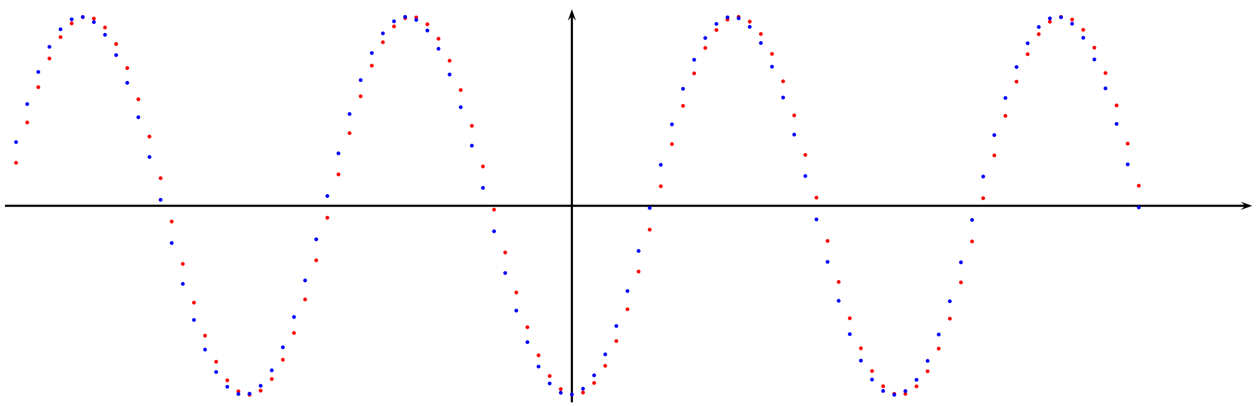
An elliptic lattice is a sequence $x_n=\mathcal{E}(nh+u_0)$, where $\mathcal{E}$ is any elliptic function of order 2 (i.e., with 2 zeros and 2 poles in a fundamental parallelogram of periods).

Pell-like equation. Make the product

$$\begin{aligned}[B_{n-1}S_0-A_{n-1}Z_0]^2-B_{n-1}^2P &= Z_0^2(x-z_0)^{2n}f_0\ldots f_{n-1}f_{0,\text{conj.}}\ldots f_{n-1,\text{conj.}}\\ &= Z_0^2\frac{S_0-\sqrt{P}}{Z_0}\cdots\frac{S_{n-1}-\sqrt{P}}{Z_{n-1}}\frac{S_0+\sqrt{P}}{Z_0}\cdots\frac{S_{n-1}+\sqrt{P}}{Z_{n-1}}\\ &= \alpha_0^2\ldots\alpha_{n-1}^2(x-z_0)^{2n}Z_0Z_n.\end{aligned}$$

Elliptic functions, at last: $x_n = \dfrac{\alpha\,\mathsf{sn}(nh + s_0; k) + \beta}{1 + \gamma\,\mathsf{sn}(nh + s_0; k)}$

$$F(x,y) = k^2(1 - k^2 z_0^2)x^2y^2 - k^2(1 - z_0^2)(x^2 + y^2) - 2k(1 - k^2)z_0xy + 1 - k^2z_0^2. \quad (5)$$



Linear 1st order difference equations.

$$a(x)(\mathcal{D}f)(x) = c(x)[f(\varphi(x)) + f(\psi(x))] + d(x) \tag{6}$$

or

$$f(\psi(x)) = \frac{\tilde{a}(x)f(\varphi(x)) + \tilde{d}(x)}{\tilde{c}(x)}$$

where $\tilde{a}(x) = a(x) + c(x)[\psi(x) - \varphi(x)]$, $\tilde{d}(x) = d(x)[\psi(x) - \varphi(x)]$, $\tilde{c}(x) = a(x) - c(x)[\psi(x) - \varphi(x)]$

on a lattice: $f(y_{n+1}) = \dfrac{\tilde{a}(x_n)f(y_n) + \tilde{d}(x_n)}{\tilde{c}(x_n)}$

Where is b?

Riccati.

$$a(x)(\mathcal{D}f)(x) = b(x)f(\varphi(x))f(\psi(x)) + c(x)[f(\varphi(x)) + f(\psi(x))] + d(x) \quad (7)$$

or

$$f(\psi(x)) = \frac{\tilde{a}(x)f(\varphi(x)) + \tilde{d}(x)}{\tilde{c}(x) - \tilde{b}(x)\, f(\varphi(x))}$$

$$f(y_{n+1}) = \frac{\tilde{a}(x_n)f(y_n) + \tilde{d}(x_n)}{\tilde{c}(x_n) - \tilde{b}(x_n)\, f(y_n)}$$

$$f(y_{n+1}) = \frac{\tilde{a}(x_n)f(y_n) + d(x_n)}{\tilde{c}(x_n) - b(x_n)\, f(y_n)} = -\frac{\tilde{a}(x_n)}{b(x_n)} + \frac{d(x_n) + \tilde{a}(x_n)\tilde{c}(x_n)/b(x_n)}{\tilde{c}(x_n) - b(x_n)\, f(y_n)} \quad (8)$$

Elliptic logarithm.

$$(\mathcal{D}f)(x) = \frac{1}{x - A} \tag{9}$$

with, say, $f(y_0) = 0$. Values on the lattice starting with (x_0, y_0):

$$f(y_n) = \sum_{k=0}^{n-1} \frac{y_k - y_{k-1}}{x_{k-1} - A}$$

$k = 0.5, z_0 = 3, A = -7$

$f(y_{n+1}) = f(y_n) + \dfrac{y_{n+1} - y_n}{x_n - A}$:

n	x	y	f
0	0.7905694	0	0
1	0.5167233	0.9856450	0.1265177
2	-0.9529158	-0.3790235	-0.05503327
3	-0.1575301	-0.8702581	-0.1362682
4	0.9994804	0.6923469	0.09210034
5	-0.2301987	0.6400372	0.08556120
6	-0.9309638	-0.9011730	-0.1420984
7	0.5766530	-0.3102076	-0.04472454
8	0.7466518	0.9944962	0.1274760
9	-0.8303816	-0.07436214	-0.01050081
10	-0.4536000	-0.9726185	-0.1560943
11	0.9706641	0.4453679	0.06051121
...			
69	0.6604197	0.9999706	0.1280656
...			
431	0.6621363	0.9999845	0.1280671
...			
793	0.6638488	0.9999940	0.1280681

$$\Lambda_N(x) = \sum_0^N \frac{\rho_k}{x - y'_k}, \tag{10}$$

where $y'_k = y(s_1 + k)$ is an elliptic lattice in the same family, but with another starting point. When $\rho_k = (x'_k - x'_{k-1})Y_2(y'_k)$,

$$\mathcal{D}\Lambda_N(x) = \frac{(x'_{-1} - x'_N)X_2(x)}{(x - x'_{-1})(x - x'_N)} = (x'_{-1} - x'_N)c_{2,2} + \frac{X_2(x'_{-1})}{x - x'_{-1}} - \frac{X_2(x'_N)}{x - x'_N}.$$

We can solve $\mathcal{D}f(x) = \dfrac{1}{x - A}$ with $f(y_0) = 0$, if an infinity of x'_ns, starting with $x'_{-1} = A$, come as close as we want to a zero, say ζ, of $X_2(x) = c_{2,2}(x - \zeta)(x - \zeta')$, then

$$f(x) = \underbrace{-\frac{c_{2,2}(A - \zeta)}{X_2(A)}}_{1/(\zeta' - A)}(x - y_0) + \frac{1}{X_2(A)} \lim_{x'_N \to \zeta} [\Lambda_N(x) - \Lambda_N(y_0)]$$

Here is an example where $f(1)$ is estimated whenever x' comes close to ζ which happens to be -2.529822 here:

$$F(x, y) = -0.3125x^2y^2 + 2x^2 - 2.25xy + 2y^2 - 1.25, A = -7,$$

n	x'	y'	res	f(1)
-1	-7	-2.168439		
0	2.006819	3.351538	1.021789	-0.02471402
1	-19.96266	2.738246	-0.5662515	0.09425273
2	-2.068051	-2.371686	-0.3255921	0.1349691
3	4.274935	-4.641409	2.254690	0.04885997
4	2.433328	2.049464	0.09509338	0.004647754
5	-2.642949	34.53302	-141.3408	0.1267041
...				
13	-2.560190	-2.024363	-0.06545824	0.1276923
...				
56	-2.525475	-2.029866	-0.07278552	0.1281233
...				
238	-2.536399	550.1306	-35945.00	0.1279866

Elliptic hypergeometric expansions.

Let

$$\mathcal{Y}_n(x) = \frac{(x-y_0)\cdots(x-y_{n-1})}{(x-y'_0)\cdots(x-y'_{n-1})}, \mathcal{X}_n(x) = \frac{(x-x_0)\cdots(x-x_{n-1})}{(x-x'_0)\cdots(x-x'_{n-1})}.$$

See that $\mathcal{D}\mathcal{Y}_n(x) = C_n X_2(x)\dfrac{\mathcal{X}_{n-1}(x)}{(x-x'_{-1})(x-x'_{n-1})}$

Indeed, $(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})$ and $(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})$ both vanish at $x = x_0, x_1, \ldots, x_{n-2}$; $(\varphi(x)-y'_0)(\varphi(x)-y'_1)\cdots(\varphi(x)-y'_{n-1})$ vanishes at $x = x'_0, \ldots, x'_{n-1}$, whereas $(\psi(x)-y'_0)(\psi(x)-y'_1)\cdots(\psi(x)-y'_{n-1})$ vanishes at $x = x'_{-1}, \ldots, x'_{n-2}$.

Simple fractions give $\mathcal{D}\dfrac{1}{x-y'_k} = -\dfrac{X_2(x)}{Y_2(y'_k)(x-x'_{k-1})(x-x'_k)}$, as seen earlier.

The constant C_n is found through particular values of x, either x_{-1} or x_{n-1}:

$$C_n = -\frac{\mathcal{Y}_n(\varphi(x_{-1}) = y_{-1})(x_{-1} - x'_{-1})(x_{-1} - x'_{n-1})}{(y_0 - y_{-1})X_2(x_{-1})\mathcal{X}_{n-1}(x_{-1})}$$
$$= \frac{\mathcal{Y}_n(\psi(x_{n-1}) = y_n)(x_{n-1} - x'_{-1})(x_{n-1} - x'_{n-1})}{(y_n - y_{n-1})X_2(x_{n-1})\mathcal{X}_{n-1}(x_{n-1})}$$

(Of course, $C_0 = 0$). Check $\mathcal{Y}_1(x) = \dfrac{x - y_0}{x - y'_0} = 1 + \dfrac{y'_0 - y_0}{x - y'_0} \Rightarrow \mathcal{D}\mathcal{Y}_1(x) = -\dfrac{(y'_0 - y_0)X_2(x)}{Y_2(y'_0)(x - x'_{-1})(x - x'_0)}$ so that $C_1 = \dfrac{y_0 - y'_0}{Y_2(y'_0)}$ to be compared with $-\dfrac{\mathcal{Y}_1(y_{-1})(x_{-1} - x'_{-1})(x_{-1} - x'_0)}{(y_0 - y_{-1})X_2(x_{-1})\mathcal{X}_0(x_{-1})} = -\dfrac{(y_{-1} - y_0)(x_{-1} - x'_{-1})(x_{-1} - x'_0)}{(y_{-1} - y'_0)(y_0 - y_{-1})X_2(x_{-1})} = \dfrac{F(x_{-1}, y'_0)}{(y_{-1} - y'_0)Y_2(y'_0)X_2(x_{-1})} = \dfrac{(y'_0 - y_{-1})(y'_0 - y_0)}{(y_{-1} - y'_0)Y_2(y'_0)}$, OK.

Also,

$$\frac{(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})}{(\varphi(x)-y'_1)(\varphi(x)-y'_2)\cdots(\varphi(x)-y'_n)}+\frac{(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})}{(\psi(x)-y'_1)(\psi(x)-y'_2)\cdots(\psi(x)-y'_n)}$$

$$= D_n(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x'_0)(x-x'_1)\cdots(x-x'_n)},$$

where D_n is a polynomial of degree 2.

Expansion for elliptic logarithm.

Expand (10) satisfying

$$\mathcal{D}\Lambda_N(x) = \frac{(x'_{-1} - x'_N)X_2(x)}{(x - x'_{-1})(x - x'_N)}$$

as $\Lambda_N(x) = \sum_1^{N+1} \gamma_n \mathcal{Y}_n(x)$: $\sum_1^{N+1} \gamma_n C_n \frac{X_2(x)\mathcal{X}_{n-1}(x)}{(x - x'_{-1})(x - x'_{n-1})} = \frac{(x'_{-1} - x'_N)X_2(x)}{(x - x'_{-1})(x - x'_N)}$,

or $\sum_1^{N+1} \gamma_n C_n \frac{\mathcal{X}_{n-1}(x)}{x - x'_{n-1}} = \frac{x'_{-1} - x'_N}{x - x'_N}$,

$$\gamma_n C_n = \frac{(x'_{n-1} - x_{n-1})(x'_N - x_n)(x'_N - x_{n+1}) \cdots (x'_N - x_{N-1})}{(x'_N - x'_{n-1})(x'_N - x'_n) \cdots (x'_N - x'_{N-1})} \gamma_{N+1} C_{N+1}$$

$$= \frac{(x'_{n-1} - x_{n-1})(x'_{-1} - x'_N)}{(x'_N - x_{n-1})\mathcal{X}_{n-1}(x'_N)}$$

Proof:

$$\sum_{i+1}^{N+1}(x'_{n-1}-x_{n-1})\frac{(x-x_i)\cdots(x-x_{n-2})}{(x-x'_i)\cdots(x-x'_{n-1})}\frac{(x'_N-x'_i)\cdots(x'_N-x'_{n-2})}{(x'_N-x_i)\cdots(x'_N-x_{n-1})}=\frac{1}{x-x'_N}$$

is an identity.

And from $C_n=-\dfrac{\mathcal{Y}_n(y_{-1})(x_{-1}-x'_{-1})(x_{-1}-x'_{n-1})}{(y_0-y_{-1})X_2(x_{-1})\mathcal{X}_{n-1}(x_{-1})}$,

$$\gamma_n=-\frac{(x'_{n-1}-x_{n-1})(x'_{-1}-x'_N)}{(x'_N-x_{n-1})\mathcal{X}_{n-1}(x'_N)}\frac{(y_0-y_{-1})X_2(x_{-1})\mathcal{X}_{n-1}(x_{-1})}{(x_{-1}-x'_{-1})(x_{-1}-x'_{n-1})\mathcal{Y}_n(y_{-1})}$$

Return to $\mathcal{D}f(x)=\dfrac{1}{x-A}$, $x'_{-1}=A$, $f(y_0)=0$,

$$f(x)=\frac{x-y_0}{\zeta'-A}+\frac{1}{X_2(A)}\lim_{x'_N\to\zeta}[\Lambda_N(x)-\Lambda_N(y_0)]=\frac{x-y_0}{\zeta'-A}+\frac{1}{X_2(A)}\sum_1^\infty\gamma_n\mathcal{Y}_n(x),$$

with the γ_ns above, with x'_N replaced by ζ:

$$f(x) = \frac{x - y_0}{\zeta' - A} + \frac{X_2(x_{-1})}{X_2(A)} \sum_1^{\infty} (x_{n-1} - x'_{n-1}) \frac{(\zeta - x'_{-1}) \cdots (\zeta - x'_{n-2})}{(\zeta - x_0) \cdots (\zeta - x_{n-1})}$$

$$\frac{(x_{-1} - x_0) \cdots (x_{-1} - x_{n-2})(y_{-1} - y'_0) \cdots (y_{-1} - y'_{n-1})}{(x_{-1} - x'_{-1}) \cdots (x_{-1} - x'_{n-1})(y_{-1} - y_1) \cdots (y_{-1} - y_{n-1})}$$

$$\frac{(x - y_0) \cdots (x - y_{n-1})}{(x - y'_0) \cdots (x - y'_{n-1})},$$

```
k    x(k)      y(k)    x'(k-1) y'(k-1)   gk       term        f(1)                f(i)
1  0.51672  0.98564;  2.0068  3.3515 ;-0.05542  0.02356958  0.1285033 -0.00453082 + 0.1201190*I
2 -0.95291 -0.37902;-19.9626  2.7382 ;-0.21650 -0.00076035  0.1277430 -0.00000099 + 0.1495888*I
3 -0.15753 -0.87025; -2.0680 -2.3716 ; 0.30473  0.00043771  0.1281807  0.01064768 + 0.1357818*I
4  0.99948  0.69234;  4.2749 -4.6414 ;-0.23940 -0.00011400  0.1280667  0.00696292 + 0.1368071*I
5 -0.23019  0.64003;  2.4333  2.0494 ;-0.05303  0.00000740  0.1280741  0.00664251 + 0.1371257*I
```

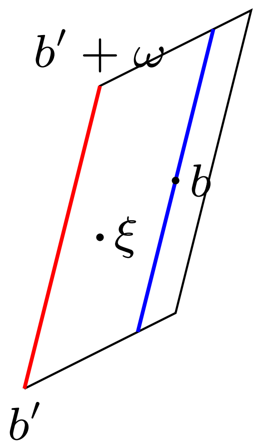
```
 k    x(k)      y(k)    x'(k-1) y'(k-1)   gk       term        f(1)              f(i)
 6 -0.93096 -0.90117; -2.6429 34.5330 ;-4.06650 -0.00000609  0.1280680  0.00686082 + 0.1382964*I
 7  0.57665 -0.31020; -3.5481 -2.0143 ; 0.77233  0.00000073  0.1280687  0.00688667 + 0.1381635*I
 8  0.74665  0.99449;  2.1363  6.1416 ;-0.07636  0.00000002  0.1280687  0.00688892 + 0.1381636*I
 9 -0.83038 -0.07436;  9.3362  2.2354 ;-0.13154 -1.411 E-10  0.1280687  0.00689104 + 0.1381629*I
10 -0.45360 -0.97261; -2.0003 -3.0677 ; 0.36320  1.029 E-10  0.1280687  0.00688980 + 0.1381615*I
15  0.90479  0.92787;  2.5007121.2785 ;-0.11543  1.975 E-15  0.1280687  0.00689019 + 0.1381625*I
20 -0.98422 -0.50887; -8.1267  3.1850 ;-0.28273 -2.400 E-19  0.1280687  0.00689019 + 0.1381625*I

42 -0.80107 -0.98273;-2.21269 -8.4391 ; 1.01356  6.075 E-38  0.1280687  0.00689019 + 0.1381625*I
43  0.77979 -0.01891;-6.58778 -2.1536 ;12.58748  4.744 E-37  0.1280687  0.00689019 + 0.1381625*I
44  0.53226  0.98828; 2.00981  3.4355 ;-0.05629  8.875 E-40  0.1280687  0.00689019 + 0.1381625*I


195 -0.3001  0.58496; 2.55791   2.024 ;-0.05612  8.02 E-173  0.1280687 0.006890192 + 0.1381625*I
196 -0.9052 -0.92747;-2.50288-130.900 ;15.57565 -7.00 E-173  0.1280687 0.006890192 + 0.1381625*I
197  0.6322 -0.24049;-3.96810  -2.033 ;0.998716 -2.85 E-174  0.1280687 0.006890192 + 0.1381625*I

361 -0.7919 -0.98528;-2.19764  -7.909 ;0.949600 -7.39 E-316  0.1280687 0.006890192 + 0.1381625*I
362  0.7892 -0.00242;-6.94413  -2.166 ;97.99943 -4.78 E-314  0.1280687 0.006890192 + 0.1381625*I
363  0.5187  0.98599; 2.00717   3.362 ;-0.05553 -1.15 E-317  0.1280687 0.006890192 + 0.1381625*I
```

```
723 -0.7932 -0.98492;-2.19981  -7.983 ; 0.95848  7.55 E-631  0.1280687 0.006890192 + 0.1381625*I
724  0.7878 -0.00484;-6.88919  -2.164 ;49.01450  2.42 E-629  0.1280687 0.006890192 + 0.1381625*I
725  0.5207  0.98634; 2.00753   3.372 ;-0.05564  1.16 E-632  0.1280687 0.006890192 + 0.1381625*I
```

Average behaviour: $\prod_1^n \frac{x - x_k}{x - x'_k} \approx \Phi(x)^n = \exp(-n\mathcal{V}(x))$,

$$\mathcal{V}(x) = \frac{2\pi i}{\omega}\xi + \text{const.}, \quad \text{where } x = \mathcal{E}(\xi)$$

$$(\gamma_n \mathcal{Y}_n(x))^{1/n} \sim \exp[\mathcal{V}(x_{-1}) - \mathcal{V}(\zeta) - \mathcal{W}(y_{-1}) + \mathcal{W}(x)]$$



Towards a conjecture on rate of convergence:

If the step h in $x_n = \mathcal{E}_1(nh+b)$, $x'_n = \mathcal{E}_1(nh+b')$, is a **real irrational** *multiple of a period ω, of bounded Lagrange-Markov constant(see papers and book by S.Khrushchev),*

$$|\gamma_n \mathcal{Y}_n(x)|^{1/n} \to \exp(-2\pi d(x)/|\omega|),$$

where $d(x)$ is the distance of ξ to the line $\{nh+b'\}$.

Walsh, 1935!

$\mathcal{D}f(x) = \dfrac{1}{x-A}$ with $f(y_0)=0$, $\zeta = -2.529822$, $F(x,y) = -0.3125x^2y^2 + 2x^2 - 2.25xy + 2y^2 - 1.25$, $A = -7$,

Here, $b=0$, $\omega = 4K$, $b' = iK'$, $k = 1/2 : K = 1.68575$, $K' = 2.15652$,

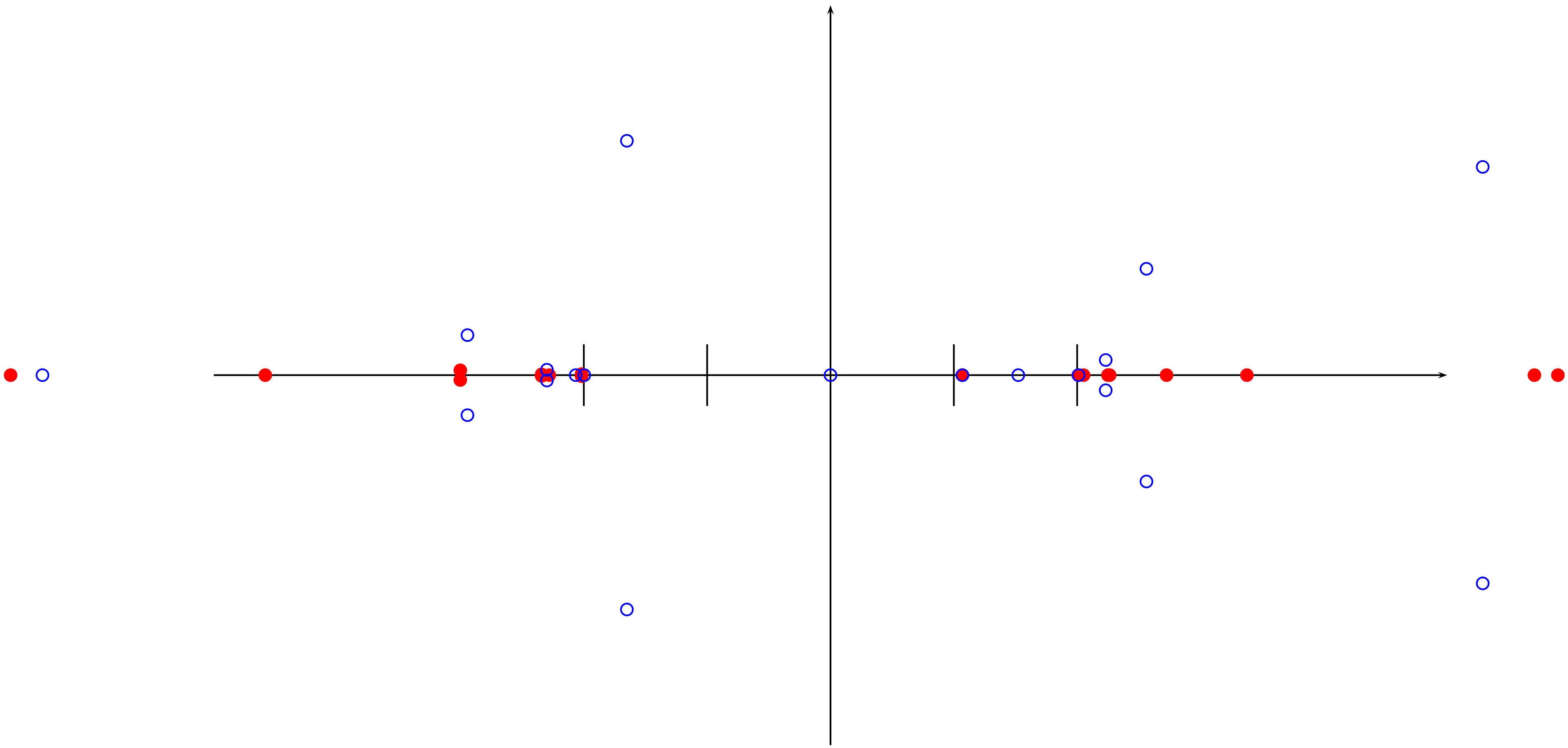
rate of convergence at $x=1$ is $\exp(-\pi K'/(2K)) = 1/7.459 = 10^{-0.8727}$.

Interpolatory continued fraction.

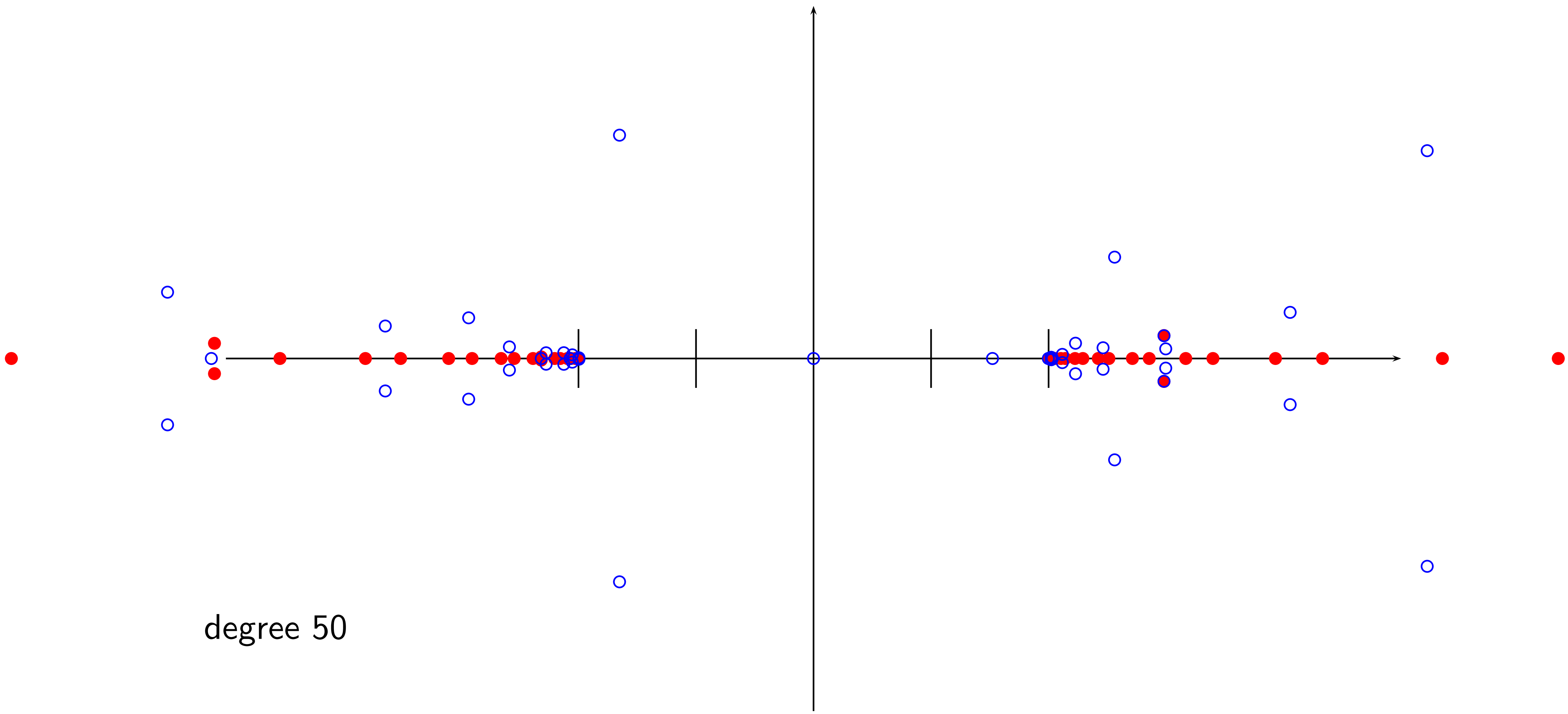
$$\cfrac{\alpha_0(x-y_0)}{1+\beta_0(x-y_1)+\cfrac{\alpha_1(x-y_1)(x-y_2)}{\cfrac{\ddots}{+\beta_{n-2}(x-y_{2n-3})+\cfrac{\alpha_{n-1}(x-y_{2n-3})(x-y_{2n-2})}{1+\beta_{n-1}(x-y_{2n-1})+\cdots}}}}$$

k	a(k-1)	b(k-1)
1	0.02342657	0.3064279
2	0.02442157	-0.5755228
3	0.2808405	0.5023201
4	0.02875770	0.3488815
5	-0.002774129	0.6898141
6	-2.635035	3.609245
7	0.05390936	0.7862154
8	0.001055843	0.2904550

```
 9  0.01638600 -0.8421168
10  0.4162262   0.5008208
...
13 -2.667423    1.473576
14  3.576408    0.6620672
16  0.01282034 -5.278312
17  3.286337    0.6235315
24 -1.406951    1.290914
31  2.713400   -4.291244
41  0.01600274  2.476500
42 -2.045183    0.8222617
49 -3.308682    4.687681
56 -1.128588    0.1804773
57  1.869775    0.6528640
59  0.01285615 -8.095352
60  5.162988    0.6383966
```

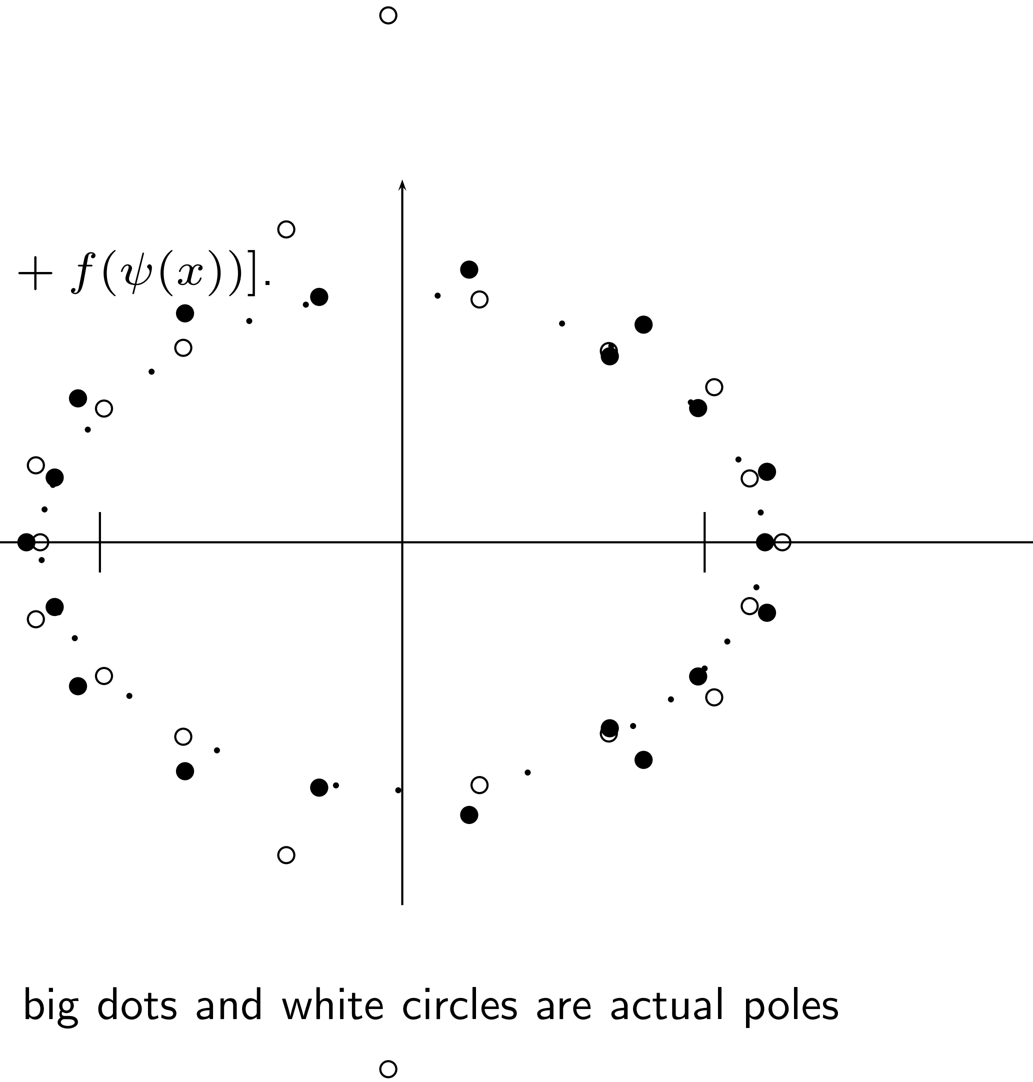


degree 20



degree 50

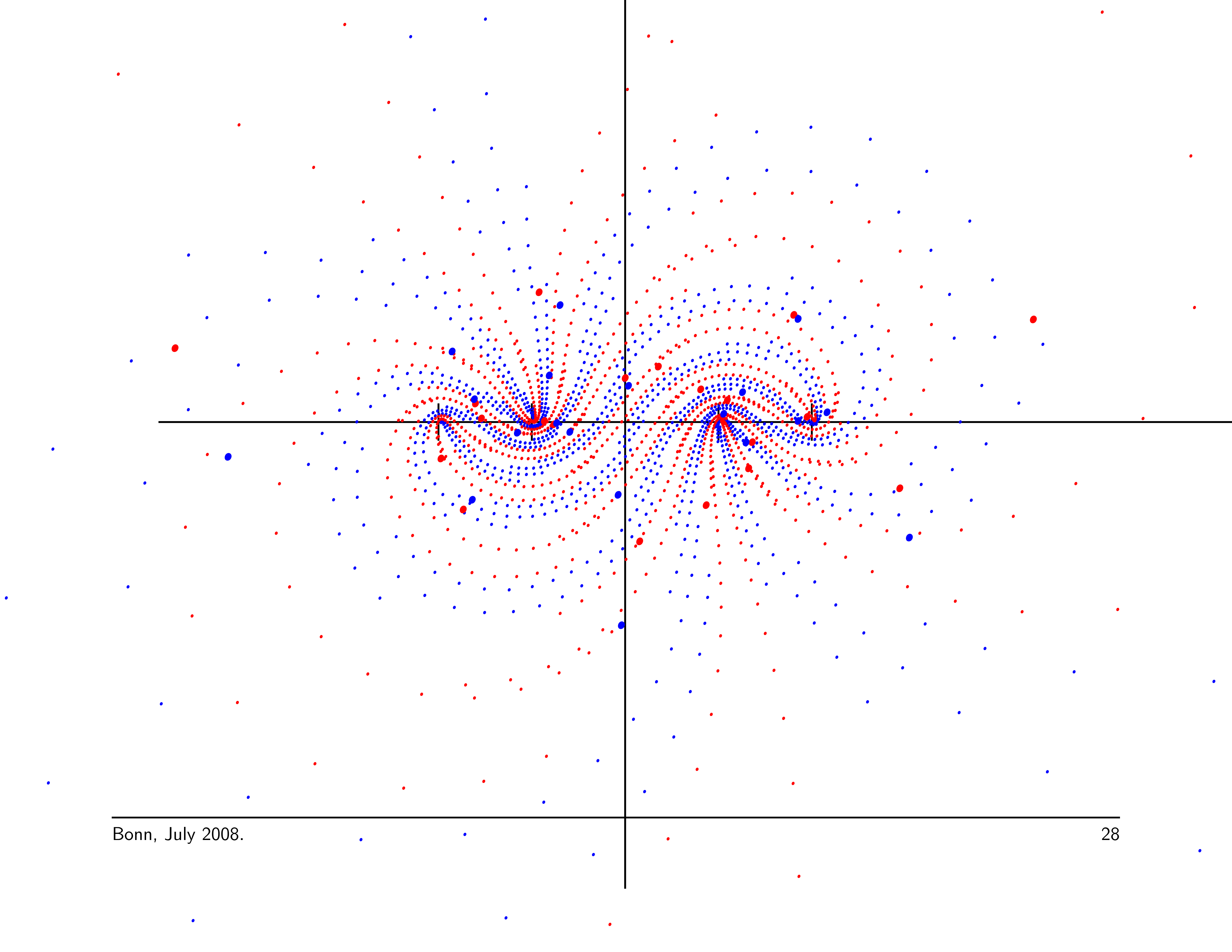
Ell. exponential: $(\mathcal{D}f)(x) = a[f(\varphi(x)) + f(\psi(x))]$.



Small black dots are $y-$sequences above; big dots and white circles are actual poles and zeros of rational interpolant of degree 20.

A less comfortable situation:

Censored



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Recommended reading.

V.P Spiridonov and A.S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, in Bustoz, Joaquin (ed.) et $al.$, *Special functions 2000: current perspective and future directions. Proceedings of the NATO Advanced Study Institute, Tempe, AZ, USA, May 29–June 9, 2000*, Dordrecht: Kluwer Academic Publishers. NATO Sci. Ser. II, Math. Phys. Chem. 30, 365-388 (2001).

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V. P. Spiridonov and A. S. Zhedanov: Elliptic grids, rational functions, and the Padé interpolation *The Ramanujan Journal* **13**, Numbers 1-3, June, 2007, p. 285–310.