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Rational interpolation to exponential-like functions on elliptic lattices.

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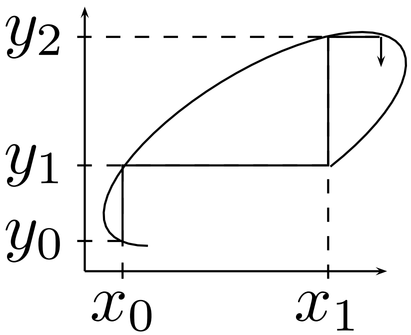
C'est le Nooooord. . .   M. Galabru

Difference equations and lattices.

Difference equations in history involve $(f(x), f(x+h))$, or $(f(x), f(x-h))$, or $f(x-h/2), f(x+h/2))$, or $f(x), f(qx))$, or $(f(q^{-1/2}x), f(q^{1/2}x))$, and more recently, values of f at $r(x) \pm \sqrt{s(x)}$, where r and s are rational functions.

Where will it end?

$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \tag{1}$$



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x, y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \qquad \text{(2a)}$$

where X_0, X_1, and X_2 are rational functions.

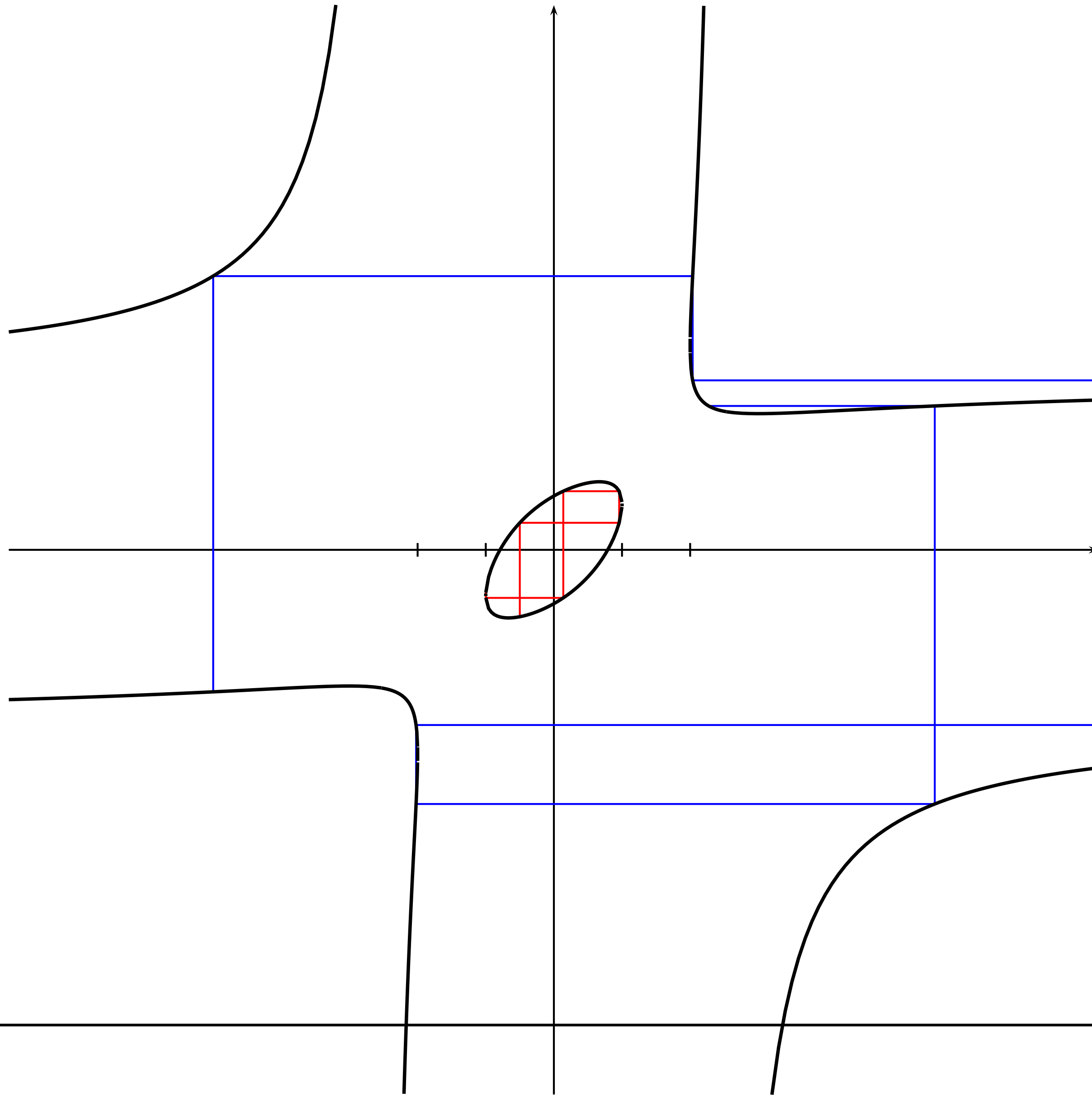
The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (2a) at x_0. We need x_1 such that y_1 is one of the two roots of (2a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \qquad \text{(2b)}$$

Both forms (2a) and (2b) hold simultaneously when F is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \qquad \text{(3)}$$

$$F(x,y) = k^2(1-k^2z_0^2)x^2y^2 - k^2(1-z_0^2)(x^2+y^2) - 2k(1-k^2)z_0xy + 1 - k^2z_0^2. \qquad \text{(4)}$$

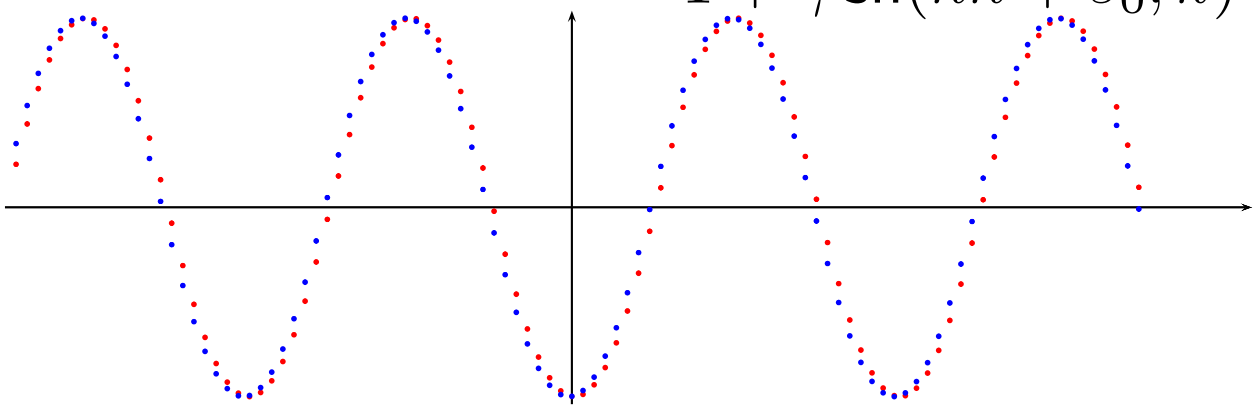


Why elliptic?

One has

$$2n\int_{z_0}^{z_1}\frac{dt}{\sqrt{P(t)}}=\int_{z_1}^{x_0}\frac{dt}{\sqrt{P(t)}}\pm\int_{z_0}^{x_n}\frac{dt}{\sqrt{P(t)}}+2\sum_j N_j\int_{z_1}^{z_{j+1}}\frac{dt}{\sqrt{P(t)}}, \tag{5}$$

Elliptic functions, at last: $x_n=\dfrac{\alpha\,\mathsf{sn}(nh+s_0;k)+\beta}{1+\gamma\,\mathsf{sn}(nh+s_0;k)}$



Exponential-like functions.

Solution of

$$(\mathcal{D}f)(x) = a[f(\varphi(x)) + f(\psi(x))] \qquad (6)$$

or

$$f(\psi(x)) = \frac{1 + a[\psi(x) - \varphi(x)]}{1 - a[\psi(x) - \varphi(x)]}\, f(\varphi(x))$$

on a lattice: $f(y_n) = \dfrac{[1 + a(y_n - y_{n-1})][1 + a(y_{n-1} - y_{n-2})] \cdots [1 + a(y_1 - y_0)]}{[1 - a(y_n - y_{n-1})][1 - a(y_{n-1} - y_{n-2})] \cdots [1 - a(y_1 - y_0)]}\, f(y$

on the arithmetic lattice, still an exponential function $f(x) = \left[\dfrac{1 + ah}{1 - ah}\right]^{(x - y_0)/h}$

on the $q-$lattice, $\varphi(x) = x$, $\psi(x) = qx$, $f(x) = \dfrac{e_q(ax)}{e_q(-ax)}$

```
 x-1= -0.7905694
 x0:    0.7905694  y0=0      f= 1
  1     0.5167233  0.9856450   2.943393
  2    -0.9529158 -0.3790235   0.5557844
  3    -0.1575301 -0.8702581   0.3365995
  4     0.9994804  0.6923469   2.741620
  5    -0.2301987  0.6400372   2.601862
...
```

```
1649 x= 0.9824965 y=0.4997582 , f= 2.165691
6729 x=-0.3972262 y=0.4999544 , f= 2.166314
9582 x=-0.3967925 y=0.5003586 , f= 2.167597
9846 x= 0.9826548 y=0.5005785 , f= 2.168296
6993 x= 0.9827326 y=0.5009825 , f= 2.169579
...
4008 x= 0.6658965 y=0.9999996 , f= 2.949004
1517 x= 0.6672618 y=0.9999997 , f= 2.949004
6861 x= 0.6662356 y=0.9999999 , f= 2.949004
9714 x= 0.6665745 y=1.000000 ,  f= 2.949005
```

Known rational approximations.

Padé for e^z, $[m/n] =$

$$\frac{1 + \frac{m}{m+n}\frac{z}{1!} + \frac{m(m-1)}{(m+n)(m+n-1)}\frac{z^2}{2!} + \cdots + \frac{m(m-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(n+1}}{1 - \frac{n}{m+n}\frac{z}{1!} + \frac{n(n-1)}{(m+n)(m+n-1)}\frac{z^2}{2!} - \cdots + (-1)^n\frac{n(n-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(m}} \tag{7}$$

$$e^z \text{ den.} - \text{num.} = \frac{(-1)^n}{(m+1)\cdots(m+n)} \sum_{k=m+n+1}^{\infty} \frac{(k-m-1)\cdots(k-m-n)}{k!} z^k$$

$$= \frac{(-1)^n}{(m+n)!}\left[\int_{-\infty}^{z} - \int_{-\infty}^{0} = \int_{0}^{z} e^t (z-t)^m t^n \, dt\right] \tag{8}$$

$$e^{ax} = 1 + \cfrac{2ax}{2 - ax + \cfrac{a^2x^2}{6 + \cfrac{a^2x^2}{10 + \cfrac{a^2x^2}{\ddots + \cfrac{a^2x^2}{a_k + \ddots}}}}} \tag{9}$$

with $a_k = 4k - 2$.

Equidistant points. [Iserles]

As far as we only need e^{Az} at $z = z_0, z_0 + h, \ldots, z_0 + (m+n)h$,

$$
\begin{aligned}
e^{Az} &= (\boldsymbol{I} + \boldsymbol{\Delta})^{(z-z_0)/h} e^{Az_0} \\
&= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \boldsymbol{\Delta}^k e^{Az_0} \\
&= \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h} \right)^k \frac{1}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h) e^{Az_0},
\end{aligned}
$$

which we multiply by the denominator

$$Q(z) = \sum_{j=0}^{n} q_j (z - z_0) \cdots (z - z_0 - (j-1)h),$$

$$Q(z)e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left(\frac{e^{Ah} - 1}{h} \right)^k \frac{C(k)}{k!}$$
$$(z - z_0)(z - z_0 - h) \cdots (z - z_0 - (k-1)h),$$

$$P(z) = e^{Az_0} \sum_{k=0}^{m} \left(\frac{e^{Ah} - 1}{h} \right)^k \binom{m}{k} (m + n - k)!$$
$$(z - z_0)(z - z_0 - h) \cdots (z - z_0 - (k-1)h),$$

$$Q(z) = \sum_{k=0}^{n} \left(\frac{e^{-Ah} - 1}{h} \right)^k \binom{n}{k} (m+n-k)!$$

$$(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

and, formally:

$$Q(z)e^{Az} - P(z) = e^{Az_0} m! (-1)^n$$

$$\sum_{k=m+n+1}^{\infty} \left(\frac{e^{Ah} - 1}{h} \right)^k \frac{(k-m-1)(k-m-2)\cdots(k-m-n)}{k!}$$

$$(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

$$(1+\alpha)^x = 1 + \cfrac{\alpha x}{1 + \cfrac{\alpha(1-x)}{2 + \cfrac{\alpha(1+x)}{3 + \cfrac{\alpha(2-x)}{\ddots + \cfrac{\alpha(k \pm x)}{a_k + \ddots}}}}} \tag{10}$$

$a_k = 2$ or $2k+1$. Already in Thiele (1909)!

$$(1+\alpha)^x = 1 + \cfrac{\alpha x}{1 + \cfrac{\alpha(1-x)}{2 + \alpha + \cfrac{\alpha(1+\alpha)(x-2)}{3 + \alpha + \cfrac{\alpha(3-x)}{\ddots + \cfrac{(-1)^k a_k(x-k)}{b_k + \ddots}}}}} \tag{11}$$

$$a_{2k} = \alpha(1+\alpha), b_{2k} = k\alpha + 2k + 1; a_{2k+1} = \alpha, b_{2k+1} = 2 + \alpha.$$

$$= 1 + \cfrac{\alpha(2+\alpha)x}{2(1+\alpha) - \alpha x + \cfrac{\alpha^2(1+\alpha)(x-1)(x-2)}{d_1 + \alpha^2 x + \cfrac{\alpha^2(1+\alpha)(x-3)(x-4)}{d_2 + \alpha^2 x + \cfrac{\alpha^2(1+\alpha)(x-3)(x-4)}{\ddots + \cfrac{c_k(x-2k+1)(x-2k)}{d_k + \alpha^2 x + \ddots}}}}} \quad (12)$$

$$c_k = \alpha^2(1+\alpha), d_k = (4k+2)(1+\alpha) - k\alpha^2.$$

Elliptic hypergeometric expansion.

$$\Phi_k(x) = \frac{(x-y_0)\cdots(x-y_{k-1})}{(x-y'_0)\cdots(x-y'_{k-1})}, \quad \Psi_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})}{(x-x'_0)\cdots(x-x'_{k-1})} \tag{13}$$

$$\begin{aligned}(\mathcal{D}\Phi_k)(x) &= C_k X_2(x) \frac{(x-x_0)\cdots(x-x_{k-2})}{(x-x'_{-1})(x-x'_0)\cdots(x-x'_{k-1})} \\ &= \frac{C_k X_2(x)\Psi_{k-1}(x)}{(x-x'_{-1})(x-x'_{k-1})}, k=1,2,\ldots \end{aligned} \tag{14}$$

The elliptic exponential-like function.

The expansion

$$f(x) = \sum_0^\infty \gamma_k \Phi_k(x) \Rightarrow (\mathcal{D}f)(x) - a[f(\varphi(x)) + f(\psi(x))]$$

$$= -2a\gamma_0 + \sum_1^\infty \gamma_k [C'_k \Psi_k(x) + C''_k \Psi_{k-1}(x)] = 0,$$

$$\gamma_k = \frac{(y_k - y'_{k-1})\Psi_{k-1}(x_{k-1})}{\Phi_{k-1}(y_k)} \frac{X_2(x'_{-1}) \cdots X_2(x'_{k-2})}{X_2(x_{-1}) \cdots X_2(x_{k-2})}$$

$$\frac{[1 + a(y'_0 - y'_{-1})] \cdots [1 + a(y'_{k-1} - y'_{k-2})]}{[1 - a(y_1 - y_0)] \cdots [1 - a(y_k - y_{k-1})]} \gamma_0 \quad (15)$$

Zeros and poles

At some $x = x_\alpha$, (6) is

$$f(y_{\alpha+1}) = \frac{1 + a[y_{\alpha+1} - y_\alpha]}{1 - a[y_{\alpha+1} - y_\alpha]} \, f(y_\alpha)$$

so that we expect poles at all the y_αs where $y_{\alpha+1-n} - y_{\alpha-n} = 1/a$, for some integer $n > 0$ and zeros when $y_{\alpha+1-n} - y_{\alpha-n} = -1/a$.

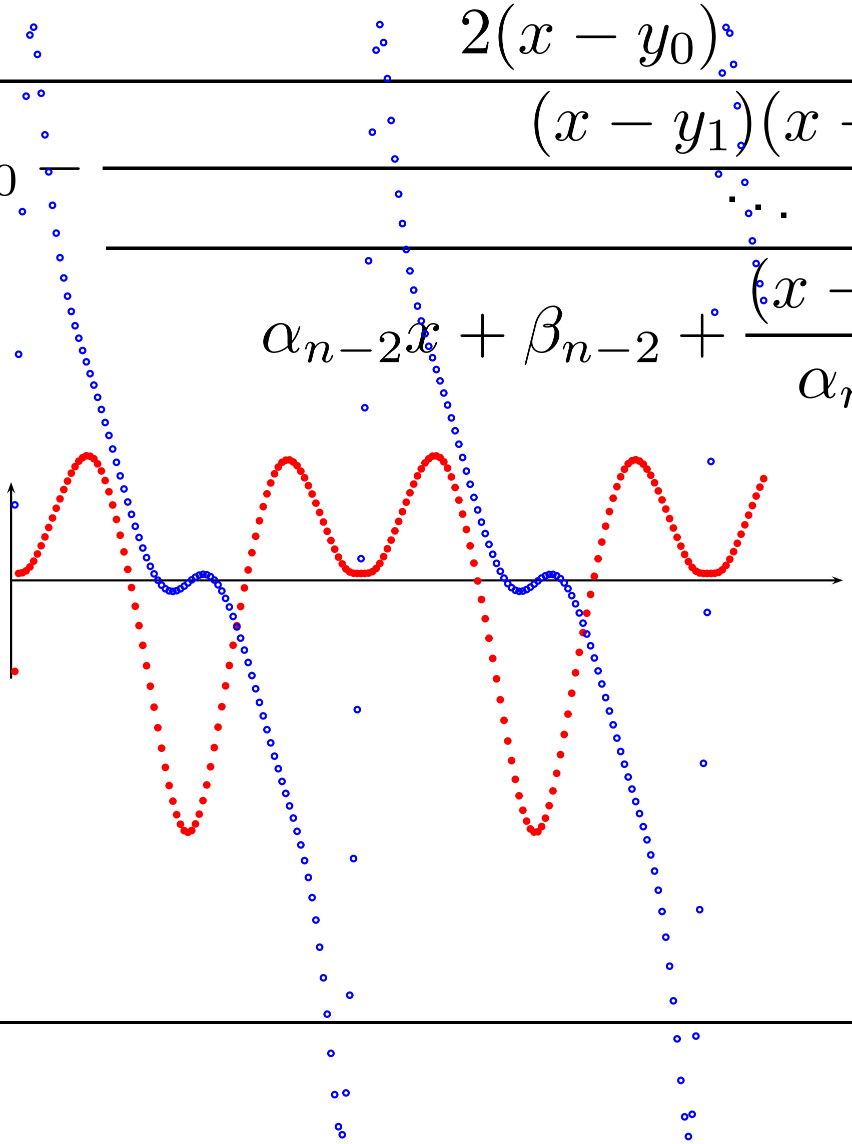
The $x-$roots of $\psi(x) - \varphi(x) = \pm 1/a$ are the 4 roots of $4a^2 P(x) - X_2^2(x) = 0$. Let x_β be such a root, then, the sequence $\{\dots, y_{\beta-1}, y_\beta, y_{\beta+1}, \dots\}$ is liable to contain zeros and poles.

For instance, with $a = 1/2$,
k=1/2, z0=3 $X_0(x) = 2x^2 - 1.25, X_1(x) = -2.25x, X_2(x) = -0.3125x^2 + 2$, so that $P(x) = X_1^2(x)/4 - X_0(x)X_2(x) = 2.5(1 - x^2)(1 - x^2/4)$, the roots are $x = \pm 2.0563 \Rightarrow \varphi$

and $\psi = \pm(4.4086, 2.4086)$; $x = \pm 0.82019i \Rightarrow \varphi$ and $\psi = \pm(1 + 0.41748i, -1 + 0.41748i)$, and the $y-$sequences are $\pm\{\dots, -4.4086, -2.4086, 2.6785, 3.4686, -2.1483, -8.6905, 2.0010, -$ and $\pm\{\dots, -1+0.41748i, 1+0.41748i, 0.69127-0.64922i, -1.1556-0.18970i, -0.21929+0.80408i, 1.1962-0.019606i, -0.31895-0.78586i, -1.1358 + 0.23132i, 0.76339 + 0.60765i, \dots\}$

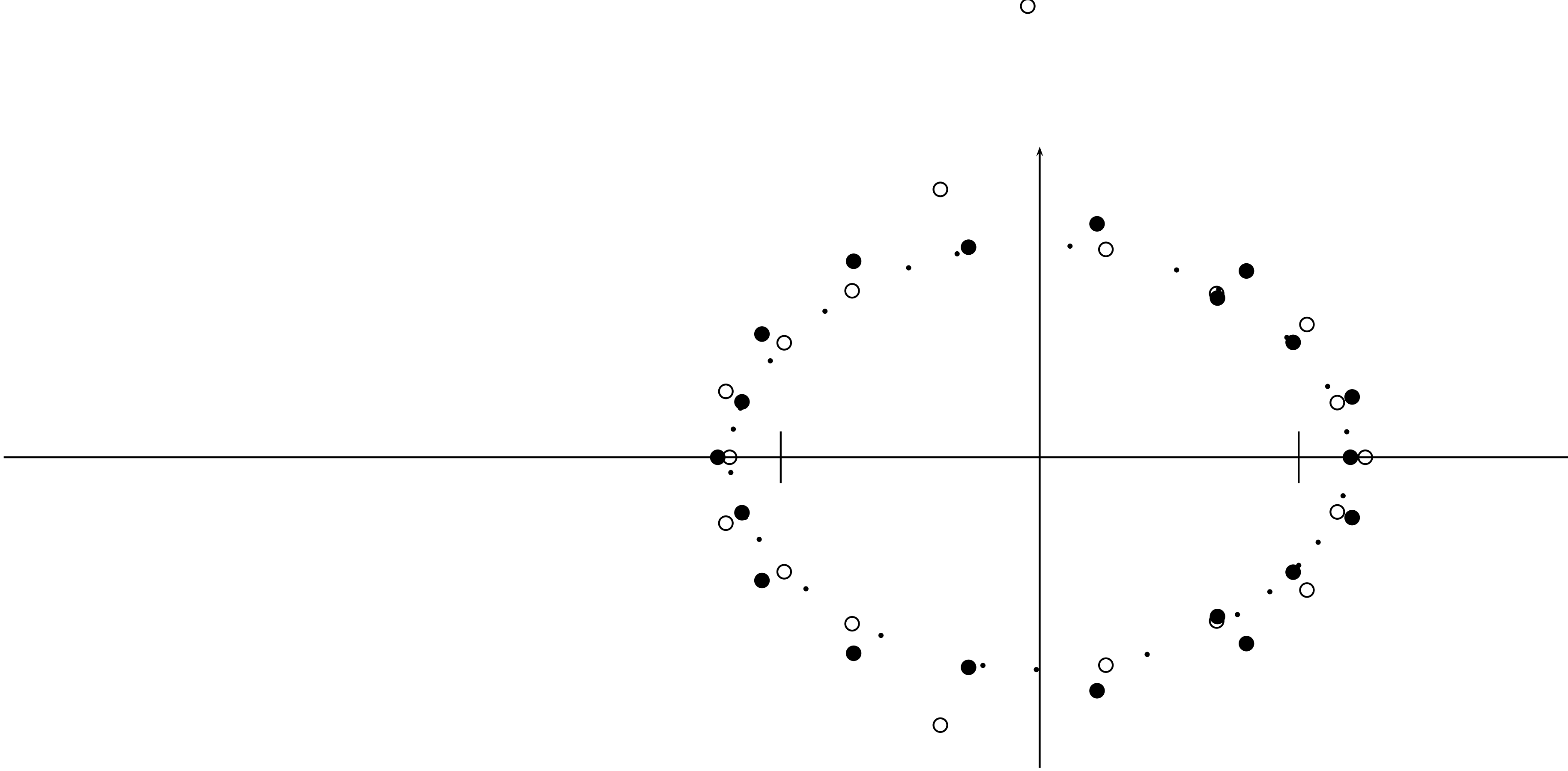
Interpolatory continued fraction expansion

$$f_0(x) = \cfrac{2(x-y_0)}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\ddots \; \cfrac{}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x + \beta_{n-1} + \cdots}}}}$$



Some values of the rational interpolants:

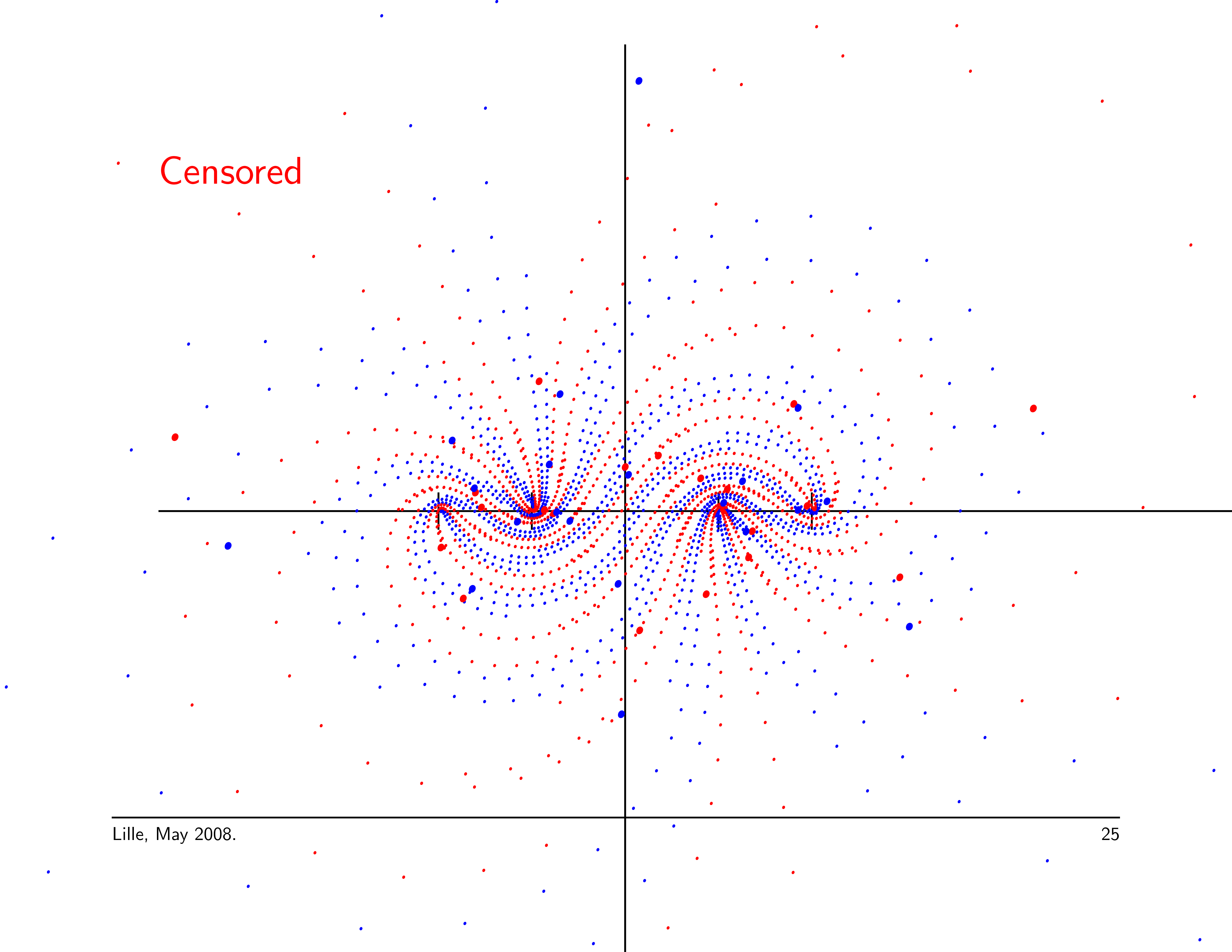
	-1.5	-1	-0.5	0	0.5	1	1.5
1	-0.3186707	0.0106012	0.4343382	1	1.793233	2.985950	4.981495
2	0.2590631	0.3097855	0.4770833	1	2.257913	2.935437	2.253881
3	-0.7447123	0.3600602	0.4684872	1	2.183663	2.918343	1.124377
4	0.6246931	0.3494623	0.4623021	1	2.107666	2.947874	3.279223
5	0.0971869	0.3404011	0.4607317	1	2.146329	2.948048	4.050295
6	0.3508917	0.3411494	0.5006247	1	2.166102	2.949342	1.083328
7	-1.800737	0.3387593	0.4615957	1	2.166283	2.949189	0.2481340
8	0.4737387	0.3403256	0.4615740	1	2.166444	2.948994	-99.08147
9	-0.1343806	0.3390977	0.4615842	1	2.166454	2.948999	-6.033262
10	0.3570745	0.3390971	0.4615828	1	2.166457	2.949005	-0.9676665
...							
20	0.3049597	0.3390975	0.4615828	1	2.166458	2.949005	1.769250
50	1.925918	0.3390975	0.4615828	1	2.166458	2.949005	-0.3163812



Small black dots are $y-$sequences above; big dots and white circles are actual poles and zeros of rational interpolant of degree 20.

A less comfortable situation:

Censored



Recommended reading.

V.P Spiridonov and A.S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, *in* Bustoz, Joaquin (ed.) et *al.*, *Special functions 2000: current perspective and future directions. Proceedings of the NATO Advanced Study Institute, Tempe, AZ, USA, May 29–June 9, 2000*, Dordrecht: Kluwer Academic Publishers. NATO Sci. Ser. II, Math. Phys. Chem. 30, 365-388 (2001).

V.P. Spiridonov, Elliptic hypergeometric functions, Abstract: This is a brief overview of the status of the theory of elliptic hypergeometric functions to the end of 2006 written as a complement to a Russian edition (to be published by the Independent University press, Moscow, 2007) of the book by G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encycl. of Math. Appl. **71**, Cambridge Univ. Press, 1999. Report number: RIMS-1589 Cite as: `http://arxiv.org/abs/0704.3099` : `arXiv:0704.3099v1 [math.CA]`

V. P. Spiridonov and A. S. Zhedanov: Elliptic grids, rational functions, and the Padé interpolation *The Ramanujan Journal* **13**, Numbers 1-3, June, 2007, p. 285–310.