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Rational interpolation to solutions of Riccati difference equations on elliptic lattices.

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... opresivo y lento y plural.
J.L. Borges

Abstract. It is shown how to define difference equations on particular lattices $\{x_n\}$, $n \in \mathbb{Z}$, where the x_n s are values of an elliptic function at a sequence of arguments in arithmetic progression (*elliptic lattice*). Solutions to special difference equations (elliptic Riccati equations) have remarkable simple (!) interpolatory continued fraction expansions.

1. DIFFERENCE EQUATIONS AND LATTICES.

Simplest difference equations relate two values of the unknown function f : say, $f(\varphi(x))$ and $f(\psi(x))$.

Most instances [19] are $(\varphi(x), \psi(x)) = (x, x + h)$, or the more symmetric $(x - h/2, x + h/2)$, or also (x, qx) in q -difference equations [7, 11, 12]. Recently, more complicated forms $(r(x) - \sqrt{s(x)}, r(x) + \sqrt{s(x)})$ have appeared [1, 3, 11, 12, 16, 17, 20, 21], where r and s are rational functions.

This latter trend will be examined here: we need, for each x , two values $f(\varphi(x))$ and $f(\psi(x))$ for f .

A first-order difference equation is $\mathcal{F}(x, f(\varphi(x)), f(\psi(x))) = 0$, or $f(\varphi(x)) - f(\psi(x)) = \mathcal{G}(x, f(\varphi(x)), f(\psi(x)))$ if we want to emphasize

the difference of f . There is of course some freedom in this latter writing. Only *symmetric* forms in φ and ψ will be considered here:

$$(\mathcal{D}f)(x) = \mathcal{F}(x, f(\varphi(x)), f(\psi(x))), \quad (1)$$

where \mathcal{D} is the divided difference operator

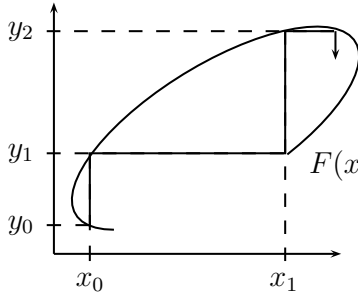
$$(\mathcal{D}f)(x) = \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)}, \quad (2)$$

and where \mathcal{F} is a symmetric function of its two last arguments.

For instance, a linear difference equation of first order may be written as $a(x)f(\varphi(x)) + b(x)f(\psi(x)) + c(x) = 0$, as well as $\alpha(x)(\mathcal{D}f)(x) = \beta(x)[f(\varphi(x)) + f(\psi(x))] + \gamma(x)$, with $\alpha(x) = [b(x) - a(x)][\psi(x) - \varphi(x)]/2$, $\beta(x) = -[a(x) + b(x)]/2$, and $\gamma(x) = -c(x)$.

2. ELLIPTIC GRID, OR LATTICE.

2.1. Definition of elliptic grid.



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two y -roots of

$$F(x, y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \quad (3a)$$

where X_0, X_1 , and X_2 are rational functions.

Remark that the sum and the product of φ and ψ are the rational functions

$$\varphi + \psi = -X_1/X_2, \quad \varphi\psi = X_0/X_2. \quad (3b)$$

When the divided difference operator \mathcal{D} of (2) is applied to a rational function, the result is still a rational function. However, polynomials are normally not sent to polynomials, for instance,

$$\mathcal{D}x^2 = -X_1(x)/X_2(x), \quad \mathcal{D}x^3 = (X_1^2(x) - X_0(x)X_2(x))/X_2^2(x).$$

But difference equations must allow the recovery of f on a whole set of points! An initial-value problem for a first order difference equation starts with a value for $f(y_0)$ at $x = x_0$, where y_0 is one root of (3a) at $x = x_0$. The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (3a) at x_0 . We need x_1 such that y_1 is one of the two roots of (3a) at x_1 , so for one of the roots of $F(x, y_1) = 0$

which is not x_0 . Here again, the simplest case is when F is of degree 2 in x :

$$F(x, y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \quad (3c)$$

Both forms (3a) and (3c) hold simultaneously when F is **biquadratic**:

$$F(x, y) = \sum_{i=0}^2 \sum_{j=0}^2 c_{i,j} x^i y^j. \quad (4)$$

Definition. A sequence $\{\dots, x_{-1}, x_0, x_1, \dots\}$ is an elliptic lattice if there exists a sequence $\{\dots, y_{-1}, y_0, y_1, \dots\}$ and a biquadratic polynomial (4) such that $F(x_n, y_n) = 0$ and $F(x_n, y_{n+1}) = 0$, for $n \in \mathbb{Z}$.

As y_n and y_{n+1} are the two roots in t of $F(x_n, t) = X_0(x_n) + X_1(x_n)t + X_2(x_n)t^2 = 0$, useful identities are

$$y_n + y_{n+1} = -\frac{X_1(x_n)}{X_2(x_n)}, \quad y_n y_{n+1} = \frac{X_0(x_n)}{X_2(x_n)}, \quad (5)$$

and the direct formula

$$y_n \text{ and } y_{n+1} = \frac{-X_1(x_n) \pm \sqrt{P(x_n)}}{2X_2(x_n)}, \quad (6)$$

where

$$P = X_1^2 - 4X_0X_2 \quad (7)$$

is a polynomial of degree 4.

Also, as x_{n+1} and x_n are the two roots in t of $F(t, y_{n+1}) = 0$,

$$x_n + x_{n+1} = -\frac{Y_1(y_{n+1})}{Y_2(y_{n+1})}, \quad x_n x_{n+1} = \frac{Y_0(y_{n+1})}{Y_2(y_{n+1})}. \quad (8)$$

Of course, the sequence $\{y_n\}$ is elliptic too.

Note that the names of the x - and y - lattices are sometimes inverted, as in [31, eq. (1.2)]

The construction above is called ‘‘T-algorithm’’ in [31, Theorem 6].

As the operators considered here are symmetric in $\varphi(x)$ and $\psi(x)$, we do not need to define precisely what φ and ψ are. However, once a starting point (x_0, y_0) is chosen, it will be convenient to define $\varphi(x_n) = y_n$ and $\psi(x_n) = y_{n+1}$, $n \in \mathbb{Z}$.

Special cases. We already encountered the usual difference operators $(\varphi(x), \psi(x)) = (x, x+h)$ or $(x-h, x)$ or $(x-h/2, x+h/2)$ corresponding to $X_2(x) \equiv 1$, X_1 of degree 1, X_0 of degree 2 with $P = X_1^2 - 4X_0X_2$ of degree 0. For the geometric difference operator, P is the square of a first

degree polynomial. For the Askey-Wilson operator [1, 3, 9, 11, 16, 17], P is an arbitrary second degree polynomial.

2.2. Equivalent definitions. The companion sequence $\{y_n\}$ is not needed in the definition of an elliptic lattice, but the definition above is best suited to the description of difference equations.

A relation involving only x_n and x_{n+1} is obtained by the elimination of y_{n+1} through the resultant of the two polynomials in y_{n+1} from (8) $P_1(y_{n+1}) = (x_n + x_{n+1})Y_2(y_{n+1}) + Y_1(y_{n+1})$ and $P_2(y_{n+1}) = x_n x_{n+1} Y_2(y_{n+1}) - Y_0(y_{n+1})$.

The form of this resultant is most easily found through interpolation at the two zeros u and v of Y_2 : let $Y_2(y) = \alpha(y-u)(y-v)$, $Y_0(y) = \beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)$, and $Y_1(y) = \gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)$, then, with $y = y_{n+1}$, $S = x_n + x_{n+1}$ and $\Pi = x_n x_{n+1}$,

$$\begin{aligned} S &= -\frac{Y_1(y)}{Y_2(y)} = -\frac{\gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)}{\alpha(y-u)(y-v)}, \\ \Pi &= \frac{Y_0(y)}{Y_2(y)} = \frac{\beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)}{\alpha(y-u)(y-v)}, \\ \alpha S + \gamma &= -\frac{\gamma'(y-u) + \gamma''(y-v)}{(y-u)(y-v)}, \quad \alpha \Pi - \beta = \frac{\beta'(y-u) + \beta''(y-v)}{(y-u)(y-v)}, \\ \beta'(\alpha S + \gamma) + \gamma'(\alpha \Pi - \beta) &= \frac{\beta''\gamma' - \beta'\gamma''}{y-u}, \\ \beta''(\alpha S + \gamma) + \gamma''(\alpha \Pi - \beta) &= \frac{\beta'\gamma'' - \beta''\gamma'}{y-v}, \end{aligned}$$

$$\frac{v-u}{\beta''\gamma' - \beta'\gamma''} = \frac{1}{\beta'(\alpha S + \gamma) + \gamma'(\alpha \Pi - \beta)} + \frac{1}{\beta''(\alpha S + \gamma) + \gamma''(\alpha \Pi - \beta)} \quad (9)$$

which leads clearly to a polynomial of degree 2 in $x_n + x_{n+1}$ and $x_n x_{n+1}$, so

2.2.1. Definition 2. An elliptic lattice, or grid, is a sequence satisfying a symmetric biquadratic relation [31, Theorem 5]

$$\begin{aligned} E(x_n, x_{n+1}) &= d_{0,0} + d_{0,1}(x_n + x_{n+1}) + d_{0,2}(x_n + x_{n+1})^2 + d_{1,1}x_n x_{n+1} \\ &\quad + d_{1,2}x_n x_{n+1}(x_n + x_{n+1}) + d_{2,2}x_n^2 x_{n+1}^2 = 0. \end{aligned} \quad (10)$$

Conversely, let us show that Definition 2 implies the main definition: from a sequence $\{x_n\}$ satisfying (10), let us build a valid sequence $\{y_n\}$. We must construct $F(x, y) = \alpha(y-u)(y-v)x^2 + [\gamma(y-u)(y-v) + \gamma'(y-u) + \gamma''(y-v)]x + \beta(y-u)(y-v) + \beta'(y-u) + \beta''(y-v)$ such

that the resultant (9) is deduced from (10) when $S = x_n + x_{n+1}$ and $\Pi = x_n x_{n+1}$, i.e.,

$$d_{0,0} + d_{0,1}S + d_{0,2}S^2 + d_{1,1}\Pi + d_{1,2}\Pi S + d_{2,2}\Pi^2 = 0. \quad (11)$$

Let us decide that $\alpha = 1$. Then, one chooses $(-\gamma, \beta) = (S, \Pi)$ as a point¹ on the conic (11). We now have

$$d_{0,2}(S+\gamma)^2 + d_{1,2}(\Pi-\beta)(S+\gamma) + d_{2,2}(\Pi-\beta)^2 + d'_{0,1}(S+\gamma) + d'_{1,1}(\Pi-\beta) = 0,$$

with $d'_{0,1} = -2d_{0,2}\gamma + d_{1,2}\beta + d_{0,1}$ and $d'_{1,1} = -d_{1,2}\gamma + 2d_{2,2}\beta + d_{1,1}$. Let ρ_1 and ρ_2 be the two roots of $d_{2,2}\rho^2 + d_{1,2}\rho + d_{0,2} = 0$, then, $d_{2,2}[\Pi - \beta - \rho_1(S + \gamma)][\Pi - \beta - \rho_2(S + \gamma)] + d'_{0,1}(S + \gamma) + d'_{1,1}(\Pi - \beta) = 0$, and we divide by $[\Pi - \beta - \rho_1(S + \gamma)][\Pi - \beta - \rho_2(S + \gamma)]$:

$$d_{2,2} + \frac{\mu}{\Pi - \beta - \rho_1(S + \gamma)} + \frac{\eta}{\Pi - \beta - \rho_2(S + \gamma)} = 0,$$

with $\mu = (d'_{1,1}\rho_1 + d'_{0,1})/(\rho_1 - \rho_2)$ and $\eta = (d'_{1,1}\rho_2 + d'_{0,1})/(\rho_2 - \rho_1)$. We now compare with (9):

$$\gamma' = \frac{\eta(v - u)}{(\rho_2 - \rho_1)d_{2,2}}, \gamma'' = \frac{\mu(v - u)}{(\rho_2 - \rho_1)d_{2,2}}, \beta' = -\rho_1\gamma', \beta'' = -\rho_2\gamma''.$$

The degrees of freedom are therefore u , v , and $(-\gamma, \beta) = (S, \Pi)$ on the conic (11).

2.2.2. Definition 3. An elliptic lattice is a sequence $x_n = \mathcal{E}(nh + u_0)$, where \mathcal{E} is any elliptic function of order 2 (i.e., with 2 zeros and 2 poles in a fundamental parallelogram of periods).

From the main definition, one may establish that the biquadratic curve $F(x, y) = 0$ in (4) has genus 1 and a parametric representation

$$x = \mathcal{E}_1(s), \quad y = \mathcal{E}_2(s),$$

with \mathcal{E}_1 and \mathcal{E}_2 elliptic functions of order 2.

Indeed, a birational transformation $(x, y) \leftrightarrow (\xi, \eta)$ sending the biquadratic curve (4) $F(x, y) = 0$ to the canonical elliptic curve $\eta^2 = Q(\xi)$, where Q is a polynomial of third degree, see [2, p.292]: from (6), choose $w =$ a square root of $P(x)$, so that $y = (-X_1(x) + w)/(2X_2(x)) \leftrightarrow w = X_1(x) + 2yX_2(x)$, and $x = z_1 + 1/\xi$, where z_1 is one of the four roots of $P(x) = 0$. Then, with $P(z_1 + 1/\xi) = Q(\xi)/\xi^4$ and $\eta = w\xi^2$, $\eta^2 = Q(\xi)$.

Then, the Weierstrass representation holds $\xi = \wp(hu + u_0)$, $\eta = \wp'(hu + u_0)$. So, $x = z_1 + 1/\wp$, $y = (-X_1 + \wp'/\wp^2)/(2X_2)$.

¹The importance of this conic has been stressed by A.Ronveaux [22].

However, the authors of [14] recommend the biquadratic setting instead of the more familiar cubic one, see [14, pp. 300-301].

Now, let s_n and s'_n correspond to the two points (x_n, y_n) and (x_n, y_{n+1}) . As $\mathcal{E}_1(s_n) = \mathcal{E}_1(s'_n)$ with y_{n+1} normally different from y_n , $s_n + s'_n = a$ constant, say γ_1 (as s_n and s'_n are integrals involving the square root of a polynomial on two paths with the same endpoints [the second endpoint being x_n], the square roots are opposite on a part of the paths). Similarly, $s'_n + s_{n+1} =$ another constant, say γ_2 . Therefore, $s_{n+1} = s_n + h$, with $h = \gamma_2 - \gamma_1$, and this establishes Definition 3 with $\mathcal{E} = \mathcal{E}_1$.

Conversely, from Definition 3, one recovers Definition 2 by recognizing (10) as an addition formula for elliptic functions [31].

The essential parameters in the description of an elliptic sequence are the modulus k and the step h . The modulus is also related to the ratio ω_1/ω_2 of periods. Finally, in a multiplicative setting, the main parameters are the nome p and the multiplier q , which are basically (i.e., up to multiplication by constants) the exponentials of the periods ratio and the step.

The modulus and the step depend only on F in (4) (or E in (10)). For each starting point (x_0, y_0) , or $s_0 = h_0$, there is a different elliptic lattice with the same k and h .

It is always possible to relate \mathcal{E}_2 to \mathcal{E}_1 through a rational transformation of first degree $\mathcal{E}_2(s) = \frac{\alpha\mathcal{E}_1(s + h/2) + \beta}{\gamma\mathcal{E}_1(s + h/2) + \delta}$ [31, p. 298].

2.3. A brief history [4, 5, 14, 31]. Elliptic lattices were developed by Baxter in the solution of special problems of statistical physics, they appear in works by Fritz John, in many treatments of a Poncelet problem [4, 5] [31, § 6], and go back to pioneering work by Euler² on the addition formulas of elliptic functions, that's why the symmetric biquadratic polynomial (10) has been called the Euler polynomial in [31, p. 294].

Even the name of our subject is not easy to choose: “elliptic sequences” seems perfect, but this name is used by other sequences related in another way to elliptic functions (sequences $\{A_n\}$ where $A_{n-1}A_{n+1}/A_n^2$ is our x_n , [32]), “elliptic lattice” may be used for the repetitions of the periods parallelogram of an elliptic function, “elliptic grid” means a convenient mesh for discretizing over ellipses, and “elliptic difference operator” is a partial difference operator extending partial differential operator of elliptic type.

²and perhaps even to Fermat [communicated by R. Askey]!

3. ELLIPTIC PEARSON EQUATION.

A famous theorem by Pearson([6, (2.25) p.152]; [8]) relates the classical orthogonal polynomials to the differential equation $w' = rw$ satisfied by the weight function, where r is a rational function of degree ≤ 2 .

Even without this constraint on the degree, the Stieltjes transform $f(z) = \int_a^b \frac{w(t) dt}{z-t}$ of w satisfies the differential equation $f'(z) = r(z)f(z) + s(z)$, where s is a rational function³too. A suitable continued fraction expansion of f leaves then important informations on the relevant orthogonal polynomials (theory of Laguerre [13]).

The Pearson equation has of course been extended to various difference calculus settings [1, 9, 18, 20, 21], here is the elliptic version:

3.1. Theorem. *Let (x'_0, y'_0) be a point on the biquadratic curve $F = 0$ of (4), $\{(x'_k, y'_k)\}$ the elliptic lattice starting from this point. If there are polynomials a and c , with*

$$a(x'_0) - (y'_1 - y'_0)c(x'_0) = a(x'_N) + (y'_{N+1} - y'_N)c(x'_N) = 0, \quad (12)$$

and a sequence $\{w_0, \dots, w_{N+1}\}$ such that

$$a(x'_k) \frac{\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1} - x'_k)} - \frac{w_k}{Y_2(y'_k)(x'_k - x'_{k-1})}}{y'_{k+1} - y'_k} = c(x'_k) \left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1} - x'_k)} + \frac{w_k}{Y_2(y'_k)(x'_k - x'_{k-1})} \right], \quad (13)$$

$k = 0, 1, \dots, N$, and $w_0 = w_{N+1} = 0$, then,

$$f(x) = \sum_{k=1}^N \frac{w_k}{x - y'_k} \quad (14)$$

satisfies

$$a(x) \mathcal{D}f(x) = a(x) \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} = c(x)[f(\varphi(x)) + f(\psi(x))] + d(x), \quad (15)$$

³ $s(z) = D(z)/A(z)$, where the polynomials A and D are related to boundary conditions for the weight function w at a and b : the product $Ar = C$ must be a polynomial and $A(t)w(t)$ must vanish at $t = a$ and $t = b$. Then $D(z) = - \int_a^b \left[\frac{A(z) - A(t) - (z-t)A'(t)}{(z-t)^2} + \frac{C(z) - C(t)}{z-t} \right] w(t) dt$. So, even in the Legendre case, where $w(t) \equiv 1$, $r = C = 0$, one must take $A(z) = 1 - z^2$, and $D(z) = 2$ follows.

where d is a polynomial too.

We already used the writing $(x_n, y_n) = (\mathcal{E}_1(s_0 + nh), \mathcal{E}_2(s_0 + nh))$ for a generic elliptic lattice, normally to be used as interpolation points. We here need a function with poles on another lattice with same modulus and step, but with another starting point, and it is written here $(x'_n, y'_n) = (\mathcal{E}_1(s'_0 + nh), \mathcal{E}_2(s'_0 + nh))$.

Remark that (13) is a recurrence relation for the $\tilde{w}_k = \frac{w_k}{Y_2(y'_k)(x'_k - x'_{k-1})}$,

which is

$$[a(x'_k) - (y'_{k+1} - y'_k)c(x'_k)]\tilde{w}_{k+1} = [a(x'_k) + (y'_{k+1} - y'_k)c(x'_k)]\tilde{w}_k, \quad (16)$$

$k = 0, \dots, N$, so that (12) ensures the boundary conditions $w_0 = w_{N+1} = 0$.

Proof:

$$\begin{aligned} \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} &= - \sum_1^N \frac{w_k}{(\varphi(x) - y'_k)(\psi(x) - y'_k)} = - \sum_1^N \frac{w_k X_2(x)}{F(x, y'_k)} \\ &= - \sum_1^N \frac{w_k X_2(x)}{Y_2(y'_k)(x - x'_{k-1})(x - x'_k)} = X_2(x) \sum_0^N \frac{\tilde{w}_{k+1} - \tilde{w}_k}{x - x'_k} \end{aligned}$$

where $\tilde{w}_k = \frac{w_k}{Y_2(y'_k)(x'_k - x'_{k-1})}$, and with $w_0 = w_{N+1} = 0$,

$$\begin{aligned} f(\psi(x)) + f(\varphi(x)) &= - \sum_1^N \frac{w_k [X_1(x) + 2y'_k X_2(x)]}{X_2(x)(\varphi(x) - y'_k)(\psi(x) - y'_k)} = - \sum_1^N \frac{w_k [X_1(x) + 2y'_k X_2(x)]}{F(x, y'_k)} \\ &= - \sum_1^N \frac{w_k [X_1(x) + 2y'_k X_2(x)]}{Y_2(y'_k)(x - x'_{k-1})(x - x'_k)} = \sum_0^N \frac{\tilde{w}_{k+1} [X_1(x) + 2y'_{k+1} X_2(x)] - \tilde{w}_k [X_1(x) + 2y'_k X_2(x)]}{x - x'_k}, \end{aligned}$$

therefore the rational functions $a\mathcal{D}f$ and $c(f(\varphi) + f(\psi))$ differ by a polynomial if all the residues are equal:

$a(x'_k)X_2(x'_k)(\tilde{w}_{k+1} - \tilde{w}_k) = c(x'_k)(\tilde{w}_{k+1}[X_1(x'_k) + 2y'_{k+1}X_2(x'_k)] - \tilde{w}_k[X_1(x'_k) + 2y'_kX_2(x'_k)])$ for $k = 0, 1, \dots, N$. Or, as $X_1(x'_k) = -(y'_k + y'_{k+1})X_2(x'_k)$, $a(x'_k)(\tilde{w}_{k+1} - \tilde{w}_k) = c(x'_k)(y'_{k+1} - y'_k)(\tilde{w}_{k+1} + \tilde{w}_k)$, which is exactly (16).

4. INTERPOLATORY CONTINUED FRACTION EXPANSION AND RICCATI EQUATIONS.

Let $f_0(x) = f(x) - f(y_0)$ be expanded in an interpolatory continued fraction (R_{II} -fraction [9, 10, 30, 33], or contracted Thiele's continued fraction [19, Chap. 5])

$$f_0(x) = \frac{x - y_0}{\alpha_0 x + \beta_0 - \frac{(x - y_1)(x - y_2)}{\alpha_{n-2} x + \beta_{n-2} + \frac{(x - y_{2n-3})(x - y_{2n-2})}{\alpha_{n-1} x + \beta_{n-1} + \cdots}}}$$

making clear that the n^{th} approximant (stopped at, and including the $\alpha_{n-1}x + \beta_{n-1}$ term) is the rational function of degree n interpolating f_0 at $x = y_0, y_1, \dots, y_{2n}$.

Let p_n be the denominator of this n^{th} approximant. If f is a Stieltjes transform, it is known [10, 30] [33, § 5] [34] that $\{p_n(x)/((x - y_0)(x - y_2) \dots (x - y_{2n}))\}$ and $\{p_m(x)/((x - y_1)(x - y_3) \dots (x - y_{2m+1}))\}$ are biorthogonal sequences of rational functions.

From $f_n(x) = \frac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$, $\alpha_n x + \beta_n$ is the polynomial interpolant of degree 1 to $(x - y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2} , so we need $f_n(y_{2n+1})$ and $f_n(y_{2n+2})$ in order to find α_n and β_n .

If f_n satisfies a difference equation of first order $\mathcal{F}_n(x, f_n(\varphi(x)), f_n(\psi(x))) = 0$, we find $f_n(y_{2n+1})$ from the equation at x_{2n} , as $\varphi(x_{2n}) = y_{2n}$, $\psi(x_{2n}) = y_{2n+1}$, and $f_n(y_{2n}) = 0$. Next, the equation at $x = x_{2n+1}$ yields $f_n(y_{2n+2})$.

As seen in the section 3 on the Pearson equation, a linear difference equation of first order easily involves a weight function useful in orthogonality considerations. However, Riccati equations are better suited to continued fraction constructions [7, 15]. Of course, linear difference equations of first order are special cases of Riccati equations, that is why the coefficients in (15) are written $a(x)$, $c(x)$, and $d(x)$, whereas $b(x)$ is the coefficient of the nonlinear part of a Riccati equation.

So, if f_n satisfies the Riccati equation

$$a_n(x) \frac{f_n(\psi(x)) - f_n(\varphi(x))}{\psi(x) - \varphi(x)} = b_n(x) f_n(\varphi(x)) f_n(\psi(x)) + c_n(x) (f_n(\varphi(x)) + f_n(\psi(x))) + d_n(x), \quad (17)$$

one finds at $x = x_{2n}$, $\varphi(x) = y_{2n}$, $\psi(x) = y_{2n+1}$, and knowing that $f_n(y_{2n}) = 0$, $a_n(x_{2n}) \frac{f_n(y_{2n+1})}{y_{2n+1} - y_{2n}} = c_n(x_{2n}) f_n(y_{2n+1}) + d_n(x_{2n})$ yields $f_n(y_{2n+1}) = \frac{d_n(x_{2n})}{\frac{a_n(x_{2n})}{y_{2n+1} - y_{2n}} - c_n(x_{2n})}$ and

$$\alpha_n y_{2n+1} + \beta_n = \frac{y_{2n+1} - y_{2n}}{f_n(y_{2n+1})} = \frac{a_n(x_{2n}) - (y_{2n+1} - y_{2n})c_n(x_{2n})}{d_n(x_{2n})}, \quad (18)$$

$$\text{and at } x = x_{2n+1}, a_n(x_{2n+1}) \frac{\frac{y_{2n+2} - y_{2n}}{\alpha_n y_{2n+2} + \beta_n} - \frac{y_{2n+1} - y_{2n}}{\alpha_n y_{2n+1} + \beta_n}}{y_{2n+2} - y_{2n+1}} =$$

$$b_n(x_{2n+1}) \frac{y_{2n+2} - y_{2n}}{\alpha_n y_{2n+2} + \beta_n} \frac{y_{2n+1} - y_{2n}}{\alpha_n y_{2n+1} + \beta_n} + c_n(x_{2n+1}) \left[\frac{y_{2n+2} - y_{2n}}{\alpha_n y_{2n+2} + \beta_n} + \frac{y_{2n+1} - y_{2n}}{\alpha_n y_{2n+1} + \beta_n} \right] +$$

$$d_n(x_{2n+1}), \text{ or}$$

$$a_n(x_{2n+1})(\alpha_n y_{2n} + \beta_n) = b_n(x_{2n+1})(y_{2n+2} - y_{2n})(y_{2n+1} - y_{2n})$$

$$+ c_n(x_{2n+1})[(y_{2n+2} - y_{2n})(\alpha_n y_{2n+1} + \beta_n) + (y_{2n+1} - y_{2n})(\alpha_n y_{2n+2} + \beta_n)]$$

$$+ d_n(x_{2n+1})(\alpha_n y_{2n+1} + \beta_n)(\alpha_n y_{2n+2} + \beta_n). \quad (19)$$

which shows how to extract α_n and β_n from a_n, \dots at x_{2n} and x_{2n+1} .

Remark also that at $x = x_{2n-1}$, knowing that $f_n(\psi(x_{2n-1})) = f_n(y_{2n}) = 0$, (17) yields

$$\left[\frac{a_n(x_{2n-1})}{y_{2n} - y_{2n-1}} + c_n(x_{2n-1}) \right] f_n(y_{2n-1}) + d_n(x_{2n-1}) = 0. \quad (20)$$

And here is how the **Riccati** form is well suited to continued fraction progression:

4.1. Theorem.

If f_n satisfies the Riccati equation (17) with rational coefficients a_n, b_n, c_n , and d_n , and if $f_n(x) = \frac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$, then f_{n+1} satisfies an equation of same complexity (degree of the rational functions) of its coefficients.

(Actually, the degrees of a_n , etc. will at most exceed the degrees at $n = 0$ by 3 units).

Proof. Let us start with (17) at $n = 0$ with polynomial coefficients a_0, b_0, c_0 , and d_0 . Suppose that, at the n^{th} step, a_n , etc. are polynomials with b_n and d_n containing the factors X_2 and $x - x_{2n-1}$, and c_n containing the factor X_2 (from (3a) and (4), X_2 is a polynomial of degree ≤ 2).

Of course, if the initial coefficients a_0 , etc. do not contain such factors, we may have to multiply the four coefficients of (17) at $n = 0$ by one or several factors of $(x - x_{-1})X_2(x)$, that's why the degrees are liable to have to be augmented by up to 3 units, but this operation has to be done only at $n = 0$.

We suppose therefore that $b_n(x) = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)$, $c_n(x) = X_2(x)\tilde{c}_n(x)$, and $d_n(x) = (x - x_{2n-1})X_2(x)\tilde{d}_n(x)$ in (17), where $\tilde{b}_n, \tilde{c}_n,$

and \tilde{d}_n are polynomials, so that (17) is now

$$a_n(x) \frac{f_n(\psi(x)) - f_n(\varphi(x))}{\psi(x) - \varphi(x)} = (x - x_{2n-1})X_2(x)\tilde{b}_n(x)f_n(\varphi(x))f_n(\psi(x)) \\ + X_2(x)\tilde{c}_n(x)(f_n(\varphi(x)) + f_n(\psi(x))) + (x - x_{2n-1})X_2(x)\tilde{d}_n(x), \quad (21)$$

in which we enter $f_n(x) = \frac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$.

After multiplication by $(\alpha_n \varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi))(\alpha_n \psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi))$:

$$a_n \frac{(\psi - y_{2n})[\alpha_n \varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)] - (\varphi - y_{2n})[\alpha_n \psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]}{\psi - \varphi} \\ = a_n \left[\alpha_n y_{2n} + \beta_n + \left(\varphi \psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n} y_{2n+1} \right) \frac{f_{n+1}(\psi) - f_{n+1}(\varphi)}{\psi - \varphi} \right. \\ \left. + \frac{y_{2n+1} - y_{2n}}{2} (f_{n+1}(\varphi) + f_{n+1}(\psi)) \right] \\ = (x - x_{2n-1})X_2 \tilde{b}_n(\varphi - y_{2n})(\psi - y_{2n}) \\ + X_2 \tilde{c}_n [(\psi - y_{2n})[\alpha_n \varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)] + (\varphi - y_{2n})[\alpha_n \psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]] \\ + (x - x_{2n-1})X_2 \tilde{d}_n [\alpha_n \varphi + \beta_n - (\varphi - y_{2n+1})f_{n+1}(\varphi)][\alpha_n \psi + \beta_n - (\psi - y_{2n+1})f_{n+1}(\psi)]$$

which is the Riccati equation for f_{n+1} , $\hat{a}_{n+1} \frac{f_{n+1}(\psi) - f_{n+1}(\varphi)}{\psi - \varphi} = \hat{b}_{n+1} f_{n+1}(\varphi) f_{n+1}(\psi) + \hat{c}_{n+1} (f_{n+1}(\varphi) + f_{n+1}(\psi)) + \hat{d}_{n+1}$, where \hat{a}_{n+1} etc. are symmetric functions of φ and ψ , so are rational functions thanks to (3b):

$$\hat{a}_{n+1} = \left(\varphi \psi - \frac{(y_{2n} + y_{2n+1})(\varphi + \psi)}{2} + y_{2n} y_{2n+1} \right) a_n + \frac{y_{2n+1} - y_{2n}}{2} (\psi - \varphi)^2 X_2 \tilde{c}_n \\ + \frac{\alpha_n y_{2n+1} + \beta_n}{2} (\psi - \varphi)^2 (x - x_{2n-1}) X_2 \tilde{d}_n = \frac{\left(X_0 + \frac{(y_{2n} + y_{2n+1})X_1}{2} + y_{2n} y_{2n+1} X_2 \right)}{X_2} a_n \\ + \frac{[y_{2n+1} - y_{2n}]\tilde{c}_n + (\alpha_n y_{2n+1} + \beta_n)(x - x_{2n-1})\tilde{d}_n}{2} \frac{X_1^2 - 4X_0 X_2}{X_2} \quad (22a)$$

$$\hat{b}_{n+1} = (x - x_{2n-1})X_2 \tilde{d}_n(\varphi - y_{2n+1})(\psi - y_{2n+1}) = (x - x_{2n-1})\tilde{d}_n (X_0 + y_{2n+1}X_1 + y_{2n+1}^2 X_2) \quad (22b)$$

$$\begin{aligned}
\hat{c}_{n+1} &= -(y_{2n+1}-y_{2n})a_n - \left(\varphi\psi - \frac{(y_{2n}+y_{2n+1})(\varphi+\psi)}{2} + y_{2n}y_{2n+1} \right) X_2 \tilde{c}_n \\
&- \left(\alpha_n \left(\varphi\psi - y_{2n+1} \frac{\varphi+\psi}{2} \right) + \beta_n \left(\frac{\varphi+\psi}{2} - y_{2n+1} \right) \right) (x-x_{2n-1}) X_2 \tilde{d}_n \\
&= -(y_{2n+1}-y_{2n})a_n - \left(X_0 + \frac{(y_{2n}+y_{2n+1})X_1}{2} + y_{2n}y_{2n+1}X_2 \right) \tilde{c}_n \\
&- \left(\alpha_n \left(X_0 + y_{2n+1} \frac{X_1}{2} \right) - \beta_n \left(\frac{X_1}{2} + y_{2n+1}X_2 \right) \right) (x-x_{2n-1}) \tilde{d}_n
\end{aligned} \tag{22c}$$

$$\begin{aligned}
\hat{d}_{n+1} &= -(\alpha_n y_{2n} + \beta_n) a_n + (x-x_{2n-1}) X_2 \tilde{b}_n (\varphi-y_{2n})(\psi-y_{2n}) \\
&\quad + X_2 \tilde{c}_n [(\psi-y_{2n})(\alpha_n \varphi + \beta_n) + (\varphi-y_{2n})(\alpha_n \psi + \beta_n)] \\
&\quad + (x-x_{2n-1}) X_2 \tilde{d}_n (\alpha_n \varphi + \beta_n)(\alpha_n \psi + \beta_n) \\
&= -(\alpha_n y_{2n} + \beta_n) a_n + (x-x_{2n-1}) \tilde{b}_n (X_0 + y_{2n} X_1 + y_{2n}^2 X_2) \\
&+ \tilde{c}_n [\alpha_n (2X_0 + y_{2n} X_1) - \beta_n (X_1 + 2y_{2n} X_2)] + (x-x_{2n-1}) \tilde{d}_n (\alpha_n^2 X_0 - \alpha_n \beta_n X_1 + \beta_n^2 X_2)
\end{aligned} \tag{22d}$$

The first coefficient \hat{a}_{n+1} is not a polynomial, but a rational function of denominator X_2 . We recover polynomials by multiplying the four coefficients \hat{a}_{n+1} , \hat{b}_{n+1} , \hat{c}_{n+1} , and \hat{d}_{n+1} by X_2 . Moreover, this already restores the factor X_2 in $X_2 \hat{b}_{n+1}$, $X_2 \hat{c}_{n+1}$, and $X_2 \hat{d}_{n+1}$!

However, the degrees of the new coefficients are higher than before. The problem is settled by seeing that the four polynomials $X_2 \hat{a}_{n+1}$, \hat{b}_{n+1} , \hat{c}_{n+1} , and \hat{d}_{n+1} vanish at $x = x_{2n-1}$ and at $x = x_{2n}$. Then, we will simply divide the four of them by $(x-x_{2n-1})(x-x_{2n})$.

- (1) The most obvious case is (22b): $\hat{b}_{n+1}(x) = (x-x_{2n-1})\tilde{d}_n(x)(X_0(x) + y_{2n+1}X_1(x) + y_{2n+1}^2X_2(x)) = (x-x_{2n-1})\tilde{d}_n(x)F(x, y_{2n+1}) = (x-x_{2n-1})\tilde{d}_n(x)Y_2(y_{2n+1})(x-x_{2n})(x-x_{2n+1})$, shows indeed the factors $x-x_{2n-1}$ and $x-x_{2n}$, as well as $x-x_{2n+1}$, so that
$$\tilde{b}_{n+1}(x) = \frac{\hat{b}_{n+1}(x)}{(x-x_{2n-1})(x-x_{2n})(x-x_{2n+1})} = Y_2(y_{2n+1})\tilde{d}_n(x).$$
- (2) Next, from (22d), $\hat{d}_{n+1}(x_{2n-1}) = (\alpha_n y_{2n} + \beta_n)[-a_n(x_{2n-1}) - (y_{2n} - y_{2n-1})c_n(x_{2n-1})] = 0$ from (20), knowing that $d_n(x_{2n-1}) = 0$,
$$\hat{d}_{n+1}(x_{2n}) = (\alpha_n y_{2n} + \beta_n)[-a_n(x_{2n}) + (y_{2n+1} - y_{2n})c_n(x_{2n}) + (\alpha_n y_{2n+1} + \beta_n)d_n(x_{2n})] =$$

$$(\alpha_n y_{2n} + \beta_n)(y_{2n+1} - y_{2n}) \left[-\frac{a_n(x_{2n})}{y_{2n+1} - y_{2n}} + c_n(x_{2n}) + \frac{d_n(x_{2n})}{f_n(y_{2n+1})} \right] =$$

0, as the last factor is the Riccati equation (17) at $x = x_{2n}$ divided by $f_n(y_{2n+1})$ (see also (18)).

$$\hat{d}_{n+1}(x_{2n+1}) = -(\alpha_n y_{2n} + \beta_n) a_n(x_{2n+1}) + b_n(x_{2n+1})(y_{2n+1} - y_{2n})(y_{2n+2} - y_{2n}) + c_n(x_{2n+1})[(y_{2n+2} - y_{2n})(\alpha_n y_{2n+1} + \beta_n) + (y_{2n+1} - y_{2n})(\alpha_n y_{2n+2} + \beta_n)] + d_n(x_{2n+1})(\alpha_n y_{2n+1} + \beta_n)(\alpha_n y_{2n+2} + \beta_n) = 0, \text{ from (19).}$$

(3) In order to show that \hat{a}_{n+1} and \hat{c}_{n+1} both vanish at $x = x_{2n-1}$

$$\text{and } x_{2n}, \text{ we consider } \frac{a_n}{\psi - \varphi} \pm c_n: \frac{\hat{a}_{n+1}(x)}{\psi(x) - \varphi(x)} + \hat{c}_{n+1}(x) =$$

$$(\varphi(x) - y_{2n+1}) \left[(\psi(x) - y_{2n}) \left(\frac{a_n(x)}{\psi(x) - \varphi(x)} - c_n(x) \right) - (\alpha_n \psi(x) + \beta_n) d_n(x) \right]$$

vanishes at $x = x_{2n}$, as the big factor is $a_n(x_{2n}) - (y_{2n+1} - y_{2n})c_n(x_{2n}) - (\alpha_n y_{2n+1} + \beta_n)d_n(x_{2n}) = 0$ from (18). At $x = x_{2n-1}$, we already encountered the condition $a_n(x_{2n-1}) + (y_{2n} - y_{2n-1})c_n(x_{2n-1}) = 0$ from (20) and $d_n(x_{2n-1}) = 0$.

The obvious vanishing of the first factor at $x = x_{2n+1}$ will allow the same condition at x_{2n+1} : $a_{n+1}(x_{2n+1}) + (y_{2n+2} - y_{2n+1})c_{n+1}(x_{2n+1}) = 0$, and this will imply $f_{n+1}(y_{2n+2}) = 0$ at the next step.

$$(4) \frac{\hat{a}_{n+1}(x)}{\psi(x) - \varphi(x)} - \hat{c}_{n+1}(x) =$$

$$(\psi(x) - y_{2n+1}) \left[(\varphi(x) - y_{2n}) \left(\frac{a_n(x)}{\psi(x) - \varphi(x)} + c_n(x) \right) + (\alpha_n \varphi(x) + \beta_n) d_n(x) \right]$$

obviously vanishes now at $x = x_{2n}$; at $x = x_{2n-1}$, the big factor is $-a_n(x_{2n-1}) - (y_{2n} - y_{2n-1})c_n(x_{2n-1}) = 0$ as already encountered (in (20), together with $d_n(x_{2n-1}) = 0$).

$$\text{We proceed therefore with } a_{n+1}(x) = \frac{X_2(x)\hat{a}_{n+1}(x)}{(x - x_{2n-1})(x - x_{2n})}, b_{n+1}(x) = \frac{X_2(x)\hat{b}_{n+1}(x)}{(x - x_{2n-1})(x - x_{2n})}, c_{n+1}(x) = \frac{X_2(x)\hat{c}_{n+1}(x)}{(x - x_{2n-1})(x - x_{2n})}, d_{n+1}(x) = \frac{X_2(x)\hat{d}_{n+1}(x)}{(x - x_{2n-1})(x - x_{2n})}.$$

□

A very interesting identity is $\hat{a}_{n+1}^2 - (\psi - \varphi)^2 \hat{c}_{n+1}^2 = (\varphi - y_{2n})(\psi - y_{2n})(\varphi - y_{2n+1})(\psi - y_{2n+1})(a_n^2 - (\psi - \varphi)^2 c_n^2) - (\psi - \varphi)^2 \hat{b}_{n+1}[\hat{d}_{n+1} - (\varphi - y_{2n})(\psi - y_{2n})b_n]$, or $\hat{a}_{n+1}^2 - (\psi - \varphi)^2 (\hat{c}_{n+1}^2 - \hat{b}_{n+1}\hat{d}_{n+1}) = (\varphi - y_{2n})(\psi - y_{2n})(\varphi - y_{2n+1})(\psi - y_{2n+1})(a_n^2 - (\psi - \varphi)^2 (c_n^2 - b_n d_n))$

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)} P(x) - a_n^2(x) = C_n \frac{x - x_{2n-1}}{x - x_{-1}} \left[\frac{c_0^2(x) - b_0(x)d_0(x)}{X_2^2(x)} P(x) - a_0^2(x) \right], \quad (23)$$

where $C_n = \frac{Y_2(y_{2n-1})Y_2(y_{2n-3}) \cdots Y_2(y_1)}{Y_2(y_{2n-2})Y_2(y_{2n-4}) \cdots Y_2(y_0)}$, and P from (7).

5. CLASSICAL ELLIPTIC BIORTHOGONAL RATIONAL FUNCTIONS.

The smallest possible degree for a_n , b_n , c_n , and d_n according to the theory above, happens to be three. Then, $b_n(x) = \xi_n(x - x_{2n-1})X_2(x)$ and $d_n(x) = \zeta_n(x - x_{2n-1})X_2(x)$ are already known, up to a single constant each. The equations (22a) and (22c) should care for the evolution of a_n and c_n with n , but (23) yields directly the values of $a_n^2(x)$ at the four zeros of $P = X_1^2 - 4X_0X_2$. Equation (22a) also provides a simple relation between $\hat{a}_{n+1}(x)$ (therefore, $a_{n+1}(x)$), and $a_n(x)$ at each of these four zeros, which is enough for a full determination of the third degree polynomials a_n .

I hope to recover in this way the results of Spiridonov and Zhedanov [28, 29], obtained through elliptic hypergeometric identities [23, 24, 25, 26, 27].

Note however that Theorem 4.1 also makes room for coefficients of degree > 3 , therefore to new families of biorthogonal functions.

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