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Rational interpolation to solutions of Riccati difference equations on elliptic lattices.

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The present file is http://www.math.ucl.ac.be/membres/magnus/num3/ellric07.pdf

Abstract. An elliptic lattice, or grid, $\{x_0, x_1, \ldots\}$ is built with the help of a biquadratic curve $F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j$ by the following rules:

(1) let y_n and y_{n+1} be the two $y-$roots of $F(x_n, y) = 0$,
(2) then, x_{n+1} is found as the remaining $x-$root of $F(x, y_{n+1}) = 0$.

There is also a direct symmetric biquadratic relation $E(x_n, x_{n+1}) = 0$, see V. P. Spiridonov and A. S. Zhedanov: Elliptic grids, rational functions, and the Padé interpolation, *The Ramanujan Journal* **13**, Numbers 1-3, June 2007, p. 285–310.

Numerators and denominators of rational interpolants on such lattices satisfy interesting difference equations when the interpolated function f itself satisfies a Riccati difference equation on the same lattice:

$$a(x_n)\frac{f(y_{n+1}) - f(y_n)}{y_{n+1} - y_n} = b(x_n)f(y_n)f(y_{n+1}) + c(x_n)(f(y_n) + f(y_{n+1})) + d(x_n),$$

where a, b, c, and d are polynomials.

Contents

1. **Difference equations and lattices.**

Simplest difference equations relate two values of the unknown fuction f: say, $f(\varphi(x))$ and $f(\psi(x))$.

Most instances [29] are $(\varphi(x), \psi(x)) = (x, x+h)$, or the more symmetric $(x-h/2, x+h/2)$, or also (x, qx) in $q-$difference equations [14]. Recently, more complicated forms $(r(x)-\sqrt{s(x)}, r(x)+\sqrt{s(x)})$ have appeared [5,8,22,23,30,31], where r and s are rational functions.

This latter trend will be examined here: we need, for each x, two values $f(\varphi(x))$ and $f(\psi(x))$ for f.
A first-order difference equation is $\mathcal{F}\left(x, f(\varphi(x)), f(\psi(x))\right) = 0$, or $f(\varphi(x)) - f(\psi(x)) = \mathcal{G}\left(x, f(\varphi(x)), f(\psi(x))\right)$ if we want to emphasize the difference of f. There is of course some freedom in this latter writing. Only symmetric forms in φ and ψ will be considered here:

$$(\mathcal{D}f)(x) = \mathscr{F}\left(x, f(\varphi(x)), f(\psi(x))\right), \tag{1}$$

where $\mathcal{D}$ is the divided difference operator

$$(\mathcal{D}f)(x) = \frac{f(\psi(x) - f(\varphi(x)}{\psi(x) - \varphi(x)}, \tag{2}$$

and where $\mathscr{F}$ is a symmetric function of its two last arguments.

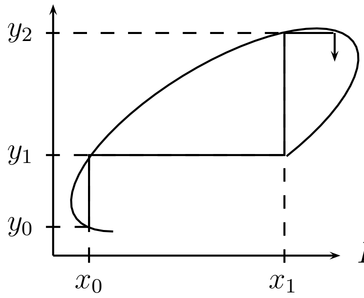
For instance, a linear difference equation of first order may be written as

$$a(x)f(\varphi(x)) + b(x)f(\psi(x)) + c(x) = 0,$$

as well as

$$\alpha(x)(\mathcal{D}f)(x) = \beta(x)[f(\varphi(x)) + f(\psi(x))] + \gamma(x),$$

with $\alpha(x) = [b(x)-a(x)][\psi(x)-\varphi(x)]/2$, $\beta(x) = -[a(x)+b(x)]/2$, and $\gamma(x) = -c(x)$.



The simplest choice for φ and ψ is to take the two determinations of an algebraic function of degree 2, i.e., the two $y-$roots of

$$F(x,y) = X_0(x) + X_1(x)y + X_2(x)y^2 = 0, \tag{3a}$$

where X_0, X_1, and X_2 are rational functions.

But difference equations must allow the recovery of f on a whole set of points! An initial-value problem for a first order difference equation starts with a value for $f(y_0)$ at $x = x_0$, where y_0 is one root of (3a) at $x = x_0$. The difference equation at $x = x_0$ relates then $f(y_0)$ to $f(y_1)$, where y_1 is the second root of (3a) at x_0. We need x_1 such that y_1 is one of the two roots of (3a) at x_1, so for one of the roots of $F(x, y_1) = 0$ which is not x_0. Here again, the simplest case is when F is of degree 2 in x:

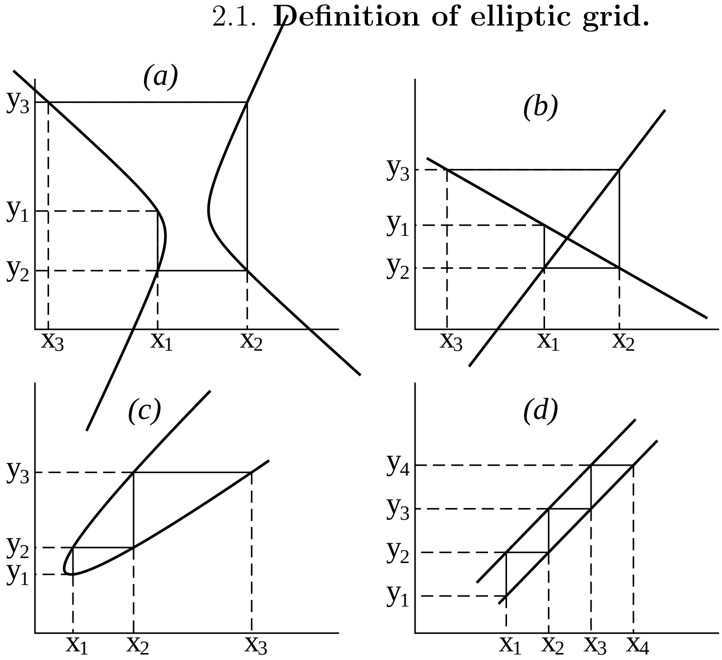
$$F(x,y) = Y_0(y) + Y_1(y)x + Y_2(y)x^2 = 0. \tag{3b}$$

Both forms (3a) and (3b) hold simultaneously when F is ***biquadratic***:

$$F(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}x^i y^j. \tag{4}$$

2. **Elliptic grid, or lattice.**

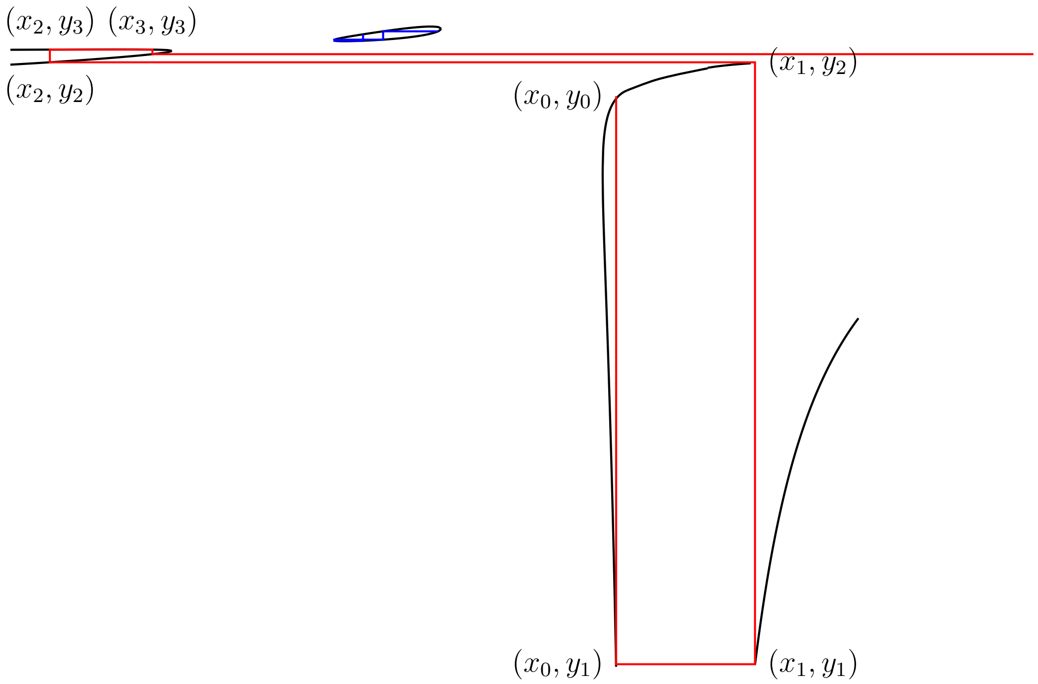
2.1. **Definition of elliptic grid.**



Various forms of the curve $F(x,y)=0$ are degenerate parabolas (two parallel lines) or hyperbolas (two lines), or generic conics, or a full biquadratic curve

$$y = \frac{-X_1(x) \pm \sqrt{X_1^2(x) - 4X_0(x)X_2(x)}}{2X_2(x)}.$$

To one $x = x_n$ correspond the two ordinates y_n and y_{n+1}.
One has $y_n + y_{n+1} = -\dfrac{X_1(x_n)}{X_2(x_n)}$, and $y_n y_{n+1} = \dfrac{X_0(x_n)}{X_2(x_n)}$.



Also, to one ordinate $y=y_n$ correspond the two abscissae x_n and x_{n-1}, and we now have

$$x_n+x_{n-1}=-\frac{Y_1(y_n)}{Y_2(y_n)},\qquad x_nx_{n-1}=\frac{Y_0(y_n)}{Y_2(y_n)}. \tag{5}$$

A relation involving only x_n and x_{n-1} is obtained by the elimination of y_n through the resultant of the two polynomials in y_n $P_1(y_n)=(x_n+x_{n-1})Y_2(y_n)+Y_1(y_n)$ and $P_2(y_n)=x_nx_{n-1}Y_2(y_n)-Y_0(y_n)$.

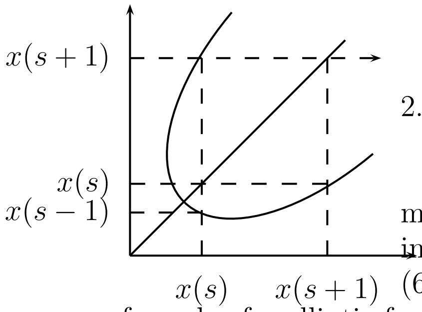
The form of this resultant is most easily found through interpolation at the two zeros u and v of Y_2: let $Y_2(t)=\alpha(t-u)(t-v)$, $Y_0(t)=\beta(t-u)(t-v)+\beta'(t-u)+\beta''$, $Y_1(t)=\gamma(t-u)(t-v)+\gamma'(t-u)+\gamma''$, then the resultant is $R=\begin{vmatrix}(x_n+x_{n-1})\alpha+\gamma & \gamma' & \gamma'' & 0\\ 0 & (x_n+x_{n-1})\alpha+\gamma & \gamma' & \gamma''\\ x_nx_{n-1}\alpha-\beta & -\beta' & -\beta'' & 0\\ 0 & x_nx_{n-1}\alpha-\beta & -\beta' & -\beta''\end{vmatrix}$, which is clearly a polynomial of degree 2 in x_n+x_{n-1} and x_nx_{n-1}, so a symmetric biquadratic relation:

2.1.1. *Definition. An elliptic lattice, or grid, is a sequence satisfying.*

$$E(x_n,x_{n-1})=d_{0,0}+d_{0,1}(x_n+x_{n-1})+d_{0,2}(x_n^2+x_{n-1}^2)+d_{1,1}x_nx_{n-1}+d_{1,2}x_nx_{n-1}(x_n+x_{n-1})+d_{2,2}x_n^2x_{n-1}^2=0. \tag{6}$$

We also get a linear recurrence relation between three x's by adding (**??**) for s and $s+1$: $x(s+1)+2x(s)+x(s-1)=\dfrac{\alpha(y(s+1)+y(s))-2\delta}{\gamma}=\dfrac{\alpha^2x(s)+\alpha\beta-2\delta}{\gamma}$, so

$$x(s+1)+\frac{2\gamma-\alpha^2}{\gamma}x(s)+x(s-1)=\frac{\alpha\beta-2\delta}{\gamma} \tag{7}$$



2.1.2. *Why elliptic?*

The learned answer is that a biquadratic curve (4) or (6) is normally of genus one, and receives therefore a parametric representation involving elliptic functions [42]. It is also known that formulas like (6) appear in Euler's pionneering work on what is now called addition formulas for elliptic functions, that's why the authors of [42] use the letter E in (6).

Here is an explanation based on special Padé approximations, periodic continued fractions, and orthogonal polynomials on several intervals:

Let S be a polynomial of degree 2 $S(z) = \gamma + \delta(z - z_0) + \xi(z - z_0)^2$, and we consider the root of

$$\zeta_0(z - z_0)(z - x_0)f^2(z) - 2S(z)f(z) + \zeta_1(z - z_0)(z - x_1) = 0 \tag{8}$$

which is regular at z_0. It is also

$$f(z) = \frac{S(z) - \sqrt{P(z)}}{\zeta_0(z - z_0)(z - x_0)},$$

where the choice of the square root of $P(z) = S^2(z) - \zeta_0\zeta_1(z - z_0)^2(z - x_0)(z - x_1) = c(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ in a neighbourhood of z_0 is such that the value of this square root is $S(z_0) = \gamma$. Actually, this regular root even vanishes at z_0, and can be represented by the continued fraction

$$f(z) = \cfrac{z - z_0}{\alpha_0 z + \beta_0 - \cfrac{(z - z_0)^2}{\alpha_1 z + \beta_1 - \cfrac{(z - z_0)^2}{\alpha_2 z + \beta_2 - \cdots}}}, \tag{9}$$

$$\text{or } f_n(z) = \frac{z - z_0}{\alpha_n z + \beta_n - (z - z_0)f_{n+1}(z)}, \quad n = 0, 1, \ldots \tag{10}$$

with $f_0 = f$. Remark that $\alpha_n z + \beta_n$ is the Taylor approximation of degree 1 to $(z - z_0)/f_n(z)$. We can therefore recover α_n and β_n from the behaviour of f_n near z_0.

The form of f is kept in all the f_n's (basically from Perron [35, § 60, eq. (5)-(14)]):

$$\zeta_n(z - z_0)(z - x_n)f_n^2(z) - 2S_n(z)f_n(x) + \zeta_{n+1}(z - z_0)(z - x_{n+1}) = 0, \tag{11}$$

and we have the

2.1.3. *Proposition. The continued fraction expansion* (9) *of the quadratic function f defined by* (8) *involves a sequence of quadratic functions defined by* (11). *The related sequence $\{x_n\}$ is an elliptic sequence.*

We first show that the quadratic equation (11) holds for all n. Indeed, if f_n is the root of (11) such that $f_n(z) = \dfrac{S_n(z) - \sqrt{P(z)}}{\zeta_n(z - z_0)(z - x_n)}$, with $S_n(z)^2 - \zeta_n\zeta_{n+1}(z - z_0)^2(z - x_n)(z - x_{n+1}) =$

$P(z)$, we have

$$f_n(z) = \frac{S_n^2(z) - P(z) = \zeta_n\zeta_{n+1}(z-z_0)^2(z-x_n)(z-x_{n+1})}{\zeta_n(z-z_0)(z-x_n)[S_n(z)+\sqrt{P(z)}]} = \cfrac{(z-z_0)}{\cfrac{S_n(z)+\sqrt{P(z)}}{\zeta_{n+1}(z-x_{n+1})}}$$

$$= \cfrac{z-z_0}{\alpha_n z+\beta_n - \cfrac{\zeta_{n+1}(\alpha_n z+\beta_n)(z-x_{n+1}) - S_n(z) - \sqrt{P(z)}}{\zeta_{n+1}(z-x_{n+1})}}$$

showing that $f_{n+1}(z) = \dfrac{S_{n+1}(z)-\sqrt{P(z)}}{\zeta_{n+1}(z-z_0)(z-x_{n+1})}$, as expected, where $S_{n+1}(z) = \zeta_{n+1}(\alpha_n z + \beta_n)(z-x_{n+1}) - S_n(z)$, $\alpha_n z+\beta_n$ being the Taylor approximant of degree 1 to $[S_n(z)+\sqrt{P(z)}]/[\zeta_{n+1}(z-x_{n+1})]$ about $z=z_0$. This way to build S_{n+1} ensures that the fourth-degree polynomial $S_{n+1}^2 - P$ has factors $(z-z_0)^2$ and $\zeta_{n+1}(z-x_{n+1})$. Let us call the remaining factor $\zeta_{n+2}(z-x_{n+2})$, and this completes the definition of f_{n+1}.

And here is how the present x_n and x_{n+1} actually satisfy the elliptic lattice equation (6):

We expand $S_n^2(z)-P(z) = \zeta_n\zeta_{n+1}(z-z_0)^2(z-x_n)(z-x_{n+1})$ with $S_n(z) = \gamma+\delta(z-z_0)+\xi_n(z-z_0)^2$ and $P(z) = \gamma^2+2\gamma\delta(z-z_0)+P''(z_0)(z-z_0)^2/2+P'''(z_0)(z-z_0)^3/6+P''''(z_0)(z-z_0)^4/24$ (so that the expansion of the square root of P starts indeed with $\gamma+\delta(z-z_0)$).

$$2\gamma\xi_n + \delta^2 - P''(z_0)/2 = \zeta_n\zeta_{n+1}(z_0-x_n)(z_0-x_{n+1}), \tag{12a}$$

$$2\delta\xi_n - P'''(z_0)/6 = \zeta_n\zeta_{n+1}(2z_0-x_n-x_{n+1}), \tag{12b}$$

$$\xi_n^2 - P''''(z_0)/24 = \zeta_n\zeta_{n+1}. \tag{12c}$$

The last equation yields $\zeta_n\zeta_{n+1}$ as a polynomial of degree 2 in ξ_n, therefore the two first equations give the sum and the product of x_n and x_{n+1} as rational functions of degree 2 of an intermediate parameter, and this is the structure of (3a)-(3b)-(4): z_0-x_n and z_0-x_{n+1} are the two roots of

$$(z_0-x)^2 - \frac{2\delta\xi_n - P'''(z_0)/6}{\xi_n^2-P''''(z_0)/24}(z_0-x) + \frac{2\gamma\xi_n+\delta^2-P''(z_0)/2}{\xi_n^2-P''''(z_0)/24} = 0,$$

so

$$F(x,y) = [y^2-P''''(z_0)/24](z_0-x)^2 - [2\delta y - P'''(z_0)/6](z_0-x) + 2\gamma y + \delta^2 - P''(z_0)/2 = 0.$$

So, the continued fraction expansion (9)-(11) of a quadratic algebraic function leads to an elliptic lattice.

Conversely, can we find P, z_0,etc. from a given elliptic lattice (6)? From a solution $x_n = x_{n+1} = z_0$ of (12a)-(12c) (when $\xi_n=\infty$), z_0 is one of the four roots of

$$E(z_0,z_0) = d_{0,0} + 2d_{0,1}z_0 + (2d_{0,2}+d_{1,1})z_0^2 + 2d_{1,2}z_0^3 + d_{2,2}z_0^4 = 0.$$

Moreover, (6) yields $y = x_{n\pm1}$ as a quadratic function of $x = x_n$ as

$$y = \frac{-d_{1,2}x^2 - d_{1,1}x - d_{0,1} \pm \sqrt{(d_{1,2}x^2+d_{1,1}x+d_{0,1})^2 - 4(d_{2,2}x^2+d_{1,2}x+d_{0,2})(d_{0,2}x^2+d_{0,1}x+d_{0,0})}}{2(d_{2,2}x^2+d_{1,2}x+d_{0,2})}.$$

Let us look now at the $(S_n, P, \zeta_n(z-x_n))$ construction as a way to find x_{n+1} from x_n: if x_n is known, S_n^2-P must vanish at $x=x_n \Rightarrow S_n(x_n) = \gamma+\delta(x_n-z_0)+\xi_n(x_n-z_0)^2 = \pm\sqrt{P(x_n)}$, giving two possible values for ξ_n. Then, as already seen, we factor S_n^2-P as a constant times $(z-z_0)^2(z-x_n)$ times a last factor which must be a constant times $z-x_{n+1}$, yielding for x_{n+1} an expression containing $\sqrt{P(x_n)}$, so that we obtain P from

$$(d_{1,2}x^2+d_{1,1}x+d_{0,1})^2-4(d_{2,2}x^2+d_{1,2}x+d_{0,2})(d_{0,2}x^2+d_{0,1}x+d_{0,0}) = \text{ const. } P(x).$$

This allows to interpret any elliptic lattice in terms of the continued fraction expansion of a quadratic algebraic function.

2.1.4. *The elliptic functions, at last.* Let

$$\frac{A_n(z)}{B_n(z)} = \cfrac{z-z_0}{\alpha_0 z+\beta_0 - \cfrac{(z-z_0)^2}{\alpha_1 z+\beta_1 - \cfrac{(z-z_0)^2}{\alpha_2 z+\beta_2 - \cfrac{\ddots}{\alpha_{n-1}z+\beta_{n-1}}}}}, \tag{13}$$

with $A_0=0$, $A_1(z)=z-z_0$, $A_{k+1}(z)=(\alpha_k z+\beta_k)A_k(z)-(z-z_0)^2A_{k-1}(z)$, $B_0=1$, $B_1(z)=\alpha_0 z+\beta_0$, $B_{k+1}(z)=(\alpha_k z+\beta_k)B_k(z)-(z-z_0)^2B_{k-1}(z)$, or also $B_{-1}=0$, $A_{-1}(z)=-1/(z-z_0)$.

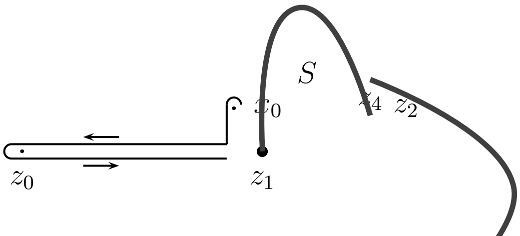
Ratios $g_{k+1} = \dfrac{\lambda B_{k+1}-A_{k+1}}{\lambda B_k - A_k}$ satisfy $g_{k+1}=\alpha_k z+\beta_k-(z-z_0)^2/g_k$, or $\dfrac{g_k}{z-z_0} = \dfrac{z-z_0}{\alpha_k z+\beta_k-g_{k+1}}$, i.e., the recurrence (10) of the f_k's, which correspond to $\lambda=f$ (as $g_0=-\lambda/A_{-1}=(z-z_0)\lambda$, so $g_0(z)/(z-z_0)=f_0(z)=f(z)$ if $\lambda=f$).

The quadratic function $B_nf-A_n=(z-z_0)^nf_0f_1\cdots f_n$ has an extremely peculiar set of zeros and poles: let f^{conj} be the conjugate function to f, i.e., the quadratic function $\dfrac{S+\sqrt{P}}{\zeta_0(z-z_0)(z-x_0)}$ where the sign of the square root of P has been changed, an elementary way to deal with two-sheeted Riemann surfaces. Then, the set of zeros and poles of B_nf-A_n is a part of the set for $(B_nf-A_n)(B_nf^{\text{conj}}-A_n)$

$$\begin{aligned}(B_nf-A_n)(B_nf^{\text{conj}}-A_n) &= (z-z_0)^{2n}f_0\cdots f_nf_0^{\text{conj}}\cdots f_n^{\text{conj}}\\ &= (z-z_0)^{-2}\frac{S_0-\sqrt{P}}{\zeta_0(z-x_0)}\cdots\frac{S_n-\sqrt{P}}{\zeta_n(z-x_n)}\frac{S_0+\sqrt{P}}{\zeta_0(z-x_0)}\cdots\frac{S_0+\sqrt{P}}{\zeta_n(z-x_n)}\\ &= \frac{(z-z_0)^{2n}(z-x_{n+1})}{\zeta_0\zeta_n(z-x_0)},\end{aligned}$$

a very limited set! Actually, most zeros and poles are concentrated at z_0 and ∞: B_nf-A_n has a zero of order $2n+1$ at z_0 (Padé property), $B_nf^{\text{conj}}-A_n$ a simple pole; both functions have a pole of order n at ∞.

2.1.5. *What is (are) the period(s)?*



Consider the integral of $\dfrac{dt}{\pi i\sqrt{P(t)}}\log\left[\dfrac{B_{n-1}(t)f(t)-A_{n-1}(t)}{B_{n-1}(t)f^{\text{conj}}(t)-A_{n-1}(t)}\right]$ on a big contour, result is 0, as everything is regular when t is large; and we shrink the contour about z_0 and the zero and pole x_0 and x_n:

$$0 = 2n\int_{z_0}^{z_1}\frac{dt}{\sqrt{P(t)}} + \int_{z_1}^{x_0}\frac{dt}{\sqrt{P(t)}} \pm \int_{z_1}^{x_n}\frac{dt}{\sqrt{P(t)}} + \underbrace{2\sum_j N_j\int_{z_j}^{z_{j+1}}\frac{dt}{\sqrt{P(t)}}}_{\text{periods}} \tag{14}$$

It happens that, knowing n, P, and x_0, (14) allows to find the remaining unknowns, including the $\pm$ signs (***Jacobi problem***, [1, 3, 26, 32, 33, **?**]).

There is absolutely no need for n to be an integer in the description (14) of the Jacobi problem. To see how this x_n is a function of n, we …take the derivative of (14) with respect to n (!!):

$$h = \pm\frac{1}{\sqrt{P(x_n)}}\frac{dx_n}{dn},$$

where $h = -2\displaystyle\int_{z_0}^{z_1}\frac{dt}{\sqrt{P(t)}}$. We have a differential equation for x_n. An initial condition consists of x_0 ***and*** a sign (= a place on the ***Riemann surface*** of $\sqrt{P}$).

Generalization.

Hyperelliptic case, generalization of Padé approximation and continued fraction (recurrence relations) constructions: see [**?**, 4, 33, 44]. We then have a vector of length g (***genus***) $[x_n^{(1)},\dots,x_n^{(g)}]$ of unknowns which is a well defined function (***Jacobi-Abel function***) of the left-hand side $[nh_0,\dots,nh_{g-1}]$.

2.2. **Periodicity, theta functions.**

x_n is kept unchanged if $nh+h_0$ in (14) is augmented by integers times the integrals $2\omega_j = 2\displaystyle\int_{z_j}^{z_{j+1}}\frac{t^k}{\sqrt{P(t)}}\,dt$ (***periods***).

So, x_n is some periodic function (***elliptic function***) of $nh+h_0$. There are two zeros and two poles in a fundamental parallelogram of periods $(2\omega_1, 2\omega_2)$, they are given by $nh+h_0 = \pm\displaystyle\int_{z_1}^{0}\frac{dt}{\sqrt{P(t)}}$ and $\pm\displaystyle\int_{z_1}^{\infty}\frac{dt}{\sqrt{P(t)}}$, say, $\pm\zeta_0$ and $\pm\zeta_\infty$.

We expect x_n to involve standard functions of $nh+h_0\pm\zeta_0$ and $nh+h_0\pm\zeta_\infty$.

With the $p-$theta function

$$\theta(u;p) = \prod_{j=0}^{\infty}(1-p^j u)(1-p^{j+1}/u),$$

which vanishes at $\log u =$ all the integer multiples of $\log p$ plus all the integer multiples of $2\pi i$, we consider [37, § 2]

$$x_n = C\frac{\theta\left(\exp\left(i\pi\dfrac{nh+h_0-\zeta_0}{\omega_1}\right)\right)\theta\left(\exp\left(i\pi\dfrac{nh+h_0+\zeta_0}{\omega_1}\right)\right)}{\theta\left(\exp\left(i\pi\dfrac{nh+h_0-\zeta_\infty}{\omega_1}\right)\right)\theta\left(\exp\left(i\pi\dfrac{nh+h_0+\zeta_\infty}{\omega_1}\right)\right)}$$

see also [42, § 4], there is a simple relation between the $p-$theta function and the Jacobi theta functions.

Let $q = \exp(i\pi h/\omega_1)$, $q_0 = \exp(i\pi h_0/\omega_1)$, $\eta_0 = \exp(i\pi\zeta_0/\omega_1)$, $\eta_\infty = \exp(i\pi\zeta_\infty/\omega_1)$, so

$$x_n = C\frac{\theta(q^n q_0\eta_0)\theta(q^n q_0/\eta_0)}{\theta(q^n q_0\eta_\infty)\theta(q^n q_0/\eta_\infty)}$$

$x_n - x_m$, where n and m need not be integers, is another (much more interesting) elliptic function of n with the same poles, but with zeros such that $n = m$ is one of them. This leads to replace η_0 by $1/(q_0 q^m)$:

$$x_n - x_m = C\frac{\theta(q^{n-m})\theta(q^{n+m}q_0^2)}{\theta(q^n q_0\eta_\infty)\theta(q^n q_0/\eta_\infty)} \tag{15}$$

For the y's, one has

$$y_n - y_m = C'\frac{\theta(q^{n-m})\theta(q^{n+m}q_0'^2)}{\theta(q^n q_0'\eta_\infty')\theta(q^n q_0'/\eta_\infty')} \tag{16}$$

with the same θ function and the same q

2.3. **A special product.**

We will have to considerate the special algebraic function

$$\mathcal{X}_n(x) = \frac{(\psi(x)-y_1)(\psi(x)-y_3)\cdots(\psi(x)-y_{2n-1})}{(\varphi(x)-y_0)(\varphi(x)-y_2)\cdots(\varphi(x)-y_{2n-2})} \tag{17}$$

and its conjugate

$$\mathcal{X}_n^{\text{conj.}}(x) = \frac{(\varphi(x)-y_1)(\varphi(x)-y_3)\cdots(\varphi(x)-y_{2n-1})}{(\psi(x)-y_0)(\psi(x)-y_2)\cdots(\psi(x)-y_{2n-2})}$$

the product is $\dfrac{F(x,y_1)F(x,y_3)\cdots F(x,y_{2n-1})}{F(x,y_0)F(x,y_2)\cdots F(x,y_{2n-2})} = \dfrac{Y_2(y_1)Y_2(y_3)\cdots Y_2(y_{2n-1})}{Y_2(y_0)Y_2(y_2)\cdots Y_2(y_{2n-2})}\dfrac{x-x_{2n-1}}{x-x_{-1}}$.

The value of (17) is well defined when x is some x_m:

$$\mathcal{X}_n(x_m) = \frac{(y_{m+1}-y_1)(y_{m+1}-y_3)\cdots(y_{m+1}-y_{2n-1})}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n-2})}$$

Now, following (15), $y_r - y_s$ is a ratio of products with numerator $\theta(q^{r-s})\theta(q^{r+s}q_0'^2)$, so

$$\mathcal{X}_n(x_m) = \frac{\theta(q^m)\theta(q^{m+2}q_0'^2)\theta(q^{m-2})\theta(q^{m+4}q_0'^2)\cdots\theta(q^{m-2n+2})\theta(q^{m+2n}q_0'^2)}{\theta(q^m)\theta(q^m q_0'^2)\theta(q^{m-2})\theta(q^{m+2}q_0'^2)\cdots\theta(q^{m-2n+2})\theta(q^{m+2n-2}q_0'^2)}$$

and what remains is

$$\mathcal{X}_n(x_m) = \frac{\theta(q^{m+2n}q_0'^2)}{\theta(q^m q_0'^2)} \tag{18}$$

and

$$\mathcal{X}_n^{\text{conj}}(x_m) = \frac{\theta(q^{m-2n+1})}{\theta(q^{m+1})} = \frac{\theta(pq^{2n-1-m})}{\theta(pq^{-m-1})} \tag{19}$$

from the identities $\theta(pu) = \theta(1/u) = -\theta(u)/u$.

3. Elliptic Pearson's equation.

A famous theorem by Pearson relates the classical orthogonal polynomials to the differential equation $w' = rw$ satisfied by the weight function, where r is a rational function of degree $\leqslant 2$.

3.1. **Theorem.** *Let $\{(x(s_0+k), y(s_0+k))\}$ be an elliptic lattice built on the biquadratic curve* (3a)-(3b)-(4)*, with $s_0 \notin \mathbb{Z}$. If there are polynomials a and c, with $a(x(s_0)) - (y(s_0+1) - y(s_0))c(x(s_0)) = a(x(s_0+N)) - (y(s_0+N+1) - y(s_0+N))c(x(s_0+N)) = 0$, such that*

$$a(x'_k)\frac{\dfrac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)} - \dfrac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}}{y'_{k+1}-y'_k} = c(x'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)} + \frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}\right], \tag{20}$$

$k = 0, 1, \ldots, N$, where (x'_k, y'_k) is a shorthand for $(x(s_0+k), y(s_0+k))$, and $w_0 = w_{N+1} = 0$, then,

$$f(x) = \sum_{k=1}^{N} \frac{w_k}{x - y'_k} \tag{21}$$

satisfies

$$a(x)\mathcal{D}f(x) = a(x)\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x)-\varphi(x)} = c(x)[f(\varphi(x)) + f(\psi(x))] + d(x), \tag{22}$$

where d is a polynomial too.

Indeed,

$$\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x)-\varphi(x)} = -\sum_1^N \frac{w_k}{(\varphi(x)-y'_k)(\psi(x)-y'_k)} = -\sum_1^N \frac{w_k X_2(x)}{F(x,y'_k)}$$

$$= -\sum_1^N \frac{w_k X_2(x)}{Y_2(y'_k)(x-x'_{k-1})(x-x'_k)} = X_2(x)\sum_0^N \frac{\dfrac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)} - \dfrac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}}{x-x'_k}$$

with $w_0 = w_{N+1} = 0$,

$$f(\psi(x)) + f(\varphi(x)) = -\sum_1^N \frac{w_k[X_1(x) + 2y'_k X_2(x)]}{X_2(x)(\varphi(x)-y'_k)(\psi(x)-y'_k)} = -\sum_1^N \frac{w_k[X_1(x)+2y'_kX_2(x)]}{F(x,y'_k)}$$

$$= -\sum_1^N \frac{w_k[X_1(x)+2y'_kX_2(x)]}{Y_2(y'_k)(x-x'_{k-1})(x-x'_k)} = \sum_0^N \frac{\dfrac{w_{k+1}[X_1(x)+2y'_{k+1}X_2(x)]}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)} - \dfrac{w_k[X_1(x)+2y'_kX_2(x)]}{Y_2(y'_k)(x'_k-x'_{k-1})}}{x-x'_k},$$

therefore the rational functions $a\mathcal{D}f$ and $c(f(\varphi) + f(\psi))$ differ by a polynomial if all the residues are equal:

$$a(x'_k)X_2(x'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}-\frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}\right]$$
$$=c(x'_k)\left[\frac{w_{k+1}[X_1(x'_k)+2y'_{k+1}X_2(x'_k)]}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}-\frac{w_k[X_1(x'_k)+2y'_kX_2(x'_k)]}{Y_2(y'_k)(x'_k-x'_{k-1})}\right]$$

for $k=0,1,\ldots,N$. Or, as $X_1(x'_k)=-(y'_k+y'_{k+1})X_2(x'_k)$,

$$a(x'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}-\frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}\right]$$
$$=c(x'_k)(y'_{k+1}-y'_k)\left[\frac{w_{k+1}}{Y_2(y'_{k+1})(x'_{k+1}-x'_k)}+\frac{w_k}{Y_2(y'_k)(x'_k-x'_{k-1})}\right]$$

3.2. **"Elliptic logarithm".**

We extend $f(x)=\log\dfrac{x-a}{x-b}$ which satisfies $f'(x)=\dfrac{a-b}{(x-a)(x-b)}$ by looking for a function whose divided difference is a rational function of low degree.

Answer: $w_k=(x'_k-x'_{k-1})Y_2(y'_k)$,

$$\mathcal{D}f(x)=\frac{(x'_N-x'_0)X_2(x)}{(x-x'_0)(x-x'_N)}. \tag{23}$$

4. Recurrences of biorthogonal rational functions.

From excerpts of Spiridonov & Zhedanov [40], also

A. Zhedanov, Biorthogonal rational functions and generalized eigenvalue problem, *J. Approx. Theory* **101** (1999), no. 2, 303–329, and [51].

Also Brezinski, Iserles, Ismail, Masson, Norsett.

4.1. Padé and interpolatory continued fractions.

4.1.1. *Padé.* $\cfrac{\alpha_0}{x-\beta_0+\cfrac{\alpha_1}{x-\beta_1+\cfrac{\ddots}{+\cfrac{\alpha_{n-1}}{x-\beta_{n-1}}}}}$ matches a given Laurent expansion $c_0/x+c_1/x^2+\cdots$ at ∞ up to the c_{2n+1}/x^{2n+2} term. Numerators and denominators satisfy the recurrence relation $P_{n+1}(x)=(x-\beta_n)P_n(x)+\alpha_nP_{n-1}(x)$, suggesting some kind of (formal?) orthogonality. This is even more obvious in the matrix-vector setting

$$\begin{bmatrix}\beta_0 & \sqrt{-\alpha_1} & & \\ \sqrt{-\alpha_1} & \beta_1 & \sqrt{-\alpha_2} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots\end{bmatrix}\begin{bmatrix}P_0(x)\\P_1(x)\\ \vdots \\ \vdots\end{bmatrix}=x\begin{bmatrix}P_0(x)\\P_1(x)\\ \vdots \\ \vdots\end{bmatrix}.$$

If one wants to approximate a Taylor expansion about the origin, just take $z = 1/x$ and rewrite the continued fraction as $\dfrac{\alpha_0 z}{1 - z\beta_0 + \dfrac{\alpha_1 z^2}{1 - \beta_1 z + \dfrac{\ddots}{+\dfrac{\alpha_{n-1} z^2}{1 - \beta_{n-1} z}}}}$ which matches a given Taylor-Maclaurin expansion up to the z^{2n} term.

4.1.2. *Interpolation.* Rational interpolations to a given set of values at $x = y_0, y_1, \ldots$ (yes, the relevant set will be a $y-$lattice) are achieved by

$$\frac{q_n(x)}{p_n(x)} = \alpha'_0 + \cfrac{x - y_0}{\alpha_0 x + \beta_0 - \cfrac{(x-y_1)(x-y_2)}{\cfrac{\ddots}{\alpha_{n-2}x + \beta_{n-2} + \cfrac{(x-y_{2n-3})(x-y_{2n-2})}{\alpha_{n-1}x + \beta_{n-1}}}}}$$

which agree with a given set up to $x = y_{2n}$.

The recurrence relations for p_n and q_n are

$$\begin{aligned} p_{n+1}(x) &= (\alpha_n x + \beta_n)p_n(x) - (x - y_{2n-1})(x - y_{2n})p_{n-1}(x), \\ q_{n+1}(x) &= (\alpha_n x + \beta_n)q_n(x) - (x - y_{2n-1})(x - y_{2n})q_{n-1}(x), \end{aligned} \tag{24}$$

with $q_0 = \alpha'_0$, $p_0 = 1$, $q_1(x) = \alpha'_0(\alpha_0 x + \beta_0) + x - y_0$, $p_1(x) = \alpha_0 x + \beta_0$. We could as well start with $q_{-1}(x) = -1/(x - y_{-1})$ and $p_{-1} = 0$.

We also have the Casorati relation

$$p_n(x)q_{n-1}(x) - p_{n-1}(x)q_n(x) = -(x - y_0)(x - y_1)\cdots(x - y_{2n-3})(x - y_{2n-2}). \tag{25}$$

Consider now rational functions $R_n(x) = \dfrac{p_n(x)}{(x - y_2)(x - y_4)\cdots(x - y_{2n})}$:

$$(x - y_{2n+2})R_{n+1}(x) = (\alpha_n x + \beta_n)R_n(x) - (x - y_{2n-1})R_{n-1}(x),$$

so that the matrix-vector setting is now

$$\begin{bmatrix} -\beta_0 & -y_2 & & & \\ -y_1 & -\beta_1 & -y_4 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ \vdots \end{bmatrix} = x \begin{bmatrix} \alpha_0 & -1 & & & \\ -1 & \alpha_1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ \vdots \end{bmatrix}.$$

So, $\{R_0, R_1, \ldots\}$ is a right eigenvector which is in some way biorthogonal to the set of left eigenvectors $\{T_0, T_1, \ldots\}$ satisfying the recurrence

$$(x - y_{2n+1})T_{n+1}(x) = (\alpha_n x + \beta_n)T_n(x) - (x - y_{2n})T_{n-1}(x),$$

which is of the same structure that the recurrence of the R_n's, but with the odd x's interchanged with the even x's. Actually, $T_n(x)$ is a constant times the same $p_n(x)$ as before, divided by $(x-y_1)(x-y_3)\dots(x-y_{2n-1})$.

That p_n is invariant under matrix transposition is clear: $p_n(x)$ is the determinant

$$\begin{vmatrix} \alpha_0 x+\beta_0 & -(x-y_2) & & & \\ (x-y_1) & \alpha_1 x+\beta_1 & -(x-y_4) & & \\ & \ddots & \ddots & \ddots & \\ & & & (x-y_{2n-3}) & \alpha_{n-1}x+\beta_{n-1} \end{vmatrix}$$

4.2. **Biorthogonality and orthogonality.**

But what is the bilinear form exhibiting the biorthogonality condition?

Let q_n/p_n interpolate a formal Stieltjes transform-like function

$$f(x)=\int_S \frac{d\mu(t)}{x-t},$$

then q_n interpolates $p_n f$ at the $2n+1$ points $y_0,\ y_1,\dots,y_{2n}$. Also, for $k<n$, $\tilde{q}(x)=q_n(x)p_k(x)(x-y_{2k+3})\cdots(x-y_{2n-1})$, still of degree $<2n$, interpolates $p_n(x)p_k(x))(x-y_{2k+3})\cdots(x-y_{2n-1})f(x)$, still has a vanishing divided difference at these $2n+1$ points:

$$[y_0,\dots,y_{2n}] \text{ of } p_n(x)p_k(x))(x-y_{2k+3})\cdots(x-y_{2n-1})f(x) \\ =\int_S \frac{p_n(t)p_k(t))(t-y_{2k+3})\cdots(t-y_{2n-1})\ d\mu(t)}{(t-y_0)\cdots(t-y_{2n})}=0,$$

as the divided difference of a rational function $A(x)/(x-t)$ is $A(t)/\{(t-y_0)\cdots(t-y_{2n})\}$ (Milne-Thomson [29, ...]).

So, R_n is orthogonal to T_k with respect to the formal scalar product

$$\langle g_1,g_2\rangle=\int_S g_1(t)g_2(t)\ d\mu(t).$$

Even simpler: if $f(x)=\sum_0^{N-1}\dfrac{\rho_k}{x-x'_k}$,

$$\langle g_1,g_2\rangle=\sum_0^{N-1}\rho_k g_1(x'_k)g_2(x'_k).$$

4.3. **Example: the exponential function (Iserles [18]).**

Rational interpolations to e^{ax} at $x=0,h,2h,\dots$

The hypergeometric expansions are

$$\begin{aligned} p_n(x) &= C_n \sum_{k=0}^{n} \frac{(-1)^k (2n-k)(2n-k-1)\cdots(n-k+1)t^k}{h^k(1+t)^k k!} x(x-h)\cdots(x-(k-1)h), \\ p_n(x)e^{ax} &= C_n \sum_{k=0}^{\infty} \frac{(2n-k)(2n-k-1)\cdots(n-k+1)t^k}{h^k k!} x(x-h)\cdots(x-(k-1)h), \\ &= q_n(x) + O(x(x-h)\cdots(x-2nh)), \end{aligned} \tag{26}$$

where $t = e^{ah} - 1$. The constant C_n is needed to allow the form of the recurrence relation (24).

5. Elliptic Riccati equations.

5.1. Definition.

An elliptic Riccati equation is

$$a(x)\frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} = b(x)f(\varphi(x))f(\psi(x)) + c(x)(f(\varphi(x)) + f(\psi(x))) + d(x). \tag{27}$$

If $x = x_m$, some point of our $x-$lattice, then $\varphi(x) = y_m$ and $\psi(x) = y_{m+1}$.

A first-order difference equation of the kind (27) relates $f(y_0)$ to $f(y_1)$ when $x = x_0$; $f(y_1)$ to $f(y_2)$ when $x = x_1$, etc. The direct relation is

$$f(\psi) = \frac{\left[\dfrac{a}{\psi - \varphi} + c\right] f(\varphi) + d}{\dfrac{a}{\psi - \varphi} - c - bf(\varphi)}.$$

It is sometimes easier to write (27) as

$$e(x)f(\varphi(x))f(\psi(x)) + g(x)f(\varphi(x)) + h(x)f(\psi(x)) + k(x) = 0, \tag{28}$$

where $e = -b$, $g = -\dfrac{a}{\psi - \varphi} - c$, $h = \dfrac{a}{\psi - \varphi} - c$, and $k = -d$.

However, if a, b, c, and d are rational functions, g and h are conjugate algebraic functions: $h+g$ and hg are symmetric functions of φ and ψ, hence rational functions. This also happens with $2a = (h-g)(\psi - \varphi)$.

5.2. Rational interpolation.

We consider now rational interpolation according to the setting of § 4.1.2 above. Why is this relevant?

From $f_n(x) = \dfrac{x - y_{2n}}{\alpha_n x + \beta_n - (x - y_{2n+1})f_{n+1}(x)}$, $\alpha_n x + \beta_n$ is the polynomial interpolant of degree 1 to $(x - y_{2n})/f_n(x)$ at y_{2n+1} and y_{2n+2}, so we need $f_n(y_{2n+1})$ and $f_n(y_{2n+2})$ in order to find α_n and β_n.

Now, if f_n satisfies the Riccati equation

$$a_n(x)\frac{f_n(\psi(x))-f_n(\varphi(x))}{\psi(x)-\varphi(x)} = b_n(x)f(\varphi(x))f_n(\psi(x)) + c_n(x)(f(\varphi(x))+f(\psi(x))) + d_n(x), \tag{29}$$

or equivalently

$$e_n(x)f_n(\varphi(x))f_n(\psi(x)) + g_n(x)f_n(\varphi(x)) + h_n(x)f_n(\psi(x)) + k_n(x) = 0, \tag{30}$$

where $e_n = -b_n$, $g_n = -\dfrac{a_n}{\psi-\varphi} - c_n$, $h_n = \dfrac{a_n}{\psi-\varphi} - c_n$, and $k_n = -d_n$, one finds at $x = x_{2n}$, $\varphi(x) = y_{2n}$, $\psi(x) = y_{2n+1}$, and $f_n(y_{2n}) = 0$, so

$$f_n(y_{2n+1}) = \frac{y_{2n+1}-y_{2n}}{\alpha_n y_{2n+1}+\beta_n} = \frac{(y_{2n+1}-y_{2n})d_n(x_{2n})}{a_n(x_{2n})-(y_{2n+1}-y_{2n})c_n(x_{2n})} = -\frac{k_n(x_{2n})}{h_n(x_{2n})}, \tag{31}$$

and at $x = x_{2n+1}$, keeping the $e-g-h-k$ form,

$$f_n(y_{2n+2}) = \frac{y_{2n+2}-y_{2n}}{\alpha_n y_{2n+2}+\beta_n} = \frac{g_n(x_{2n+1})k_n(x_{2n})-k_n(x_{2n+1})h_n(x_{2n})}{-e_n(x_{2n+1})k_n(x_{2n})+h(x_{2n+1})h(x_{2n})}., \tag{32}$$

which shows how to extract α_n and β_n from $a_n,\dots$ at x_{2n} and x_{2n+1}. Another form of (32) is

$$(y_{2n+2}-y_{2n})[(y_{2n+1}-y_{2n})e_n(x_{2n+1}) + (\alpha_n y_{2n+1}+\beta_n)h_n(x_{2n+1})]$$
$$+ (\alpha_n y_{2n+2}+\beta_n)[(y_{2n+1}-y_{2n})g_n(x_{2n+1}) + (\alpha_n y_{2n+1}+\beta_n)k_n(x_{2n+1})] = 0. \tag{33}$$

Furthermore, the ***Riccati*** form is well suited to continued fraction progression:

5.3. **Theorem.**

If f_n satisfies the Riccati equation (29) *with rational coefficients a_n, b_n, c_n, and d_n, and if $f_n(x) = \dfrac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$, then f_{n+1} satisfies an equation of same complexity (degree of the rational functions) of its coefficients.*

Indeed, enter $f_n(x) = \dfrac{x-y_{2n}}{\alpha_n x+\beta_n-(x-y_{2n+1})f_{n+1}(x)}$ in the Riccati equation (27) for f_n, actually using the (30) form:

$$e_n(x)\frac{\varphi(x)-y_{2n}}{\alpha_n\varphi(x)+\beta_n-(\varphi(x)-y_{2n+1})f_{n+1}(\varphi(x))}\frac{\psi(x)-y_{2n}}{\alpha_n\psi(x)+\beta_n-(\psi(x)-y_{2n+1})f_{n+1}(\psi(x))}$$
$$+g_n(x)\frac{\varphi(x)-y_{2n}}{\alpha_n\varphi(x)+\beta_n-(\varphi(x)-y_{2n+1})f_{n+1}(\varphi(x))}+h_n(x)\frac{\psi(x)-y_{2n}}{\alpha_n\psi(x)+\beta_n-(\psi(x)-y_{2n+1})f_{n+1}(\psi(x))}+k_n(x)=0.$$

or

$$e_n(x)(\varphi(x)-y_{2n})(\psi(x)-y_{2n}) + g_n(x)(\varphi(x)-y_{2n})[\alpha_n\psi(x)+\beta_n-(\psi(x)-y_{2n+1})f_{n+1}(\psi(x))]$$
$$+h_n(x)(\psi(x)-y_{2n})[\alpha_n\varphi(x)+\beta_n-(\varphi(x)-y_{2n+1})f_{n+1}(\varphi(x))]$$
$$+k_n(x)[\alpha_n\psi(x)+\beta_n-(\psi(x)-y_{2n+1})f_{n+1}(\psi(x))][\alpha_n\varphi(x)+\beta_n-(\varphi(x)-y_{2n+1})f_{n+1}(\varphi(x))] = 0$$

We therefore have a relation between $f_{n+1}(\varphi(x))$ and $f_{n+1}(\psi(x))$ of the form

$$\tilde{e}_{n+1}(x)f_{n+1}(\varphi(x))f_{n+1}(\psi(x)) + \tilde{g}_{n+1}(x)f_{n+1}(\varphi(x)) + \tilde{h}_{n+1}(x)f_{n+1}(\psi(x)) + \tilde{k}_{n+1}(x) = 0, \tag{34}$$

where

$$\begin{aligned}
\tilde{e}_{n+1}(x) &= (\psi(x) - y_{2n+1})(\varphi(x) - y_{2n+1})k_n(x), \\
\tilde{g}_{n+1}(x) &= -(\varphi(x) - y_{2n+1})\left[(\psi(x) - y_{2n})h_n(x) + (\alpha_n\psi(x) + \beta_n)k_n(x)\right], \\
\tilde{h}_{n+1}(x) &= -(\psi(x) - y_{2n+1})\left[(\varphi(x) - y_{2n})g_n(x) + (\alpha_n\varphi(x) + \beta_n)k_n(x)\right], \\
\tilde{k}_{n+1}(x) &= (\varphi(x) - y_{2n})(\psi(x) - y_{2n})e_n(x) + (\varphi(x) - y_{2n})(\alpha_n\psi(x) + \beta_n)g_n(x) \\
&\quad + (\psi(x) - y_{2n})(\alpha_n\varphi(x) + \beta_n)h_n(x) + (\alpha_n\psi(x) + \beta_n)(\alpha_n\varphi(x) + \beta_n)k_n(x).
\end{aligned} \tag{35}$$

The tilde $\tilde{e}$,etc. notation is needed because (34) is not yet the equation (30) with n replaced by $n+1$: the coefficients of (34) will have to be divided by common factors first.

The equation (34) is already a Riccati equation, introducing the difference and the sum of $f_{n+1}(\varphi(x))$ and $f_{n+1}(\psi(x))$ as

$$f_{n+1}(\psi(x)), f_{n+1}(\varphi(x)) = \frac{f_{n+1}(\varphi(x)) + f_{n+1}(\psi(x))}{2} \pm (\psi(x) - \varphi(x))\frac{f_{n+1}(\psi(x)) - f_{n+1}(\varphi(x))}{2(\psi(x) - \varphi(x))},$$

then (34) takes the form of (27)

$$\begin{aligned}
(\tilde{h}_{n+1}(x) - \tilde{g}_{n+1}(x))(\psi(x) - \varphi(x))\frac{f_{n+1}(\psi(x)) - f_{n+1}(\varphi(x))}{2(\psi(x) - \varphi(x))} &= -\tilde{e}_{n+1}(x)f_{n+1}(\varphi(x))f_{n+1}(\psi(x)) \\
-(h_{n+1}(x) + g_{n+1}(x))\frac{f_{n+1}(\varphi(x)) + f_{n+1}(\psi(x))}{2} - k_{n+1}(x).
\end{aligned}$$

The coefficients are now symmetric functions of φ and ψ, therefore rational functions, as $\varphi + \psi = -X_1/X_2$ and $\varphi\psi = X_0/X_2$.

$$\begin{aligned}
2\tilde{a}_{n+1}(x) = (h_{n+1}(x) - g_{n+1}(x))(\psi(x) - \varphi(x)) &= [(\psi(x) - y_{2n+1})(\varphi(x) - y_{2n}) + (\varphi(x) - y_{2n+1})(\psi(x) - y_{2n})]a_n(x) \\
&+ (\psi(x) - \varphi(x))^2[(y_{2n+1} - y_{2n})c_n(x) + (\alpha_n y_{2n+1} + \beta_n)d_n(x)],
\end{aligned}$$

$$\begin{aligned}
-2\tilde{c}_{n+1}(x) = h_{n+1}(x) + g_{n+1}(x) &= (y_{2n+1} - y_{2n})a_n(x) + [(\psi(x) - y_{2n+1})(\varphi(x) - y_{2n}) + (\varphi(x) - y_{2n+1})(\psi(x) - y_{2n})] \\
&+ [(\varphi(x) - y_{2n+1})(\alpha_n\psi(x) + \beta_n) + (\psi(x) - y_{2n+1})(\alpha_n\varphi(x) + \beta_n)]d_n(x)
\end{aligned}$$

We must now be sure that no increase of complexity occurs in the new Riccati equation!

From $\varphi + \psi = -X_1/X_2$ and $\varphi\psi = X_0/X_2$, where the X's are second degree polynomials, we see that the new coefficients $(h_{n+1}(x) - g_{n+1}(x))(\psi(x) - \varphi(x))$, $e_{n+1}(x)$, $h_{n+1}(x) + g_{n+1}(x)$, and $k_{n+1}(x)$ are rational functions of denominator X_2, and sometimes X_2^2. This problem is settled by multiplying the four coefficients by X_2, assuming that b_n, c_n, and d_n already have the factor X_2.

The new coefficients are now polynomials, but of higher degree than before! Fortunately, they have convenient common factors, which can be removed:

(1) *The four coefficients of* (34) *vanish at* $x = x_{2n}$.
Indeed, at $x = x_{2n}$, $\psi(x) = y_{2n+1}$, so that $\tilde{e}_{n+1}$ and $\tilde{h}_{n+1}$ do already vanish. Moreover, with $\varphi(x) = y_{2n}$, one sees that $\tilde{g}_{n+1}$ and $\tilde{k}_{n+1}$ are products containing the factor $(y_{2n+1} - y_{2n})h_n(x_{2n}) + (\alpha_n y_{2n+1} + \beta_n)k_n(x_{2n})$, which must vanish, according to (31).
Remark that $x - x_{2n+1}$ is an obvious factor of $\tilde{e}_{n+1}$ and $\tilde{g}_{n+1}$; also a (much less obvious) factor of $\tilde{k}_{n+1}$, as $\tilde{k}_{n+1}$ at $x = x_{2n+1}$ from (35) gives (33).

(2) *When $n>0$, the four coefficients of* (34) *vanish at $x=x_{2n-1}$.*
According to the remark just above, if $n\geqslant 1$, e_n, g_n, and k_n already vanish at x_{2n-1}, so $\tilde e_{n+1}$ and $\tilde h_{n+1}$ vanish too. The values of $\tilde g_{n+1}$ and $\tilde k_{n+1}$ at x_{2n-1} from (35) contain the same remaining term $(\psi(x)-y_{2n})h_n(x)$ which vanishes at $x=x_{2n-1}$ too.

If a_n, b_n, c_n, and d_n are polynomials, we recover polynomials without increasing the degrees by multiplying $(\tilde h_{n+1}(x)-\tilde g_{n+1}(x))(\psi(x)-\varphi(x))/2$, $-\tilde e_{n+1}(x)$, $-[\tilde h_{n+1}(x)+\tilde g_{n+1}(x)]$, and $-\tilde k_{n+1}(x)$ by $\dfrac{X_2(x)}{(x-x_{2n-1})(x-x_{2n})}$, or, considering that

$$(\varphi(x)-y_{2n})(\psi(x)-y_{2n})=\frac{X_0(x)+X_1(x)y_{2n}+X_2(x)y_{2n}^2}{X_2(x)}=\frac{F(x,y_{2n})}{X_2(x)}=Y_2(y_{2n})\frac{(x-x_{2n})(x-x_{2n-1})}{X_2(x)},$$

we may as well divide by $(\varphi(x)-y_{2n})(\psi(x)-y_{2n})$:

$$[e_{n+1},g_{n+1},h_{n+1},k_{n+1}]=\frac{1}{(\varphi-y_{2n})(\psi-y_{2n})}[\tilde e_{n+1},\tilde g_{n+1},\tilde h_{n+1},\tilde k_{n+1}],$$

and the division of (35) by $(\varphi(x)-y_{2n})(\psi(x)-y_{2n})$ yields at last the Riccati coefficients at the $(n+1)^{\text{th}}$ step:

$$\begin{bmatrix} g_{n+1}\\ h_{n+1}\\ e_{n+1}\\ k_{n+1}\end{bmatrix}=\begin{bmatrix} 0, & -\dfrac{\varphi-y_{2n+1}}{\varphi-y_{2n}}, & 0, & -\dfrac{(\varphi-y_{2n+1})(\alpha_n\psi+\beta_n)}{(\varphi-y_{2n})(\psi-y_{2n})}\\ -\dfrac{\psi-y_{2n+1}}{\psi-y_{2n}}, & 0, & 0, & -\dfrac{(\psi-y_{2n+1})(\alpha_n\varphi+\beta_n)}{(\varphi-y_{2n})(\psi-y_{2n})}\\ 0, & 0, & 0, & \dfrac{(\varphi-y_{2n+1})(\psi-y_{2n+1})}{(\varphi-y_{2n})(\psi-y_{2n})}\\ \dfrac{\alpha_n\psi+\beta_n}{\psi-y_{2n}}, & \dfrac{\alpha_n\varphi+\beta_n}{\varphi-y_{2n}}, & 1, & \dfrac{(\alpha_n\psi+\beta_n)(\alpha_n\varphi+\beta_n)}{(\varphi-y_{2n})(\psi-y_{2n})}\end{bmatrix}\begin{bmatrix} g_n\\ h_n\\ e_n\\ k_n,\end{bmatrix} \tag{36}$$

or $\boldsymbol{g}_{n+1}=\boldsymbol{M}_n\boldsymbol{g}_n$.

A very interesting identity is

$$g_{n+1}h_{n+1}-e_{n+1}k_{n+1}=\frac{(\varphi-y_{2n+1})(\psi-y_{2n+1})}{(\varphi-y_{2n})(\psi-y_{2n})}[g_nh_n-e_nk_n],$$

which, by $\dfrac{(\varphi(x)-y_{2n+1})(\psi(x)-y_{2n+1})}{(\varphi(x)-y_{2n})(\psi(x)-y_{2n})}=\dfrac{F(x,y_{2n+1})}{F(x,y_{2n})}=\dfrac{Y_2(y_{2n+1})}{Y_2(y_{2n})}\dfrac{(x-x_{2n})(x-x_{2n+1})}{(x-x_{2n-1})(x-x_{2n})}$, becomes

$$g_n(x)h_n(x)-e_n(x)k_n(x)=\frac{Y_2(y_{2n-1})Y_2(y_{2n-3})\cdots Y_2(y_1)}{Y_2(y_{2n-2})Y_2(y_{2n-4})\cdots Y_2(y_0)}\frac{x-x_{2n-1}}{x-x_{-1}}[g_0(x)h_0(x)-e_0(x)g_0(x)]. \tag{37}$$

The polynomial coefficients are then recovered by $a_{n+1}=(h_{n+1}-g_{n+1})(\psi-\varphi)/2$, $b_{n+1}=-e_{n+1}$, $c_{n+1}=-(h_{n+1}+g_{n+1})/2$, and $d_{n+1}=-k_{n+1}$, using the rational functions $\varphi+\psi=-X_1/X_2$, $\varphi\psi=X_0/X_2$, $(\psi-\varphi)^2=(X_1^2-4X_0X_2)/X_2^2$, $(\varphi(x)-y_m)(\psi(x)-y_m)=(X_0(x)+y_mX_1(x)+y_m^2X_2(x))/X_2(x)=F(x,y_m)/X_2(x)=Y_2(y_m)(x-x_m)(x-x_{m-1})/X_2(x)$:

$$g_{n+1}\pm h_{n+1} = -\frac{\varphi - y_{2n+1}}{\varphi - y_{2n}}h_n - \frac{(\varphi - y_{2n+1})(\alpha_n\psi+\beta_n)}{(\varphi - y_{2n})(\psi - y_{2n})}k_n \mp \frac{\psi - y_{2n+1}}{\psi - y_{2n}}g_n \mp \frac{(\psi - y_{2n+1})(\alpha_n\varphi+\beta_n)}{(\varphi - y_{2n})(\psi - y_{2n})}k_n$$

$$= -\frac{1}{2}\left[\frac{\varphi - y_{2n+1}}{\varphi - y_{2n}} + \frac{\psi - y_{2n+1}}{\psi - y_{2n}}\right](h_n \pm g_n) - \frac{1}{2}\left[\frac{\varphi - y_{2n+1}}{\varphi - y_{2n}} - \frac{\psi - y_{2n+1}}{\psi - y_{2n}}\right](h_n \mp g_n)$$

$$-\frac{(\varphi - y_{2n+1})(\alpha_n\psi+\beta_n) \pm (\psi - y_{2n+1})(\alpha_n\varphi+\beta_n)}{(\varphi - y_{2n})(\psi - y_{2n})}k_n$$

$$= -\frac{1}{2}\left[\frac{2X_0 + (y_{2n}+y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X_2}{X_2F(x,y_{2n})}\right](h_n \pm g_n) - \frac{(y_{2n+1}-y_{2n})(\varphi-\psi)}{2F(x,y_{2n})}(g_n \mp h_n)$$

$$-\frac{\alpha_n[(1\pm 1)\varphi\psi - y_{2n+1}(\psi\pm\varphi)] + \beta_n[\varphi\pm\psi - y_{2n+1}(1\pm 1)]}{F(x,y_{2n})}k_n$$

so,

$$a_{n+1} = (h_{n+1} - g_{n+1})(\psi-\varphi)/2 = \frac{2X_0 + (y_{2n}+y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X_2}{2F(x,y_{2n})}a_n$$

$$+(\psi-\varphi)^2\left[\frac{y_{2n+1}-y_{2n}}{2F(x,y_{2n})/X_2}c_n + \frac{\alpha_n y_{2n+1}+\beta_n}{F(x,y_{2n})/X_2}d_n\right]$$

or

$$a_{n+1} = \frac{2X_0 + (y_{2n}+y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X_2}{2F(x,y_{2n})}a_n$$

$$+ (X_1^2 - 4X_0X_2)\frac{[(y_{2n+1}-y_{2n})c_n + 2(\alpha_n y_{2n+1}+\beta_n)d_n]/X_2}{2F(x,y_{2n})}. \quad (38)$$

$$c_{n+1} = -(h_{n+1}+g_{n+1})/2 = -\frac{2X_0 + (y_{2n}+y_{2n+1})X_1 + 2y_{2n}y_{2n+1}X_2}{2F(x,y_{2n})}c_n$$

$$+\frac{y_{2n+1}-y_{2n}}{2F(x,y_{2n})}X_2a_n - \frac{\alpha_n(2X_0 + y_{2n+1}X_1) - \beta_n(X_1 + 2y_{2n+1}X_2)}{2F(x,y_{2n})}d_n$$

$$b_{n+1} = \frac{(\varphi - y_{2n+1})(\psi - y_{2n+1})}{(\varphi - y_{2n})(\psi - y_{2n})}d_n = \frac{F(x,y_{2n+1})}{F(x,y_{2n}}d_n,$$

$$d_{n+1} = -\frac{\alpha_n y_{2n}+\beta_n}{F(x,y_{2n})}X_2a_n + b_n$$

$$+\frac{[\alpha_n(2X_0 + y_{2n}X_1) - \beta_n(X_1 + 2y_{2n}X_2)]c_n + (\alpha_n^2X_0 - \alpha_n\beta_nX_1 + \beta_n^2X_2)d_n}{F(x,y_{2n})}$$

From the theorem of § 5.3, $a_{n+1},\dots,d_{n+1}$ must remain polynomials of fixed degree. This can be rechecked, knowing that X_2 is a factor of b_n, c_n, and d_n, that $b_n(x_{2n-1}) = d_n(x_{2n-1}) = 0$.

And (37) becomes, using $g_n = -a_n/(\psi-\varphi) - c_n$, $h_n = a_n/(\psi-\varphi) - c_n$, and $(\psi-\varphi)^2 = P/X_2^2 = (X_1^2 - 4X_0X_2)/X_2^2$

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)}P(x) - a_n^2(x) = C_n\frac{x - x_{2n-1}}{x - x_{-1}}\left[\frac{c_0^2(x) - b_0(x)d_0(x)}{X_2^2(x)}P(x) - a_0^2(x)\right], \quad (39)$$

where $C_n = \dfrac{Y_2(y_{2n-1})Y_2(y_{2n-3})\cdots Y_2(y_1)}{Y_2(y_{2n-2})Y_2(y_{2n-4})\cdots Y_2(y_0)}$.

6. **Classical case.**

We keep the lowest possible degree, which is 3, considering that b_n and d_n must be $X_2(x)$ times a polynomial containing the factor $x - x_{2n-1}$.

Let $d_n(x) = \zeta_n(x - x_{2n-1})X_2(x)$, a_n of degree 3, and $c_n = X_2$ times a polynomial of degree 1.

$$b_{n+1}(x) = \frac{F(x, y_{2n+1})}{F(x, y_{2n})}\zeta_n(x - x_{2n-1})X_2(x) = \frac{Y_2(y_{2n+1})}{Y_2(y_{2n})}\zeta_n(x - x_{2n+1})X_2(x)$$

From the Riccati equation (29) at $x = x_{2n-1}$ and $f_n(y_{2n}) = 0$, we have

$$\frac{a_n(x_{2n-1})}{y_{2n} - y_{2n-1}} = c_n(x_{2n-1}),$$

allowing the divison of the left-hand side of (39), leaving

$$\frac{c_n^2(x) - b_n(x)d_n(x)}{X_2^2(x)}P(x) - a_n^2(x) = C_n(x - x_{2n-1})Q(x),$$

where Q is a fixed polynomial of degree 5.

At each of the four zeros $z_1, \dots, z_4$ of P,

$$a_n(z_j) = \pm\sqrt{-C_n(z_j - x_{2n-1})Q(z_j)},$$

allowing to recover the third degree polynomial a_n from four values... should the square roots be determined! Square root-free relations come from (38) at z_j, knowing that $\varphi(z_j) = \psi(z_j)$, which we call φ_j:

$$a_{n+1}(z_j) = \frac{\varphi_j - y_{2n+1}}{\varphi_j - y_{2n}}a_n(z_j)$$

Remark that, from (39), $Q(z_j) = -a_0^2(z_j)/(z_j - x_{-1})$, so there is a subtle relation between the product of the $(\varphi_j - y_{2n+1})/(\varphi_j - y_{2n})$'s and a square root of $(z_j - x_{2n-1})/(z_j - x_{-1})$. Indeed, as explained in § 2.3, eq. (17), the product is the value of the algebraic function $\mathcal{X}_n$ at z_j, where $\mathcal{X}$ = its conjugate $\mathcal{X}^{\text{conj}}$, so that the value is a square root of $\mathcal{X}(z_j)\mathcal{X}^{\text{conj}}(z_j) = C_n(z_j - y_{2n+1})/(z_j - x_{-1})$.

Well, from (14), $x_m = z_j$ means $mh + h_0$ = a sum of ***half-periods***, so $q^m q'_0 = -1$ or $\pm p^{1/2}$, and we have from (18)-(19)

$$a_n(z_j) = \frac{\theta(q^{2n}q'_0\sigma_j)}{\theta(q'_0\sigma_j)}a_0(z_j), \quad j = 1, \dots, 4$$

where $\sigma_j = \pm 1$ or $\pm p^{1/2}$.

7. **Linear difference relations and equations for the numerators and the denominators of the interpolants.**

, the recurrence relations for p_n and q_n being now (§ 4.1.2, p. 13)
$p_{n+1}(x) = (\alpha_n x + \beta_n)p_n(x) - (x - y_{2n-1})(x - y_{2n})p_{n-1}(x)$,
or $p_{n-1}(x) = \dfrac{(\alpha_n x + \beta_n)p_n(x) - p_{n+1}(x)}{(x - y_{2n-1})(x - y_{2n})}$.

*We now consider the **linear** recurrence satisfied by combinations of such products*, i.e., by combinations of

$$p_n(\varphi)p_n(\psi), p_n(\varphi)q_{n-1}(\psi), q_{n-1}(\varphi)p_n(\psi), \text{ and } q_{n-1}(\varphi)q_{n-1}(\psi).$$

We just have to consider a product $r_n s_n$, kowing that
$r_{n-1} = \dfrac{(\alpha_n\varphi+\beta_n)r_n - r_{n+1}}{(\varphi-y_{2n+1})(\varphi-y_{2n+2})}$, and
$s_{n-1} = \dfrac{(\alpha_n\psi+\beta_n)s_n - s_{n+1}}{(\psi-y_{2n-1})(\psi-y_{2n})}$.
Simplest way is to write it as matrix-vector recurrence.

$$\begin{bmatrix} r_n s_{n-1} \\ r_{n-1}s_n \\ r_n s_n \\ r_{n-1}s_{n-1} \end{bmatrix} = \begin{bmatrix} & -A(\psi) & & B(\psi) \\ -A(\varphi) & & & B(\varphi) \\ & & & 1 \\ -A(\varphi)B(\psi) & -A(\psi)B(\varphi) & A(\varphi)A(\psi) & B(\varphi)B(\psi) \end{bmatrix} \begin{bmatrix} r_{n+1}s_n \\ r_n s_{n+1} \\ r_{n+1}s_{n+1} \\ r_n s_n \end{bmatrix} \tag{40}$$

where $A(t) = \dfrac{1}{(t-y_{2n-1})(t-y_{2n})}$ and $B(t) = \dfrac{\alpha_n t+\beta_n}{(t-y_{2n-1})(t-y_{2n})}$.
The matrix is $\boldsymbol{D}_n^{-1}\boldsymbol{M}_n^T\boldsymbol{E}_n^{-1}$, where

$$\boldsymbol{D}_n = \begin{bmatrix} \psi-y_{2n-1} & & & \\ & \varphi-y_{2n-1} & & \\ & & 1 & \\ & & & (\varphi-y_{2n-1})(\psi-y_{2n-1}) \end{bmatrix},$$

$$\boldsymbol{E}_n = \begin{bmatrix} \varphi-y_{2n+1} & & & \\ & \psi-y_{2n+1} & & \\ & & (\varphi-y_{2n+1})(\psi-y_{2n+1}) & \\ & & & 1 \end{bmatrix},$$

and where $\boldsymbol{M}_n^T$ is the transposed of the matrix of (36).
This means that the recurrence of the $r_n s_n$'s is basically the ***adjoint*** of the recurrence (36) of the (e_n, g_n, h_n, k_n)'s
Remark that $\boldsymbol{E}_n\boldsymbol{D}_{n+1} = (\varphi-y_{2n+1})(\psi-y_{2n+1})\boldsymbol{I}$.
So, (40) is $\boldsymbol{\rho}_{n-1} = \boldsymbol{D}_n^{-1}\boldsymbol{M}_n^T\boldsymbol{E}_n^{-1}\boldsymbol{\rho}_n$, $\boldsymbol{g}_n\boldsymbol{D}_n\boldsymbol{\rho}_{n-1} = \boldsymbol{g}_n\boldsymbol{M}_n^T\boldsymbol{E}_n^{-1}\boldsymbol{\rho}_n = \boldsymbol{g}_{n+1}\boldsymbol{E}_n^{-1}\boldsymbol{\rho}_n$ (from (36)), or $(\varphi-y_{2n+1})(\psi-y_{2n+1})\boldsymbol{g}_n\boldsymbol{D}_n\boldsymbol{\rho}_{n-1} = \boldsymbol{g}_{n+1}\boldsymbol{D}_{n+1}\boldsymbol{\rho}_n$:

$$\boldsymbol{g}_{n+1}\boldsymbol{D}_{n+1}\boldsymbol{\rho}_n = (\varphi-y_1)(\psi-y_1)(\varphi-y_3)(\psi-y_3)\cdots(\varphi-y_{2n+1})(\psi-y_{2n+1})\boldsymbol{g}_0\boldsymbol{D}_0\boldsymbol{\rho}_{-1}$$

so that, for any choice of r_n and s_n (p_n or q_n),

$$(\psi-y_{2n+1})g_{n+1}r_{n+1}s_n+(\varphi-y_{2n+1})h_{n+1}r_ns_{n+1}+e_{n+1}r_{n+1}s_{n+1}+(\varphi-y_{2n+1})(\psi-y_{2n+1})k_{n+1}r_ns_n$$
$$=(\varphi-y_1)(\psi-y_1)(\varphi-y_3)(\psi-y_3)\cdots(\varphi-y_{2n+1})(\psi-y_{2n+1})$$
$$\times[(\psi-y_{-1})g_0r_0s_{-1}+(\varphi-y_{-1})h_0r_{-1}s_0+e_0r_0s_0+(\varphi-y_{-1})(\psi-y_{-1})k_0r_{-1}s_{-1}]$$

With the two choices $(r,s)=(p,p)$ and (q,p):

$$(\psi-y_{2n+1})g_{n+1}p_{n+1}(\varphi)p_n(\psi)+(\varphi-y_{2n+1})h_{n+1}p_n(\varphi)p_{n+1}(\psi)+e_{n+1}p_{n+1}(\varphi)p_{n+1}(\psi)+(\varphi-y_{2n+1})(\psi-y_{2n+1})k_{n+1}p_n(\varphi)p_n(\psi)$$
$$=e_0(\varphi-y_1)(\psi-y_1)(\varphi-y_3)(\psi-y_3)\cdots(\varphi-y_{2n+1})(\psi-y_{2n+1}),$$

$$(\psi-y_{2n+1})g_{n+1}q_{n+1}(\varphi)p_n(\psi)+(\varphi-y_{2n+1})h_{n+1}q_n(\varphi)p_{n+1}(\psi)+e_{n+1}q_{n+1}(\varphi)p_{n+1}(\psi)+(\varphi-y_{2n+1})(\psi-y_{2n+1})k_{n+1}q_n(\varphi)p_n(\psi)$$
$$=(\varphi-y_{-1})h_0q_{-1}(\varphi)(\varphi-y_1)(\psi-y_1)(\varphi-y_3)(\psi-y_3)\cdots(\varphi-y_{2n+1})(\psi-y_{2n+1}),$$

Multiply the first equation by $q_n(\varphi)$, the second one by $p_n(\varphi)$, and subtract:

$$[q_n(\varphi)p_{n+1}(\varphi)-p_n(\varphi)q_{n+1}(\varphi)][(\psi-y_{2n+1})g_{n+1}p_n(\psi)+e_{n+1}p_{n+1}(\psi)]$$
$$=[q_n(\varphi)e_0-p_n(\varphi)(\varphi-y_{-1})h_0q_{-1}(\varphi)](\varphi-y_1)(\psi-y_1)(\varphi-y_3)(\psi-y_3)\cdots(\varphi-y_{2n+1})(\psi-y_{2n+1}),$$

and using the Casorati relation (25)

$$(\psi-y_{2n+1})g_{n+1}p_n(\psi)+e_{n+1}p_{n+1}(\psi)=[q_n(\varphi)e_0+p_n(\varphi)h_0]\mathcal{X}_{n+1}, \tag{41a}$$

where $\mathcal{X}_{n+1}=\dfrac{(\psi-y_1)(\psi-y_3)\cdots(\psi-y_{2n+1})}{(\varphi-y_0)(\varphi-y_2)\cdots(\varphi-y_{2n})}$.

Similarly, with $(r,s)=(p,p)$ and (p,q),

$$(\varphi-y_{2n+1})h_{n+1}p_n(\varphi)+e_{n+1}p_{n+1}(\varphi)=[q_n(\psi)e_0+p_n(\psi)g_0]\mathcal{X}^{\text{conj}}_{n+1}, \tag{41b}$$

where $\mathcal{X}^{\text{conj}}_{n+1}=\dfrac{(\varphi-y_1)(\varphi-y_3)\cdots(\varphi-y_{2n+1})}{(\psi-y_0)(\psi-y_2)\cdots(\psi-y_{2n})}$ is the conjugate of the algebraic function $\mathcal{X}_{n+1}$.

7.1. **Difference equation for the denominator p_n.**

Here, $e_0(x)\equiv 0$.

Take (41a) at $\psi^{-1}(y)$:

$$(y-y_{2n+1})g_{n+1}(\psi^{-1}(y))p_n(y)+e_{n+1}(\psi^{-1}(y))p_{n+1}(y)=h_0(\psi^{-1}(y))p_n(\varphi(\psi^{-1}(y)))\mathcal{X}_{n+1}(\psi^{-1}(y)),$$

and (41b) at $\varphi^{-1}(y)$:

$$(y-y_{2n+1})h_{n+1}(\varphi^{-1}(y))p_n(y)+e_{n+1}(\varphi^{-1}(y))p_{n+1}(y)=g_0(\varphi^{-1}(y))p_n(\psi(\varphi^{-1}(y)))\mathcal{X}^{\text{conj}}_{n+1}(\varphi^{-1}(y)),$$

so,

$$p_{n+1}(y)=\frac{h_0(\psi^{-1}(y))p_n(\varphi(\psi^{-1}(y)))\mathcal{X}_{n+1}(\psi^{-1}(y))-(y-y_{2n+1})g_{n+1}(\psi^{-1}(y)))p_n(y)}{e_{n+1}(\psi^{-1}(y))}$$
$$=\frac{g_0(\varphi^{-1}(y))p_n(\psi(\varphi^{-1}(y)))\mathcal{X}^{\text{conj}}_{n+1}(\varphi^{-1}(y))-(y-y_{2n+1})h_{n+1}(\varphi^{-1}(y))p_n(y)}{e_{n+1}(\varphi^{-1}(y))}$$

possible only if

$$\frac{h_0(\psi^{-1}(y))p_n(\varphi(\psi^{-1}(y)))\mathcal{X}_{n+1}(\psi^{-1}(y))}{e_{n+1}(\psi^{-1}(y))} - \frac{g_0(\varphi^{-1}(y))p_n(\psi(\varphi^{-1}(y)))\mathcal{X}^{\rm conj}_{n+1}(\varphi^{-1}(y))}{e_{n+1}(\varphi^{-1}(y))}$$
$$= \left[\frac{g_{n+1}(\psi^{-1}(y)))}{e_{n+1}(\psi^{-1}(y))} - \frac{h_{n+1}(\varphi^{-1}(y))}{e_{n+1}(\varphi^{-1}(y))}\right](y - y_{2n+1})p_n(y)$$

At $y =$ some y_m:

$$\frac{h_0(x_{m-1})p_n(y_{m-1})\mathcal{X}_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})} - \frac{g_0(x_m)p_n(y_{m+1})\mathcal{X}^{\rm conj}_{n+1}(x_m)}{e_{n+1}(x_m)}$$
$$= \left[\frac{g_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})} - \frac{h_{n+1}(x_m)}{e_{n+1}(x_m)}\right](y_m - y_{2n+1})p_n(y_m)$$

Remark that $\mathcal{X}_{n+1}(x_{m-1})$ and $\mathcal{X}^{\rm conj}_{n+1}(x_m)$ have the same numerator $(y_m-y_1)(y_m-y_3)\cdots(y_m-y_{2n+1})$, so,

$$\frac{g_0(x_m)}{e_{n+1}(x_m)}\frac{p_n(y_{m+1})}{(y_{m+1}-y_0)(y_{m+1}-y_2)\cdots(y_{m+1}-y_{2n})} - \frac{h_0(x_{m-1})}{e_{n+1}(x_{m-1})}\frac{p_n(y_{m-1})}{(y_{m-1}-y_0)(y_{m-1}-y_2)\cdots(y_{m-1}-y_{2n})}$$
$$= \left[\frac{h_{n+1}(x_m)}{e_{n+1}(x_m)} - \frac{g_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})}\right]\frac{p_n(y_m)}{(y_m-y_1)(y_m-y_3)\cdots(y_m-y_{2n-1})},$$

Therefore, $R_n(x) = \dfrac{p_n(x)}{(x-y_0)(x-y_2)\cdots(x-y_{2n})}$ satisfies

$$\frac{g_0(x_m)}{e_{n+1}(x_m)}R_n(y_{m+1}) - \frac{h_0(x_{m-1})}{e_{n+1}(x_{m-1})}R_n(y_{m-1})$$
$$= \left[\frac{h_{n+1}(x_m)}{e_{n+1}(x_m)} - \frac{g_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})}\right]\frac{p_n(y_m)}{(y_m-y_1)(y_m-y_3)\cdots(y_m-y_{2n-1})},$$

?

If $\mathcal{D}^\dagger$ means

$$(\mathcal{D}^\dagger p)(y) = \frac{p(\psi^{-1}(y)) - p(\varphi^{-1}(y))}{\psi^{-1}(y) - \varphi^{-1}(y)},$$

then

$$(\mathcal{D}^\dagger(r\mathcal{D}p))(y) = \frac{r(\psi^{-1}(y))\dfrac{p(y) - p(\varphi(\psi^{-1}(y)))}{y - \varphi(\psi^{-1}(y))} - r(\varphi^{-1}(y))\dfrac{p(\psi(\varphi^{-1}(y))) - p(y)}{\psi(\varphi^{-1}(y)) - y}}{\psi^{-1}(y) - \varphi^{-1}(y)}$$

At $y =$ some y_m:

$$(\mathcal{D}^\dagger(r\mathcal{D}p))(y_m) = \frac{r(x_{m-1})\dfrac{p(y_m) - p(y_{m-1})}{y_m - y_{m-1}} - r(x_m)\dfrac{p(y_{m+1}) - p(y_m)}{y_{m+1} - y_m}}{x_{m-1} - x_m}$$

match if $\dfrac{r(x_m)(y_m-y_{m-1})}{r(x_{m-1})(y_{m+1}-y_m)} = -\dfrac{g_0(x_m)e_{n+1}(x_{m-1})}{e_{n+1}(x_m)h_0(x_{m-1})}$. We already encountered a function satisfying a similar difference equation, from Pearson's equation $\dfrac{w(x_m)}{(x_m-x_{m-1})Y_2(y_m)} = -\dfrac{g_0(x_{m-1})}{h_0(x_{m-1})}\dfrac{w(x_{m-1})}{(x_{m-1}-x_{m-2})Y_2(y_{m-1})}$. So, $\dfrac{r(x_m)e_{n+1}(x_m)}{(y_{m+1}-y_m)g_0(x_m)} = \dfrac{w(x_m)}{Y_2(y_m)(x_m-x_{m-1})}$.

$$(\mathcal{D}^\dagger(r\mathcal{D}))\frac{p_n(y_m)}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})}$$
$$= \frac{1}{x_{m-1}-x_m}\left[r(x_{m-1})\frac{\dfrac{p_n(y_m)}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})} - \dfrac{p_n(y_{m-1})}{(y_{m-1}-y_0)(y_{m-1}-y_2)\cdots(y_{m-1}-y_{2n})}}{y_m-y_{m-1}} \right.$$
$$\left. -r(x_m)\frac{\dfrac{p_n(y_{m+1})}{(y_{m+1}-y_0)(y_{m+1}-y_2)\cdots(y_{m+1}-y_{2n})} - \dfrac{p_n(y_m)}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})}}{y_{m+1}-y_m} \right]$$

$$= -\frac{r(x_m)e_{n+1}(x_m)}{(y_{m+1}-y_m)(x_{m-1}-x_m)g_0(x_m)}$$
$$\left[\frac{h_0(x_{m-1})}{e_{n+1}(x_{m-1})}\left[\frac{p_n(y_m)}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})} - \frac{p_n(y_{m-1})}{(y_{m-1}-y_0)(y_{m-1}-y_2)\cdots(y_{m-1}-y_{2n})}\right]\right.$$
$$\left. +\frac{g_0(x_m)}{e_{n+1}(x_m)}\left[\frac{p_n(y_{m+1})}{(y_{m+1}-y_0)(y_{m+1}-y_2)\cdots(y_{m+1}-y_{2n})} - \frac{p_n(y_m)}{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})}\right]\right]$$

Therefore, $R_n(x) = \dfrac{p_n(x)}{(x-y_0)(x-y_2)\cdots(x-y_{2n})}$ satisfies

$$(\mathcal{D}^\dagger(r\mathcal{D}))R_n(y_m) = -\frac{r(x_m)e_{n+1}(x_m)}{(y_{m+1}-y_m)(x_{m-1}-x_m)g_0(x_m)}$$
$$\left[\frac{h_0(x_{m-1})}{e_{n+1}(x_{m-1})} - \frac{g_0(x_m)}{e_{n+1}(x_m)} + \left[\frac{h_{n+1}(x_m)}{e_{n+1}(x_m)} - \frac{g_{n+1}(x_{m-1})}{e_{n+1}(x_{m-1})}\right]\frac{(y_m-y_0)(y_m-y_2)\cdots(y_m-y_{2n})}{(y_m-y_1)(y_m-y_3)\cdots(y_m-y_{2n-1})}\right]R_n(y_m)$$

8. **Hypergeometric expansions.**

From:

David R. Masson: The last of the hypergeometric continued fractions, Report-no: OP-SF 12 Sep 1994 `http://arxiv.org/abs/math.CA/9409229`

Dharma P. Gupta; David R. Masson: Contiguous relations, continued fractions and orthogonality *Trans. Amer. Math. Soc.* **350** (1998), 769-808. This article is available free of charge `http://www.ams.org/tran/1998-350-02/S0002-9947-98-01879-0/home.html`

Building blocks:

$$\mathcal{D}\frac{(x-y_0)(x-y_1)\cdots(x-y_{n-1})}{(x-y'_1)(x-y'_2)\cdots(x-y'_n)} = C_n X_2(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x'_0)(x-x'_1)\cdots(x-x'_n)}.$$

Indeed, $(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})$ and $(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})$ both vanish at $x=x_0,x_1,\dots,x_{n-2}$, and similarly for the $\{x'_k\}$s and the $\{y'_k\}$s.

The common denominator is $(\varphi(x)-y'_1)(\psi(x)-y'_1)\varphi(x)-y'_2)(\psi(x)-y'_2)\cdots = [F(x,y'_1)F(x,y'_2)\cdots F(x,y'_n)]/X_2^n(x) = Y_2(y'_1)\cdots Y_2(y'_n)(x-x'_0)(x-x'_1)^2\cdots(x-x'_{n-1})^2(x-x'_n)/X_2(x)^2$, and the numerator is $[(\psi^n-(y_0+\cdots+y_{n-1})\psi^{n-1}+\cdots)(\varphi^n-(y'_1+\cdots+y'_n)\varphi^{n-1}+\cdots)-(\varphi^n-(y_0+\cdots+y_{n-1})\varphi^{n-1}+\cdots)(\psi^n-(y'_1+\cdots+y'_n)\psi^{n-1}+\cdots)]/(\psi-\varphi) = (y_0+\cdots y_{n-1}-y'_1-\cdots-y'_n)\varphi^{n-1}\psi^{n-1}+\cdots$, a symmetric polynomial of degree $2n-2$ vanishing at $x=x_0,x_1,\dots,x_{n-2}$ and $x=x'_1,x'_2,\dots,x'_{n-1}$.

The constant C_n is found through particular values of x, either x_{-1} or x_{n-1}:

$$\begin{aligned} C_n &= \frac{(y_{-1}-y_1)\cdots(y_{-1}-y_{n-1})}{(y_{-1}-y'_1)(y_{-1}-y'_2)\cdots(y_{-1}-y'_n)}\frac{(x_{-1}-x'_0)(x_{-1}-x_1)\cdots(x_{-1}-x'_n)}{X_2(x_{-1})(x_{-1}-x_0)(x_{-1}-x_1)\cdots(x_{-1}-x_{n-2})}\\ &= \frac{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-2})}{(y_n-y'_1)(y_n-y'_2)\cdots(y_n-y'_n)}\frac{(x_{n-1}-x'_0)(x_{n-1}-x_1)\cdots(x_{n-1}-x'_n)}{X_2(x_{n-1})(x_{n-1}-x_0)(x_{n-1}-x_1)\cdots(x_{n-1}-x_{n-2})}\end{aligned}$$

(Of course, $C_0=0$).

Also,

$$\frac{(\varphi(x)-y_0)(\varphi(x)-y_1)\cdots(\varphi(x)-y_{n-1})}{(\varphi(x)-y'_1)(\varphi(x)-y'_2)\cdots(\varphi(x)-y'_n)} + \frac{(\psi(x)-y_0)(\psi(x)-y_1)\cdots(\psi(x)-y_{n-1})}{(\psi(x)-y'_1)(\psi(x)-y'_2)\cdots(\psi(x)-y'_n)}$$

$$= D_n(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}{(x-x'_0)(x-x'_1)\cdots(x-x'_n)},$$

where D_n is a polynomial of degree 2.

Now let us consider the difference equation (23) of the elliptic logarithm with $f(x) = \sum_0^N \gamma_k \frac{(x-y_0)(x-y_1)\cdots(x-y_{k-1})}{(x-y'_1)(x-y'_2)\cdots(x-y'_k)}$, as we know that the poles are the y's and that f in interpolated at y_0, $y_1,\dots$ (cf. Zhedanov [51]). Here, $a(x)=(x-x'_0)(x-x'_N)$, $c=0$, and d is a constant time X_2 in (22):

$$\frac{(x-x'_0)(x-x'_N)}{X_2(x)}\mathcal{D}f(x) = x'_N - x'_0.$$

$$(x-x'_N)\sum_{k=1}^N \gamma_k C_k \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-2})}{(x-x'_1)(x-x'_2)\cdots(x-x'_k)} = x'_N - x'_0.$$

Use $x-x'_N = \dfrac{x'_N-x'_k}{x_{k-1}-x'_k}(x-x_{k-1}) + \dfrac{x_{k-1}-x'_N}{x_{k-1}-x'_k}(x-x'_k)$:

$$\sum_{k=0}^N \left[\gamma_k C_k \frac{x'_N-x'_k}{x_{k-1}-x'_k} + \gamma_{k+1}C_{k+1}\frac{x_k-x'_N}{x_k-x'_{k+1}}\right]\frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})}{(x-x'_1)(x-x'_2)\cdots(x-x'_k)} = x'_N - x'_0.$$

So, $\gamma_1 C_1 \dfrac{x_0 - x'_N}{x_0 - x'_1} = x'_N - x'_0$,

$$\gamma_k = -\frac{x_{k-1} - x'_k}{C_k} \frac{(x'_0 - x'_N)(x'_1 - x'_N)\cdots(x'_{k-1} - x'_N)}{(x_0 - x'_N)(x_1 - x'_N)\cdots(x_{k-1} - x'_N)}, \qquad k = 1, \cdots, N.$$

$p_n f - q_n$ vanishes at $x = y_0, y_1, \cdots y_{2n}$. Let

$$p_n(x) = \sum_0^n \delta_j (x - y_0)(x - y_1)\cdots(x - y_{j-1})$$

.

We shall manage to represent $p_n(x)f(x)$ as a polynomial of degree n (which will be q_n) plus a sum of terms $\dfrac{(x - y_0)(x - y_1)\cdots(x - y_{k+n-1})}{(x - y'_1)(x - y'_2)\cdots(x - y'_k)}$, $k = 1, 2, \ldots, N$, with vanishing coefficients when $k = 1, 2, \ldots, n$.

At $n = 1$, $p_1(x) = \alpha_0 x + \beta_0$ which interpolates $\dfrac{x - y_0}{f(x) - f(y_0)}$ at $x = y_1$ and y_2

$$\alpha_0 y_1 + \beta_0 = \frac{y_1 - y'_1}{\gamma_1} = \frac{(x_1 - x'_0)(x_2 - x'_N)}{(x'_N - x'_0)X_2(x_1)},$$

$$\alpha_0 y_2 + \beta_0 = \frac{y_2 - y_0}{\gamma_1 \dfrac{y_2 - y_0}{y_2 - y'_1} + \gamma_2 \dfrac{(y_2 - y_0)(y_2 - y_1)}{(y_2 - y'_1)(y_2 - y'_2)}} = \frac{y_2 - y'_1}{\gamma_1 + \gamma_2 \dfrac{y_2 - y_1}{y_2 - y'_2}}$$

whence $\alpha_0 = \dfrac{1}{\gamma_1} \dfrac{\gamma_1 - \gamma_2 \dfrac{y_1 - y'_1}{y_2 - y'_2}}{\gamma_1 + \gamma_2 \dfrac{y_2 - y_1}{y_2 - y'_2}}$

$$p_1(x)f(x) = \gamma_0(\alpha_0 x + \beta_0) + \gamma_1(\alpha_0 x + \beta_0)\frac{x - y_0}{x - y'_1} + \gamma_2(\alpha_0 x + \beta_0)\frac{(x - y_0)(x - y_1)}{(x - y'_1)(x - y'_2)} + \cdots$$

Use $\alpha_0 x + \beta_0 = \dfrac{\alpha y_k + \beta_0}{y_k - y'_k}(x - y'_k) + \dfrac{\alpha y'_k + \beta_0}{y'_k - y_k}(x - y_k)$:

$$p_1(x)f(x) = \underbrace{\gamma_0(\alpha_0 x + \beta_0) + \gamma_1 \frac{\alpha_0 y_1 + \beta_0}{y_1 - y'_1}(x - y_0)}_{q_1(x)} + \gamma_1 \frac{\alpha_0 y'_1 + \beta_0}{y'_1 - y_1} \frac{(x - y_0)(x - y_1)}{x - y'_1} + \gamma_2 \frac{\alpha_0 y_2 + \beta_0}{y_2 - y'_2} \frac{(x - y_0)(x - y_1)}{x - y'_1}$$

$$+ \gamma_2 \frac{\alpha_0 y'_2 + \beta_0}{y'_2 - y_2} \frac{(x - y_0)(x - y_1)(x - y_2)}{(x - y'_1)(x - y'_2)} + \cdots$$

and we have to check that

$$\gamma_1 \frac{\alpha_0 y'_1 + \beta_0}{y'_1 - y_1} + \gamma_2 \frac{\alpha_0 y_2 + \beta_0}{y_2 - y'_2} =$$

$$\gamma_1 \alpha_0 + \gamma_1 \frac{\alpha_0 y_1 + \beta_0}{y'_1 - y_1} + \gamma_2 \frac{\alpha_0 y_2 + \beta_0}{y_2 - y'_2}$$

$$= \frac{\gamma_1(y_2 - y'_2) - \gamma_2(y_1 - y'_1)}{\gamma_1(y_2 - y'_2) + \gamma_2(y_2 - y_1)} - 1 + \gamma_2 \frac{y_2 - y'_1}{\gamma_1(y_2 - y'_2) + \gamma_2(y_2 - y_1)}$$

$$= 0.$$

$$p_1 f(x) - q_1(x) = \sum_{k=2}^{N} \left[\gamma_k \frac{\alpha_0 y'_k + \beta_0}{y'_k - y_k} + \gamma_{k+1} \frac{\alpha_0 y_{k+1} + \beta_0}{y_{k+1} - y'_{k+1}} \right] \frac{(x-y_0)(x-y_1)\cdots(x-y_k)}{(x-y'_1)(x-y'_2)\cdots(x-y'_k)}$$

For a general n, we represent the unknown denominator as

$$p_n(x) = \sum_{j=0}^{n} \delta_j (x-y_0)(x-y_1)\cdots(x-y_{j-1})(x-y'_{j+1})\cdots(x-y'_n)$$

.

Then, in each term of $p_n(x)f(x) = \sum_{j=0}^n \delta_j f(x)(x-y_0)(x-y_1)\cdots(x-y_{j-1})(x-y'_{j+1})\cdots(x-y'_n)$, we expand f as $f(x) = \sum_0^N \gamma_{k,j} \frac{(x-y_j)(x-y_{j+1})\cdots(x-y_{j+k-1})}{(x-y'_1)(x-y'_2)\cdots(x-y'_k)}$, so

$$p_n(x)f(x) = \sum_{j=0}^{n} \delta_j \sum_{k=0}^{N} \gamma_{k,j} \frac{(x-y_0)(x-y_1)\cdots(x-y_{j+k-1})}{(x-y'_1)(x-y'_2)\cdots(x-y'_k)}$$

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