Seminar series on

Rational approximations and systems theory.

February-March 2002

Asymptotic convergence rates of rational interpolation to exponential functions.

The present slides file is
http://www.math.ucl.ac.be/~magnus/num3/rslides.ps and **pdf**
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and references therein.

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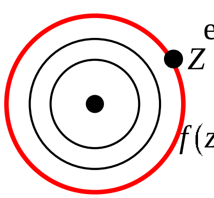
1. Taylor, polynomial, and rational interpolation.

1.1. Taylor expansions.

The Taylor series expansion of a function with finite convergence domain shows “typically” almost circular level lines of equal approximation, explained by a convenient representation of the error



$$f(z) - \sum_0^n c_k z^k = \frac{1}{2\pi i}\int_{|t|=r} f(t)\frac{z^{n+1}}{t^{n+1}(t-z)}\,dt, \qquad (1)$$

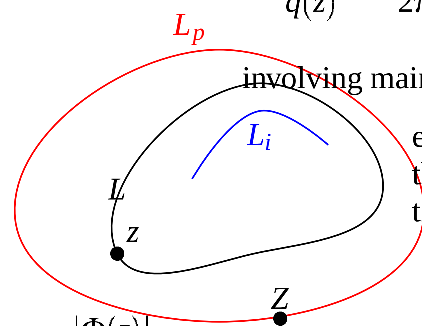
$|z| < |Z| = R,$

1.2. General rational interpolation with given poles. Condenser capacity.

. . . , tout entier à une idée
qui lui était venue sur les potentiels.
Alphonse Allais (from *Madrigal manqué*)

The error at z is

$$f(z) - \frac{p(z)}{q(z)} = \frac{\prod(z-z_j)}{2\pi i q(z)}\int_C \frac{q(t)\,f(t)\,dt}{(t-z)\prod(t-z_j)} \qquad (2)$$



involving mainly $\left(\dfrac{\Phi(z)}{\Phi(Z)}\right)^n$, with $\Phi(z) = \exp\mathcal{V}(z) = \exp[\mathcal{V}_p(z) - \mathcal{V}_i(z)]$, where $\mathcal{V}_p$ and $\mathcal{V}_i$ are the (complex) ***potentials*** of the distributions of the poles and interpolation points: $\mathcal{V}_{p,i}(z) := \int_C \log\dfrac{1}{z-t}\,d\mu_{p,i}(t).$

$$\frac{|\Phi(z)|}{|\Phi(Z)|} = \exp[\mathrm{Re}\,(\mathcal{V}(z) - (\mathrm{Re}\,\mathcal{V}\text{ on } L_p)] = \exp\left(\frac{-1}{\mathrm{cap}\,(L, L_p)}\right).$$

1.3. The problem of rational interpolation at $m+n+1$ points, orthogonal polynomials.

Numerator interpolates $q_n f$ at $m+n+1$ points: $p_m(z) =$

$$\frac{1}{2\pi i}\int_{C_f} q_n(t)\sum_{j=0}^{m+n}\frac{L_j(z)}{t-z_j} f(t)\,dt = \frac{1}{2\pi i}\int_{C_f} q_n(t)\left[\frac{1}{t-z} - \frac{\prod_0^{m+n}(z-z_j)}{(t-z)\prod_0^{m+n}(t-z_j)}\right] f(t)\,dt$$

So, p_m is only $O(z^m)$ as it should if $\displaystyle\int_{C_f}\frac{q_n(t)\,f(t)}{(t-z)\prod_0^{m+n}(t-z_j)}\,dt$ is $O(z^{-n-1})$,
so, as $(t-z)^{-1} = -z^{-1} - tz^{-2} - \cdots + t^n z^{-n}(z-t)^{-1}$, if

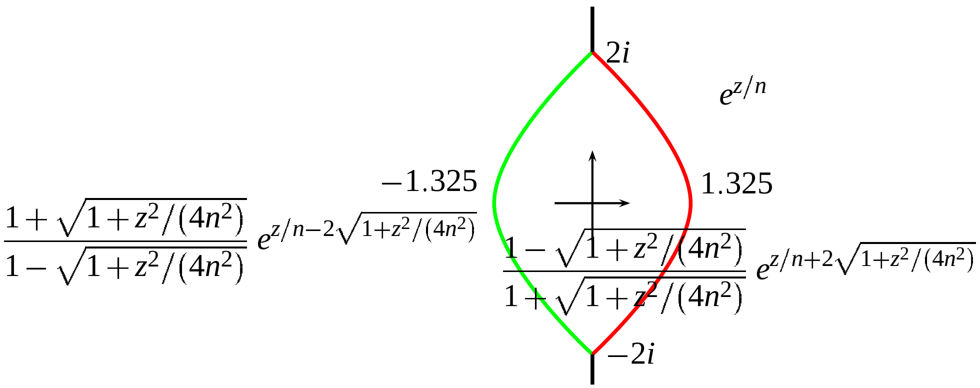
$$\int_{C_f} q_n(t)t^j w_n(t)\,dt = 0, \qquad j = 0,\ldots,n-1, \qquad (3)$$

where $w_n(t) = \dfrac{f(t)}{\prod_0^{m+n}(t-z_j)}$: formal ***orthogonality***!

2. Known rational interpolations to the exponential function.

2.1. Padé.

For the error $e^z -$ approximant, we have the n^{th} powers of



2.2. Rational interpolation at equidistant points (Iserles). [3]

Interpolation of $\exp(Az)$ at $z_0, z_0+h, \ldots, z_0+(m+n)h$:
Rough asymptotics.

If $m \sim n$, one finds that the numerator, denominator, and the error behave like the n^{th} powers of

$$\exp\left[\zeta \log\left(e^{Ah/2}\frac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\zeta-1}\right)+\frac{Ah}{2}+\log(\gamma+\sqrt{\sigma^2\zeta^2+1})\right]$$

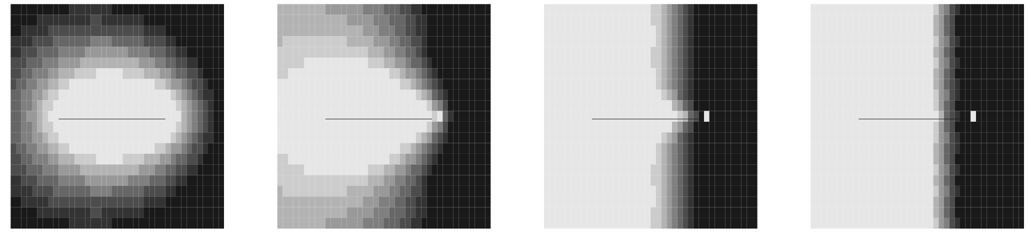
$$\exp\left[\zeta \log\left(e^{-Ah/2}\frac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\zeta-1}\right)-\frac{Ah}{2}+\log(\gamma+\sqrt{\sigma^2\zeta^2+1})\right]$$

$$\exp\left[\zeta \log\left(e^{Ah}\frac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\gamma\zeta+\sqrt{\sigma^2\zeta^2+1}}\right)+Ah+\log\left(\frac{\gamma+\sqrt{\sigma^2\zeta^2+1}}{\gamma-\sqrt{\sigma^2\zeta^2+1}}\right)\right]$$

where $\zeta = [2(z-z_0)/((m+n)h)]-1$, $\gamma = \cosh Ah/2$, $\sigma = \sinh Ah/2$.

We look at the performance of some examples of the region of good approximation in the complex plane, coloured in light gray:

exp(5(0.250z)) -12.6 | exp(5(1.00z)) -0.0897 | exp(5(2.00z)) 3.84 | exp(5(3.00z)) 6.57



3. Asymptotic features of rational interpolation.

3.1. According to Gončar-Rahmanov-Stahl (a sloppy rendering).

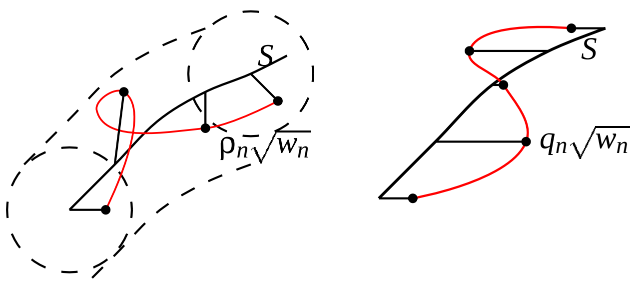
Interpolation to $f_n(z) = \displaystyle\int_{C_f} \frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt$ at $z_0, \ldots, z_{m+n}$ by p_m/q_n yields

$$f_n(z) - \frac{p_m(z)}{q_n(z)} = \frac{\prod_0^{m+n}(z-z_j)}{q_n^2(z)} \int_{C_f} \frac{q_n^2(t)}{\prod_0^{m+n}(t-z_j)} \frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt,$$

where q_n is (formally) orthogonal with respect to $w_n(t) := \dfrac{\varphi_0(t)\varphi^n(t)}{\prod_0^{m+n}(t-z_j)}$ on C_f, as in (3).

Well, we expect that most of the poles of q_n will tend to a set $S \subseteq C_f$.

♩ *On the support of* μ_p, q_n *is almost a Szegő orthogonal polynomial! which means that* $\pm q_n(t)\sqrt{w_n(t)}$ *has slowly varying phase and absolute value there.*



$$q_n(t)\frac{\varphi^{n/2}(t)}{\sqrt{\prod_0^{m+n}(t-z_j)}} \approx \exp\left(n\left[\frac{\log\varphi(t)}{2}+\mathcal{V}_i(t)-\frac{\mathcal{V}_{p,+}(t)+\mathcal{V}_{p,-}(t)}{2}\right]\right)$$

$$\cos n\left(\frac{\mathcal{V}_{p,+}(t)-\mathcal{V}_{p,-}(t)}{i}\right)$$

on S, or:

$$\log\varphi(t)/2+\mathcal{V}_i(t)-[\mathcal{V}_{p,+}(t)+\mathcal{V}_{p,-}(t)]/2 = \text{constant}, \qquad (4)$$

the same real constant on all the arcs of S, has a real part smaller than this constant on $C_f\setminus S$.

For derivatives:

$$(\log\varphi(z))'/2+\mathcal{V}_i'(z)+\fint_S\frac{d\mu_p(t)}{z-t}=0 \text{ on } z\in S. \qquad (5)$$

3.2. Conditions on a single arc.

Suppose that we know that $\fint_\alpha^\beta \frac{d\mu_p(t)}{z-t} = g(z)$, with g analytic in some domain (the arc $[\alpha,\beta]$ is not yet known).

3.2.1. *A little bit of Chebyshev polynomials calculus.* N.B. Ullman

$\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{-1/2}$ is the constant term of the Chebyshev expansion of $g(t)/(z-t)$.

Let $g_0/2+\sum_1^\infty g_nT_n$ be the expansion of g. $g_0=0$, $\qquad g_1=\dfrac{4}{\beta-\alpha}$.

$$\mathcal{V}_p'(z)=\sum_{n=1}^\infty g_n\rho^n. \qquad (6)$$

$$\frac{\rho+\rho^{-1}}{2}=\frac{2z-\alpha-\beta}{\beta-\alpha}, \qquad (7)$$

4. Rational interpolation to $\exp(nB_1z+nB_2z^2)$.

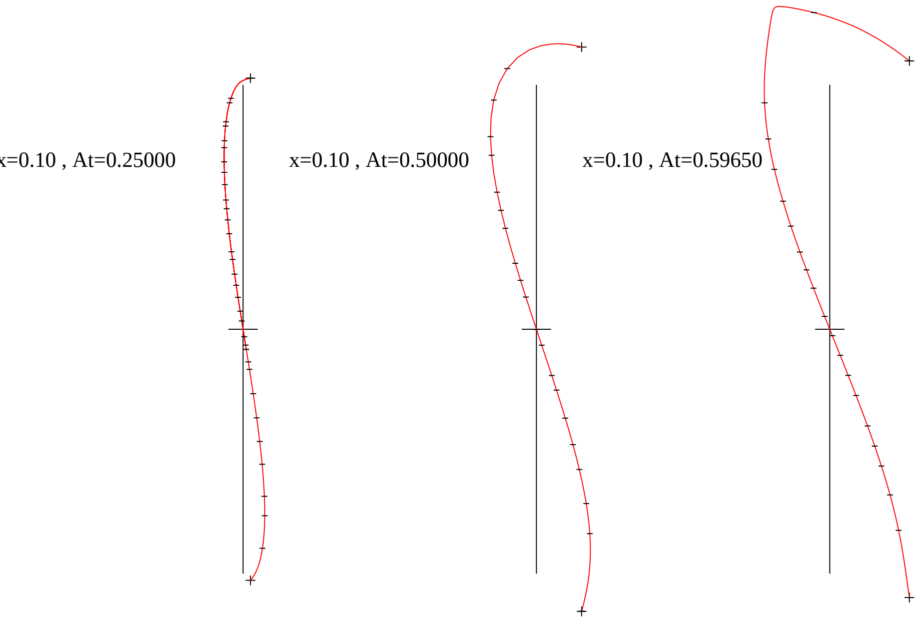
This very interesting rational interpolation appears in special nonlinear Schrödinger problems ([6, 8] and remarks by J. Nuttall in [4]) .

4.1. The single arc case.

$$\frac{\rho_k+\rho_k^{-1}}{2}=\frac{2I_k-\alpha-\beta}{\beta-\alpha}, k=1,2, \qquad (8)$$

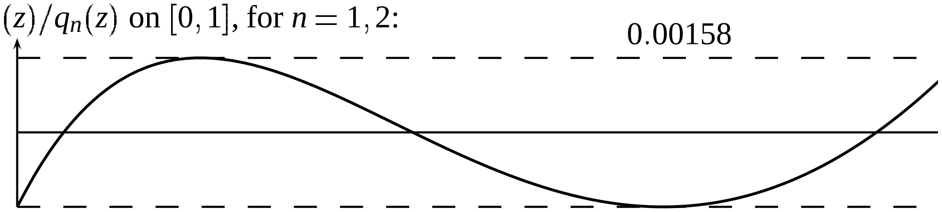
$$\begin{aligned}\mathcal{V}_p(z) &= \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho)-(z-I_2)\log(1-\rho_2\rho)]\\ &-\frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2}{2}-\log\rho.\end{aligned} \qquad (9)$$

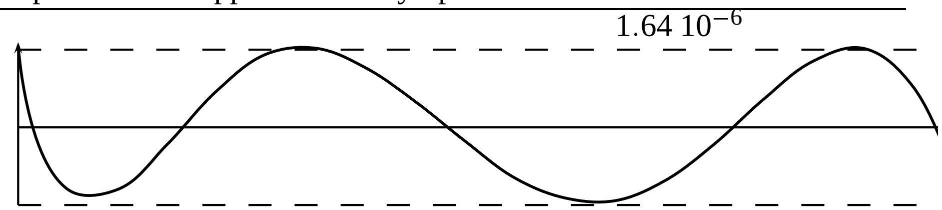
4.2. First caustic.



5. Best rational approximation to $e^{-(An+B)z}$ on a real interval

Best rational approximation to $\exp(-z)$ on a given real interval, say $[0,c]$ has a strict equioscillating error function, as seen here with $e^z - p_n(z)/q_n(z)$ on $[0,1]$, for $n=1,2$:





5.1. Root asymptotics.

Finally, the error decreases like ρ^n, with

$$\log\frac{1}{\rho} = \pi\frac{\alpha\gamma(\mathsf{K}-\mathsf{E})(\mathsf{K}'-\mathsf{E}')-\mathsf{E}\mathsf{E}'}{(\alpha\gamma-1)\mathsf{E}(\mathsf{K}-\mathsf{E})} \tag{10}$$

5.2. Strong asymptotics .

Consider rational approximants to functions $f^n g$, and suppose that the Hermite-Walsh error formula can already be written as

$$f^n(z)g(z) - \frac{p_n(z)}{q_n(z)} \sim e^{\mathcal{W}_n(z)}\frac{1}{2\pi i}\int_C f^n(t)g(t)e^{-\mathcal{W}_n(t)}\frac{dt}{z-t},$$

Aptekarev [1] established in some cases a more accurate picture $\mathcal{W}_n = 2n\mathcal{V} + \widetilde{\mathcal{V}} + o(1)$ (strong asymptotics, also called first order asymptotics by Nuttall). I give here a probably very sloppy account of Aptekarev's wonderful results (to be available soon):

> *Also sprache Aptekarev*: $\widetilde{\mathcal{V}}$ is (multivalued) analytic outside $E\bigcup F$, with a period $2\pi i$ about F, and $-2\pi i$ about E, corresponding to a positive unit charge on F, and a negative unit charge on E, with $\widetilde{\mathcal{V}}_+ + \widetilde{\mathcal{V}}_-$ constant on E, $\widetilde{\mathcal{V}}(z)_+ + \widetilde{\mathcal{V}}(z)_- + 2\log g(z) =$ another constant on F, and finally $\widetilde{\mathcal{V}}(z) =$ const. $+o(1)$ when $z\to\infty$ (if E and F are bounded).

Moreover, the error norm is $E_n \sim 2\rho^n\tilde{\rho}$, where $2\log\tilde{\rho} = \mathrm{Re}\,\{(\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z))_E - [\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z)+2\log g(z)]_F\}$.

$\rho_0 = \exp\left(-\dfrac{\pi}{2}\dfrac{\mathsf{K}'}{\mathsf{K}}\right)$. And for any B, $\mathcal{V}_B = \dfrac{2B}{A}\mathcal{V} + \left(1-\dfrac{2B}{A}\right)\mathcal{V}_0$ does the trick, see Meinguet [7] for such relations. So,

$$\rho_B = \rho^{B/A}\rho_0^{(1-2B/A)} .$$

and we just have to get $\rho_0 = \exp(-1/C)$, where C is the plain condenser capacity of $(E,\widetilde{F})$.

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