Seminar series on

Rational approximations and systems theory.

February-March 2002

Asymptotic convergence rates of rational interpolation to exponential functions.

The present file is **http://www.math.ucl.ac.be/~magnus/num3/rsummary.ps** and **pdf**
For more, see **http://www.math.ucl.ac.be/~magnus/num3/m3xxx00.pdf** and **ps** and references therein.

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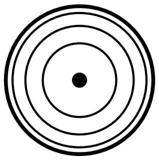
Various instances of behaviour of error of rational approximations in the complex plane.

CONTENTS

1. Taylor, polynomial, and rational interpolation.

1.1. Taylor expansions.

The Taylor series expansion of a function with finite convergence domain shows “typically” almost circular level lines of equal approximation:



explained by a convenient representation of the error

$$f(z) - \sum_0^n c_k z^k = K_n(z)(z/Z)^n, \qquad |z| < |Z| = R, \tag{1}$$

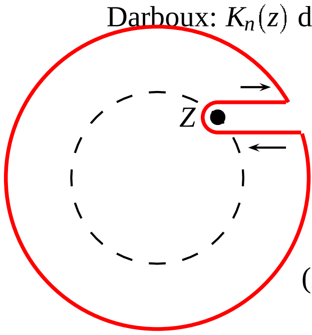
where K_n is "typically" slowly variable in n. What is meant by "typical" must be estimated on particular classes of functions. The only general truth here is that K_n is bounded by a slowly variable[1] function of n when $|z| < |Z| = R$:

$$K_n(z) = \left(\frac{Z}{z}\right)^n \frac{1}{2\pi i} \int_{|t|=r} f(t) \frac{z^{n+1}}{t^n(t-z)}\, dt,$$

with $|z| < r < R$, and r arbitrarily close to R. For some functions f, an infinite subset $\{K_{n_i}\}_i$ may be much smaller than expected, that's why some special classes of functions f will be described (sometimes a *single* function…) when accurate asymptotic estimates of (1) will be needed.

So, we stretch integration contours as far as possible, so to have smallest possible integrands, in order to catch an idea of the integral. This is a way to cope with the additive logic of integration, where the same result may be achieved with many terms of different phases. But if there are not many terms, each of them has to cooperate in the same direction, more or less.

Darboux: $K_n(z)$ depends on the behaviour of $f_+(t) - f_-(t)$ in neighbourhoods of the singular points



of f on the convergence circle. The main part of the error integral is

$$\frac{z^{n+1}}{2\pi i} \int_0^\varepsilon \frac{f_+(Z+u) - f_-(Z+u)}{(Z+u)^n(Z+u-z)}\, du$$

$$\sim \frac{z^{n+1}}{2\pi i Z^n(Z-z)} \int_0^\varepsilon [f_+(Z+u) - f_-(Z+u)] \exp(-nu/Z)\, du$$

(Laplace transform)

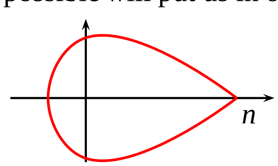
For instance, if $f(t) \sim A(t-Z)^\alpha$ near Z,

$$f(z) - \sum_0^n c_k z^k \sim \frac{z^{n+1} A(1 - e^{2\pi i\alpha}) Z^{\alpha+1} \Gamma(\alpha+1)}{2\pi i Z^n (Z-z) n^{\alpha+1}}$$

(Watson's lemma for Laplace transforms).

So, refined asymptotics will have to deal with differences $f_+(t) - f_-(t)$ on the the two sides of cuts.

For the ***exponential*** function, there is no finite convergence radius! Stretching the contour as far as possible will put us in orbit! Of course, the size of e^t will now have to be considered too.



The error integral of $z^{n+1} e^t t^{-n} (t-z)^{-1} dt$ is best analysed on a contour of radius $|t| = n$, whence the main behaviour $z^{n+1}(e/n)^n$ (when $|z|$ is much smaller than n). Remark the famous 100% relative error curve $|e^z| = |z|^n (e/n)^n$ where zeros of the approximants can be found (Szegő).

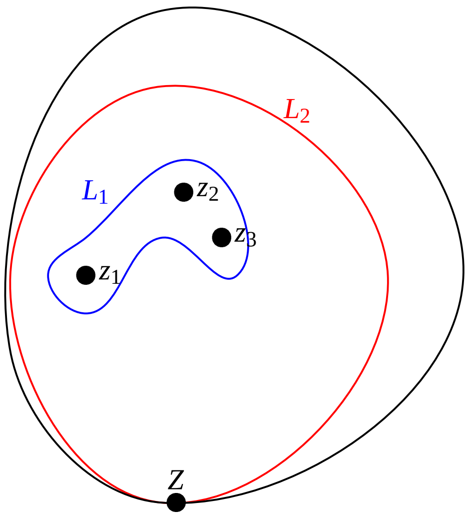
1.2. **Interpolatory (Jacobi) expansions.**

We consider repeated (confluent) interpolation at a finite number of points $z_1, \ldots, z_k$, amounting in a modified polynomial series (Jacobi expansion)

$$\sum_{m=0}^\infty d_m(z)[(z-z_1)\ldots(z-z_k)]^m,$$

where d_m is a polynomial of degree $< k$.

[1] slowly variable = less than exponentially variable.



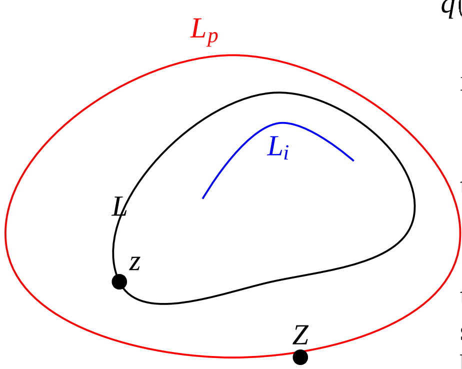
What we have is similar to a power expansion, with powers of $(z-z_1)\dots(z-z_k)$. And the relevant contour integral is examined after stretching the contour so to have $|(t-z_1)\dots(t-z_k)|$ as large as possible on it: $f(z)-$ interpolant of degree $n=$
$$\frac{[(z-z_1)\dots(z-z_k)]^{n'}\rho_n(z)}{2\pi i}\int_C \frac{(t-z)^{-1}f(t)\,dt}{[(t-z_1)\dots(t-z_k)]^{n'}\rho_n(t)}$$
where $n'=\lfloor n/k\rfloor$, and where ρ_n is the product of less than k factors $z-z_j$. The main n^{th} power involved in the error at z is $\left(\dfrac{\Phi(z)}{\Phi(Z)}\right)^n$, with $\Phi(z)=[(t-z_1)\dots(t-z_k)]^{1/k}$, and where $|\Phi(Z)|=\min|\Phi(t)|$ on $t\in C$. We also see that $|\Phi(t)|=|\Phi(Z)|$ is the largest ***lemniscate*** L_2 within the contour C.
Let L_1 be the lemniscate $\{u:|\Phi(u)|=\text{constant}=|\Phi(z)|\}$ containing z. The error level lines are the lemniscates $|\Phi(z)|=\text{constant}$.

1.3. **General rational interpolation with given poles. Condenser capacity.**

…, tout entier à une idée
qui lui était venue sur les *potentiels*.

Alphonse Allais (from *Madrigal manqué*)

At the end of this line of thought, one puts $m+1$ interpolation points on a locus E, and n poles on a locus F.
The error at z is
$$f(z)-\frac{p(z)}{q(z)}=\frac{\prod(z-z_j)}{2\pi i q(z)}\int_C \frac{q(t)\,f(t)\,dt}{(t-z)\prod(t-z_j)} \tag{2}$$



involving mainly $\left(\dfrac{\Phi(z)}{\Phi(t^*)}\right)^n$, with $\Phi(z)=\exp\mathcal{V}(z)=\exp[\mathcal{V}_p(z)-\mathcal{V}_i(z)]$, where $\mathcal{V}_p$ and $\mathcal{V}_i$ are the (complex) ***potentials*** of the distributions of the poles and interpolation points: $\mathcal{V}_p(z):=\displaystyle\int_C\log\frac{1}{z-t}\,d\mu_p(t)$, $\mathcal{V}_i(z):=\displaystyle\int_I\log\frac{1}{z-t}\,d\mu_i(t)$. We now have flexibility enough [10] to make the set of interpolation points L_i and the set of poles L_p sets where Re $\mathcal{V}=\log|\Phi|$ take two constant values (Dirichlet boundary-value problem for Re $\mathcal{V}$). Then,
$$\frac{|\Phi(z)|}{|\Phi(Z)|}=\exp[\mathrm{Re}\,(\mathcal{V}(z)-(\mathrm{Re}\mathcal{V}\text{ on }L_i)]=\exp\left(\frac{-1}{\mathrm{cap}\,(L,L_i)}\right),$$
where cap (L,L_i) is the ***capacity*** of the ***condenser*** (L,L_i).

1.4. **The problem of rational interpolation at** $m+n+1$ **points, orthogonal polynomials.**

The rational interpolation setting above, with given poles and interpolation points, is a well-conditioned problem, somewhat like the boundary value problems with Dirichlet data on a part of the boundary, and Neumann data on the complementary part.

The seemingly inocuous alternative interpolation problem, to construct a numerator of degree $\leqslant m$, and a denominator of degree $\leqslant n$, such that the ratio interpolates at $m+n+1$ points, is an ill-conditioned problem (when m and n are large), somewhat like an elliptic boundary value problem with Dirichlet ***and*** Neumann data on the same part of the boundary.

Then, why venture such dangerous hasards? Because we don't have to look for the boundary of a region of analyticity, we expect that the interpolant will find it through the locus of its poles! So, full interpolation is a tool of discovery. This was expecially clear when ***Padé*** approximants (the full confluent case) were heavily used some decades ago.

We look for p_m and q_n such that $\dfrac{p_m(z_i)}{q_n(z_i)} = f(z_i)$ for $i=0,\dots,m+n$. OK, suppose the denominator q_n known (that's the hard part), then p_m interpolates $q_n f$. But a polynomial interpolant at $m+n+1$ points will normally have a degree as high as $m+n$! That the degree of p_m is actually $\leqslant m$ represents the set of conditions for the denominator q_n. For analytic functions, the interpolant is

$$p_m(z) = \frac{1}{2\pi i}\int_{C_f} q_n(t)\sum_{j=0}^{m+n}\frac{L_j(z)}{t-z_j} f(t)\,dt = \frac{1}{2\pi i}\int_{C_f} q_n(t)\left[\frac{1}{t-z} - \frac{\prod_0^{m+n}(z-z_j)}{(t-z)\prod_0^{m+n}(t-z_j)}\right] f(t)\,dt,$$

where C_f is a valid contour with the z_j's as interior points. So, p_m is normally $O(z^{m+n})$ for large z, it only be $O(z^m)$ as it should if $\displaystyle\int_{C_f}\frac{q_n(t)f(t)}{(t-z)\prod_0^{m+n}(t-z_j)}\,dt$ is $O(z^{-n-1})$, so, as $(t-z)^{-1} = -z^{-1} - tz^{-2} - \cdots + t^n z^{-n}(z-t)^{-1}$, if

$$\int_{C_f} q_n(t)t^j w_n(t)\,dt = 0, \qquad j=0,\dots,n-1, \tag{3}$$

where $w_n(t) = \dfrac{f(t)}{\prod_0^{m+n}(t-z_j)}$.

What we have are conditions of ***orthogonality***, although the complex facors in (3) represent by no means "nice" orthogonality.

2. Known rational interpolations to the exponential function.

2.1. Padé.

for e^z,

$$[m/n] = \frac{1 + \dfrac{m}{m+n}\dfrac{z}{1!} + \dfrac{m(m-1)}{(m+n)(m+n-1)}\dfrac{z^2}{2!} + \cdots + \dfrac{m(m-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(n+1)}\dfrac{z^m}{m!}}{1 - \dfrac{n}{m+n}\dfrac{z}{1!} + \dfrac{n(n-1)}{(m+n)(m+n-1)}\dfrac{z^2}{2!} - \cdots + (-1)^n\dfrac{n(n-1)\cdots 2.1}{(m+n)(m+n-1)\cdots(m+1)}\dfrac{z^n}{n!}} \tag{4}$$

$$\begin{aligned} e^z\,\text{den.} - \text{num.} &= \frac{(-1)^n}{(m+1)\cdots(m+n)}\sum_{k=m+n+1}^{\infty}\frac{(k-m-1)\cdots(k-m-n)}{k!}z^k \\ &= \frac{(-1)^n}{(m+n)!}\left[\int_{-\infty}^{z} - \int_{-\infty}^{0} = \int_0^z e^t(z-t)^m t^n\,dt\right] \end{aligned} \tag{5}$$

Exponential behaviour of numerator and denominator has been much worked, especially the distribution of zeros and poles. Saff & Varga remark that, when $m\sim n$, these distributions had already been examined by Olver in a study of Bessel functions.

Integrals of the form (5) behave for large n as value at saddlepoint. With $m\sim n$, saddlepoint is a root of

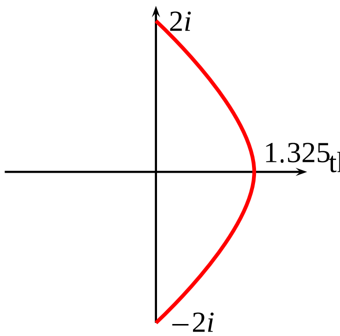
$$1 + \frac{n}{t-z} + \frac{n}{t} = 0, \text{ whence } t = \frac{z}{2} - n + \sqrt{\frac{z^2}{4} + n^2},$$

with some choice for the square root (see later), and $(z-t)t = 2n^2\left(\sqrt{1+\dfrac{z^2}{4n^2}}-1\right)$.

Denominator behaves like n^{th} power of

$$\frac{1+\sqrt{1+\dfrac{z^2}{4n^2}}}{2}\ \exp\left(1-\frac{z}{2n}-\sqrt{1+\frac{z^2}{4n^2}}\right), \tag{6}$$

where the value of the square root is 1 at $z=0$. When z increases, the value of (6) becomes very small and must be replaced by the same formula with the other choice of the square root as soon as the new formula has an absolute value which is larger than the former one. This happens near $z/n = 1.3255$. In the

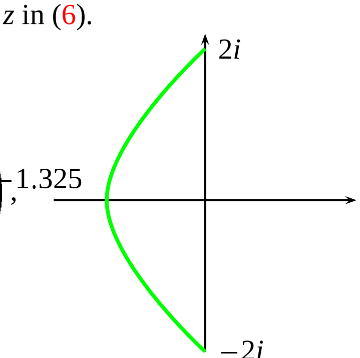
$z/n-$plane: 

the formula (6) holds with a continuous square root outside the shown arc, which is the locus where the two formulas have the same absolute value, also the limit of the ***poles*** of the approximant. The equation of the arc is $|w(z)| = 1$, where

$$w(z) = \sqrt{\frac{\sqrt{1+\frac{z^2}{4n^2}}-1}{\sqrt{1+\frac{z^2}{4n^2}}+1}}\ \exp\left(\sqrt{1+\frac{z^2}{4n^2}}\right) = \frac{(z/2n)\exp\left(\sqrt{1+z^2/(4n^2)}\right)}{1+\sqrt{1+\frac{z^2}{4n^2}}}$$

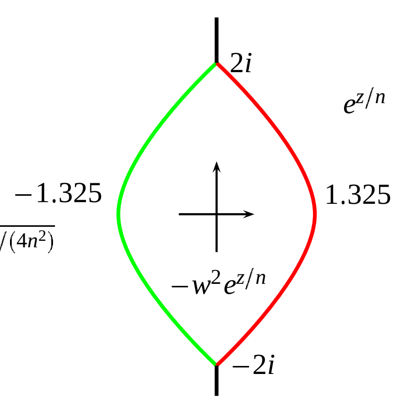
Remark the square root behaves like $-z/(2n)$ for large z in (6).

Numerator: $\dfrac{1+\sqrt{1+\dfrac{z^2}{4n^2}}}{2}\ \exp\left(1+\dfrac{z}{2n}-\sqrt{1+\dfrac{z^2}{4n^2}}\right)$,



Finally, for the error e^z- approximant, the powers are

$$\frac{1+\sqrt{1+z^2/(4n^2)}}{1-\sqrt{1+z^2/(4n^2)}}\ e^{z/n-2\sqrt{1+z^2/(4n^2)}}$$



2.2. **Rational interpolation at equidistant points (Iserles).** [3]

As far as we only need e^{Az} at $z = z_0, z_0+h, \ldots, z_0+(m+n)h$,

$$\begin{aligned} e^{Az} &= (\boldsymbol{I}+\boldsymbol{\Delta})^{(z-z_0)/h} e^{Az_0} \\ &= \sum_{k=0}^{m+n} \binom{(z-z_0)/h}{k} \boldsymbol{\Delta}^k e^{Az_0} \\ &= \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{1}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

which we multiply by the denominator $Q(z) = \sum_{j=0}^{n} q_j (z-z_0)\cdots(z-z_0-(j-1)h)$, using

$$(z-z_0)(z-z_0-h)\cdots(z-z_0-(j-1)h)e^{Az} = \\ e^{A(z_0+jh)} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^{k-j} \frac{1}{(k-j)!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

$$Q(z)e^{Az} = e^{Az_0} \sum_{k=0}^{m+n} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{C(k)}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h),$$

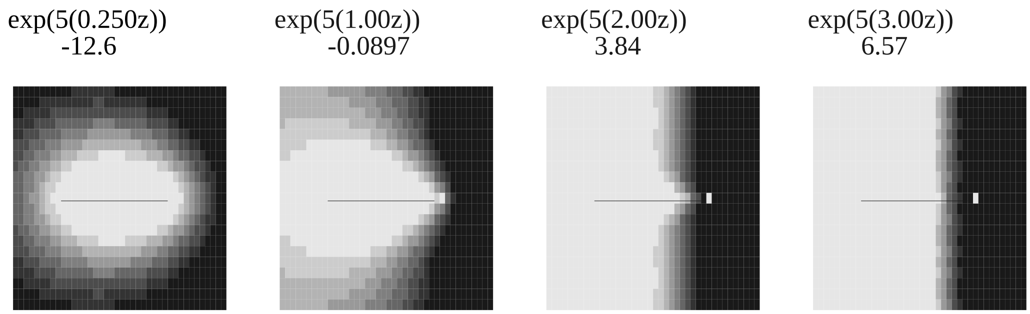
where $C(k) = \sum_{j=0}^{n} q_j e^{Ajh} \left(\frac{e^{Ah}-1}{h}\right)^{-j} \frac{1}{(k-j)!}$ is a polynomial of degree n in k, which must vanish at $k = m+1, m+2, \ldots, m+n$,

$$\begin{aligned} P(z) &= e^{Az_0} \sum_{k=0}^{m} \left(\frac{e^{Ah}-1}{h}\right)^k \binom{m}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \\ Q(z) &= \sum_{k=0}^{n} \left(\frac{e^{-Ah}-1}{h}\right)^k \binom{n}{k} (m+n-k)!(z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \end{aligned}$$

and, formally:

$$Q(z)e^{Az} - P(z) = \\ e^{Az_0} m!(-1)^n \sum_{k=m+n+1}^{\infty} \left(\frac{e^{Ah}-1}{h}\right)^k \frac{(k-m-1)(k-m-2)\cdots(k-m-n)}{k!} (z-z_0)(z-z_0-h)\cdots(z-z_0-(k-1)h), \tag{7}$$

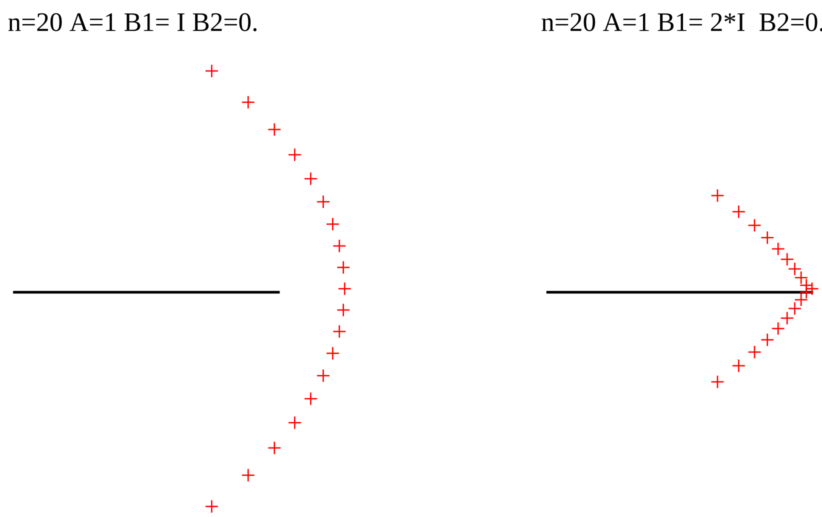
We look at the performance of some examples of the region of good approximation in the complex plane, coloured in light gray:



colouring is made with respect of the average of $\log|\exp(Az)-P(z)/Q(z)|$ in the square $[-2,2]\times[-2,2]$. The degrees of P and Q are here 5 and 4. When A is small, the region is an oval around the locus of the interpolation points (here, the interval $[-1,1]$ shown by a thin horizontal black line).

The interpolation points should appear as bright white dots, but they are hardly visible in somewhat big pixels, if colouring is made according to an arbitrary point of the pixel. This chosen point happens to be an actual interpolation point only for the endpoints, whence the rightmost interpolation point looking like a beacon in a dark environment.

Somewhat similarly poles may even enter the locus of interpolation points. Here, the poles of P/Q (degrees $20/19$) interpolating $\exp(20az)$, with $a=1$ and $a=2$:



So, the locus of poles of P/Q of degrees $m\sim n$ and n approximating $\exp(naz)$ enters the locus $[-1,1]$ of interpolation points when a becomes larger than a number slightly smaller than 2. Such features will be explained.

Rough asymptotics. Let $\zeta := \dfrac{z-z_0-(m+n)h/2}{nh}$. Then, with $m\sim n$, we intend to follow things at constant Ah and ζ, i.e., a fixed exponential and z expanding linearly with n, or A increasing linearly with n, and $2n$ interpolating points filling a fixed segment $[z_0, z_0+2nh]$. Remark that this segment of interpolation points is $-1\leqslant\zeta\leqslant 1$.

One finds that the numerator, denominator, and the error behave like the n^{th} powers of

$$\begin{array}{lll} \text{num.} \quad e^{-Az_0}P(z) & \exp\left[\zeta\log\left(e^{Ah/2}\dfrac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\zeta-1}\right)+\dfrac{Ah}{2}+\log(\gamma+\sqrt{\sigma^2\zeta^2+1})\right] \\ \text{den.} \quad Q(z) & \exp\left[\zeta\log\left(e^{-Ah/2}\dfrac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\zeta-1}\right)-\dfrac{Ah}{2}+\log(\gamma+\sqrt{\sigma^2\zeta^2+1})\right] \\ \text{err. } e^{-Az_0}\left(e^{Az}-\dfrac{P(z)}{Q(z)}\right) & \exp\left[\zeta\log\left(e^{Ah}\dfrac{\gamma\zeta-\sqrt{\sigma^2\zeta^2+1}}{\gamma\zeta+\sqrt{\sigma^2\zeta^2+1}}\right)+Ah+\log\left(\dfrac{\gamma+\sqrt{\sigma^2\zeta^2+1}}{\gamma-\sqrt{\sigma^2\zeta^2+1}}\right)\right] \\ & \text{in the convergence region} \end{array} \tag{8}$$

where $\gamma=\cosh Ah/2$, $\sigma=\sinh Ah/2$, and where determinations of the logarithms and of the square root have to be chosen appropriately.

At least for large z, we know that P and Q must behave like z^n, possible only if different choices of the square root are taken in the first two rows of (8), with $\sqrt{\sigma^2\zeta^2+1}\sim\sigma\zeta$ for P, $-\sigma\zeta$ for Q.

The region where the same determination of the square root holds in the asymptotic behaviours of P and Q is simply the region of good approximation! Indeed, one finds for the ration the n^{th} power of $\exp(Ah(\zeta+1))$, which makes $\exp(A(z-z_0))$.

When Ah is small, we almost have the Padé situation of the figure of p. 5 (take $\zeta\to\infty$ and $\sigma\zeta\sim z/(2n)$).

Various interesting situations occur, the wildest situation being $e^{Ah}=-1$: we then interpolate merely the sequence $1,-1,1,-1,\dots$ at $z_0,z_1,\dots,z_{m+n}$ by p/q of degrees m and n, without any reference to an exponential function anymore!

Moreover, the solution of the Cauchy problem (i.e., find p and q such that $q(z)f(z)-p(z)=0$ at $z_0,\dots,z_{m+n}$) is then immediate: $q+p$ must vanish at $z_1,z_3,\dots$, and $q-p$ vanishes at $z_0,z_2,\dots$:

$$\frac{p(z)}{q(z)}=\frac{c(z-z_1)(z-z_3)\cdots-c'(z-z_0)(z-z_2)\cdots}{c(z-z_1)(z-z_3)\cdots+c'(z-z_0)(z-z_2)\cdots},$$

where one of the two numbers c or c' may very well vanish if it is the only way to achieve degrees $\leqslant m$ and n!

3. **Asymptotic features of rational interpolation.** #♩

3.1. **According to Gončar-Rahmanov-Stahl (a sloppy rendering).**

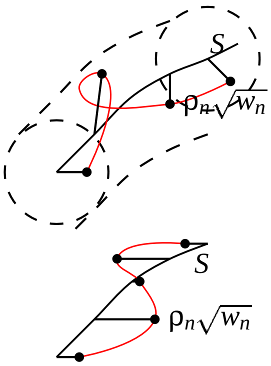
Interpolation to $f_n(z)=\displaystyle\int_{C_f}\frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt$ at $z_0,\dots,z_{m+n}$ by p_m/q_n yields

$$f_n(z)-\frac{p_m(z)}{q_n(z)}=\frac{\prod_0^{m+n}(z-z_j)}{q_n^2(z)}\int_{C_f}\frac{q_n^2(t)}{\prod_0^{m+n}(t-z_j)}\,\frac{\varphi_0(t)\varphi^n(t)}{z-t}\,dt,$$

where q_n is (formally) orthogonal with respect to $w_n(t):=\dfrac{\varphi_0(t)\varphi^n(t)}{\prod_0^{m+n}(t-z_j)}$ on C_f, as in (3).

Well, we expect that most of the poles of q_n will tend to a set of arcs S, with a limit distribution μ_p, that C_f may be modified within the closure of the domain where φ_0 and φ are analytic, so that $S\subseteq C_f$.

♩ *On the support of μ_p, q_n is almost a Szegő orthogonal polynomial! which means that $\pm q_n(t)\sqrt{w_n(t)}$ has slowly varying phase and absolute value there.*



Indeed, a monic polynomial ρ_n orthogonal in Szegő's sense with respect to $|w_n|$ on C_f minimizes its (square) norm $\int_{C_f} |\rho_n(t)|^2 |w_n(t)| |dt|$. The absolute value $|\rho_n\sqrt{w_n}|$ will remain almost constant along S, but the argument of $\rho_n\sqrt{w_n}$ will normally have fast variation. So, $|\rho_n^2 w_n|$ is normally not the same thing as $\rho_n^2 w_n$.

But if there is a particular C_f where $\rho_n\sqrt{w_n}$ happens to be –almost– **real**, then q_n may be expected to be close to ρ_n.

Sloppy asymptotic explanation with (complex) potentials $\mathcal{V}_p(z) := \int_{\text{supp}\ \mu_p} \log\dfrac{1}{z-t}\, d\mu_p(t)$ and $\mathcal{V}_i(z) := \int_{\text{supp}\ \mu_i} \log\dfrac{1}{z-t}\, d\mu_i(t)$ (interpolation points), so that $q_n(z) \sim \exp(-n\mathcal{V}_p(z))$ when $z \notin \text{supp}(\mu_p)$, $\prod_0^{m+n}(z-z_j) \sim \exp(-2n\mathcal{V}_i(z))$, and $q_n(z) \approx \exp(-n\mathcal{V}_{p,+}(z)) + \exp(-n\mathcal{V}_{p,-}(z))$ on $\text{supp}(\mu_p)$. Then,

$$q_n(t)\frac{\varphi^{n/2}(t)}{\sqrt{\prod_0^{m+n}(t-z_j)}} \approx [\exp(-n\mathcal{V}_{p,+}(t)) + \exp(-n\mathcal{V}_{p,-}(t))]\ \exp(n\log\varphi(t)/2 + n\mathcal{V}_i(t))$$

$$\approx \exp\left(n\left[\frac{\log\varphi(t)}{2} + \mathcal{V}_i(t) - \frac{\mathcal{V}_{p,+}(t)+\mathcal{V}_{p,-}(t)}{2}\right]\right)\ \cos n\left(\frac{\mathcal{V}_{p,+}(t)-\mathcal{V}_{p,-}(t)}{i}\right)$$

on S, or:

$$\log\varphi(t)/2 + \mathcal{V}_i(t) - [\mathcal{V}_{p,+}(t)+\mathcal{V}_{p,-}(t)]/2 = \text{constant}, \tag{9}$$

the same real constant on all the arcs of S, has a real part smaller than this constant on $C_f \setminus S$.

For derivatives:

$$(\log\varphi(z))'/2 + \mathcal{V}_i'(z) + \fint_S \frac{d\mu_p(t)}{z-t} = 0 \text{ on } z \in S. \tag{10}$$

Remark that the (complex conjugate of) the derivative $(\log\varphi(z))'/2 + \mathcal{V}_i'(z) - \mathcal{V}_p'(z)$ on the two sides of S gives the gradient of the real potential $\text{Re}[\log\varphi(z)/2 + \mathcal{V}_i(z) - \mathcal{V}_p(z)]$, and has ***opposite*** values $\pm\pi i\overline{\mu'(z)}$ on the two sides of S, from (10) and the ***Sokhotskyi-Plemelj*** formulas for $\mathcal{V}_p'$: ***symmetry property*** [1,2,9, ,etc.].

3.2. **Conditions on a single arc.**

Let the function (often associated to a distribution of poles) $\mathcal{V}_p'(z) = \int_\alpha^\beta \dfrac{d\mu_p(t)}{z-t}$. Suppose that we know that

$$\fint_\alpha^\beta \frac{d\mu_p(t)}{z-t} = g(z), \tag{11}$$

with g analytic in some domain (the arc $[\alpha,\beta]$ is not yet known). The trick is to multiply $\mathcal{V}_p'$ by a function $[(z-\alpha)(z-\beta)]^{\gamma/2}$ taking *opposite* values on the two sides of $[\alpha,\beta]$. We consider only $\gamma = 1$ and $\gamma = -1$. Also, $[(z-\alpha)(z-\beta)]^{\gamma/2}$ is defined to be continuous outside the arc, and behaves like z^γ for large z.

The solution is

$$\mathcal{V}_p'[(z-\alpha)(z-\beta)]^{\gamma/2} - \delta_{\gamma,1} = \frac{1}{\pi i}\int_\alpha^\beta \frac{g(t)[(t-\alpha)(t-\beta)]_-^{\gamma/2}}{z-t}\, dt, \qquad \gamma = \pm 1. \tag{12}$$

It may help to realize that the phase of $\dfrac{\beta-\alpha}{[(t-\alpha)(t-\beta)]_-^{1/2}}$ is exactly the one of $+i$ on the rectilinear segment $[\alpha,\beta]$.

Some questions: the -1 in the left-hand side of (12) when $\gamma = 1$ is needed from $\mathcal{V}_p''(z) = 1/z + O(1/z^2)$ for large z. But the two sides of (12) when $\gamma = -1$ should be $\sim 1/z^2$ for large z, everything works only if

$$\int_\alpha^\beta \frac{g(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}} = 0, \quad \int_\alpha^\beta \frac{tg(t)\,dt}{[(t-\alpha)(t-\beta)]_-^{1/2}} = \pi i. \tag{13}$$

The two forms of (12) then agree, either with $\gamma = -1$, or $\gamma = 1$. It will also be useful to check that, as (11) is a plain integral when $z = \alpha$ and $z = \beta$, one has $\mathcal{V}_p''(\alpha) = g(\alpha)$, and $\mathcal{V}_p''(\beta) = g(\beta)$.

3.2.1. *A little bit of Chebyshev polynomials calculus.* N.B. Ullman
$\mathcal{V}_p''[(z-\alpha)(z-\beta)]^{-1/2}$ is the constant term of the Chebyshev expansion of $g(t)/(z-t)$.
Let $g_0/2 + \sum_1^\infty g_n T_n$ be the expansion of g. Remark that (13) becomes

$$g_0 = 0, \qquad g_1 = \frac{4}{\beta - \alpha}. \tag{14}$$

We need the expansion of $1/(z-t) = X_0/2 + \sum_1^\infty X_n T_n$: $X_n = X_0 \rho^n$, where ρ is a root of

$$\frac{\rho + \rho^{-1}}{2} = \frac{2z - \alpha - \beta}{\beta - \alpha}, \tag{15}$$

normally with $|\rho| < 1$, but this will have to be discussed later. The value of X_0 comes from $n = 0$:

$$X_0 = \frac{8}{(\beta-\alpha)(\rho^{-1} - \rho)}.$$

Remark that $[(z-\alpha)(z-\beta)]^{1/2} = (\beta-\alpha)^2(1-\rho^2)^2/(16\rho^2)$, so that

$$\mathcal{V}_p''(z) = \sum_{n=1}^\infty g_n \rho^n. \tag{16}$$

The two determinations of $\mathcal{V}_p''$ on the two sides of the cut $[\alpha, \beta]$ are obtained with the *two* roots ρ and $1/\rho$ of (15). One checks that the arithmetic mean is indeed

$$(\mathcal{V}_{p,+}''(z) + \mathcal{V}_{p,-}''(z))/2 = \sum_1^\infty g_n(\rho^n + \rho^{-n})/2 = \sum_1^\infty g_n T_n = g(z).$$

As for the discontinuity along the cut,

$$\pm \pi i \mu_p'(z) = \mathcal{V}_{p,-}''(z) + \mathcal{V}_{p,+}''(z) = \sum_1^\infty g_n(\rho^{-n} - \rho^n) = \frac{4}{\beta-\alpha}[(z-\alpha)(z-\beta)]^{1/2} \sum_1^\infty g_n U_{n-1}(z), \tag{17}$$

it appears as a kind of harmonic conjugate to g.

3.2.2. *Check with* (8).

From (8),

$$-\mathcal{V}_p(z) = \lim_{n\to\infty} \frac{\log Q(z)}{n} = \zeta \log\left(e^{-Ah/2}\frac{\gamma\zeta - \sqrt{\sigma^2\zeta^2+1}}{\zeta - 1}\right) - \frac{Ah}{2} + \log(\gamma + \sqrt{\sigma^2\zeta^2+1})$$
$$= \zeta \log \frac{1 - ie^{-Ah/2}\rho}{1 - ie^{Ah/2}\rho} - \frac{Ah}{2} + \log(\gamma + (\rho + \rho^{-1})/(2i)\,)$$

where ζ is basically our z (managed so that the interpolation points are in $[-1, 1]$), and where $\rho + \rho^{-1} = 2i\sigma\zeta$. Then, the derivative in ζ simplifies into

$$\frac{d\mathcal{V}_p(z)}{d\zeta} = -\log \frac{1 - ie^{-Ah/2}\rho}{1 - ie^{Ah/2}\rho}$$

This matches (15) provided $\alpha = -\beta = i/\sigma = 2i/[\exp(Ah/2) - \exp(-Ah/2)]$ so that $\dfrac{d\mathcal{V}_p(z)}{d\zeta} = \log\dfrac{1-ie^{-Ah/2}\rho}{1-ie^{Ah/2}\rho}$ $= -\sum_{n=1}^{\infty}\dfrac{i^n(e^{-nAh/2}-e^{nAh/2})}{n}\rho^n$. Remark that $g_1 = -i(e^{-Ah/2}-e^{Ah/2}) = 2i\sigma = 4/(\beta-\alpha)$ as it should.

3.3. **Strong asymptotics (Aptekarev).**

Aptekarev [1] has the following accurate asymptotic formulas for formal orthogonal polynomials:

Let the set of complex functions $w_n(z) := \tilde{w}_n(z)\exp(-2nQ(z))$, with $\tilde{w}_n \to \tilde{w}_\infty$ when $n\to\infty$ be such that the boundary value problem for complex potentials

$$Q(t) + [\mathcal{V}_{p,+}(t) + \mathcal{V}_{p,-}(t)]/2 = \Gamma, \qquad t\in S$$

where Γ is a constant, has a solution with $S =$ a single analytic arc of endpoints α and β. Then the monic formal orthogonal polynomials related to w_n on S satisfy

$$q_n(z) \sim C_n\Phi^n(z)\psi(z), \qquad z\notin S,$$

when $n\to\infty$, where $\Phi(z) = \exp(-\mathcal{V}_p(z)+\Gamma)$, ψ is analytic nonvanishing outside S such that

$$\psi_+(t)\psi_-(t)\tilde{w}_\infty(t) = \frac{i}{\sqrt{(t-\alpha)(t-\beta)}_+}, \qquad t\in S,$$

and $C_n = [e^{n\Gamma}\psi(\infty)]^{-1}$. This is exactly the extension of the famous Szegő's theory. Also, $q_n(z)\sim C_n[\Phi_+^n(z)\psi_+(z) + \Phi_-^n(z)\psi_-(z)]$ on S. For the functions of the second kind $R_n(z) := \dfrac{1}{2\pi i}\displaystyle\int_S \frac{q_n(t)w_n(t)}{t-z}\,dt$,

$$R_n(z) \sim \frac{iC_n}{\Phi^n(z)\psi(z)\sqrt{(z-\alpha)(z-\beta)}}$$

for $z\notin S$, and up to the interior of S.

There are also accurate uniform estimates on the whole arc S if the product $\mu_p'(t)\sqrt{(t-\alpha)(t-\beta)}$ is known to be regular.

4. **Rational interpolation to** $\exp(nB_1z + nB_2z^2)$.

This very interesting rational interpolation appears in special nonlinear Schrödinger problems ([8, 6] and remarks by J. Nuttall in [4]) .

4.1. **The single arc case.**

Let the interpolation points be equidistant on $[I_1, I_2]$. Then,

$$g(z) = \int_{I_1}^{I_2}\frac{(I_2-I_1)^{-1}\,dt}{z-t} - \frac{B_1}{2} - B_2z = \frac{\log\dfrac{z-I_1}{z-I_2}}{I_2-I_1} - \frac{B_1}{2} - B_2\left(\frac{\beta-\alpha}{2}\frac{2z-\alpha-\beta}{\beta-\alpha} + \frac{\alpha+\beta}{2}\right) \tag{18}$$

The logarithms have the expansions

$$\log(z-I_k) = \log\frac{\alpha-\beta}{4\rho_k} - 2\sum_1^\infty\frac{\rho_k^n}{n}T_n,$$

where ρ_k is now a root of

$$\frac{\rho_k+\rho_k^{-1}}{2} = \frac{2I_k-\alpha-\beta}{\beta-\alpha}\ ,k=1,2, \tag{19}$$

where $|\rho_k| < 1$ should be the orthodox choice, but which will not be kept in the final formula. Precisely, the closed form is now

$$\mathcal{V}_p''(z) = \sum_1^\infty g_n \rho^n = \frac{2}{I_2 - I_1} \log \frac{1-\rho_1\rho}{1-\rho_2\rho} - B_2 \frac{\beta-\alpha}{2}\rho, \tag{20}$$

with the conditions (14) on g_0 and g_1

$$\frac{\log(\rho_2/\rho_1)}{I_2 - I_1} - \frac{B_1}{2} - B_2\frac{\alpha+\beta}{2} = 0, \tag{21}$$

$$g_1 = 2\frac{\rho_2-\rho_1}{I_2-I_1} - B_2\frac{\beta-\alpha}{2} = \frac{4}{\beta-\alpha}, \tag{22}$$

We integrate (20) along the lines suggested by the exercises of section 3.2.2, p. 10:

$$\mathcal{V}_p(z) = \frac{2}{I_2-I_1}[(z-I_1)\log(1-\rho_1\rho) - (z-I_2)\log(1-\rho_2\rho)] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2}{2} - \log\rho. \tag{23}$$

The *difference* of the two determinations of $\mathcal{V}_p''$ must be $\pm 2\pi i\mu'$:

$$\pm 2\pi i \mu'(z) = \frac{2}{I_2-I_1}\left[\log\frac{1-\rho_1\rho}{1-\rho_1/\rho} - \log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right] - B_2\frac{\beta-\alpha}{2}(\rho-\rho^{-1}), \tag{24}$$

(Nuttall's $\Delta\Psi_2$)

and the cut itself is the locus $\{z : \mu'(z)\,dz \text{ real }\}$, which is integrated as $\{z : \mathcal{V}_{p,+}(z) - \mathcal{V}_{p,-}(z) \text{ pure imaginary }\}$,

$$\frac{2}{I_2-I_1}\left[(z-I_1)\log\frac{1-\rho_1\rho}{1-\rho_1/\rho} - (z-I_2)\log\frac{1-\rho_2\rho}{1-\rho_2/\rho}\right] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2} - 2\log\rho \text{ pure imaginary.} \tag{25}$$

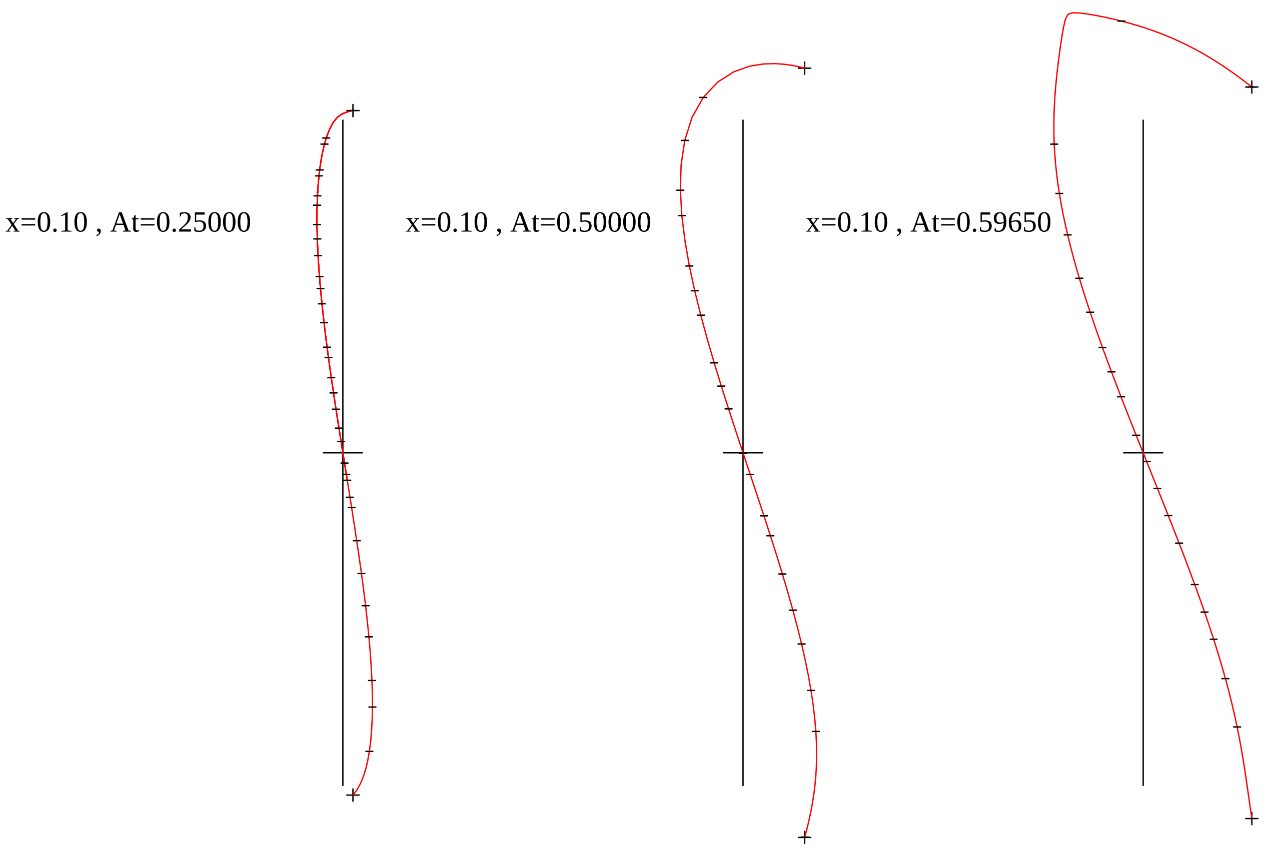
Writing (25) as a function of ρ (using (15) and (19)), we have

$$F(\rho) = \frac{2}{(\rho_2-\rho_1)(1-1/(\rho_1\rho_2))}\left[(\rho-\rho_1)\left(1-\frac{1}{\rho\rho_1}\right)L_1 - (\rho-\rho_2)\left(1-\frac{1}{\rho\rho_2}\right)L_2\right] - \frac{\rho_1\rho_2+1}{\rho_1\rho_2-1}\frac{\rho^2-\rho^{-2}}{2},$$

with $L_1 = \log\dfrac{1-\rho_1\rho}{\rho-\rho_1}$, $L_2 = \log\dfrac{1-\rho_2\rho}{\rho-\rho_2}$, and where, for given B_1, B_2, I_1, I_2, one must determine ρ_1 and ρ_2.

4.2. **First caustic.**

The present setting of the limit set of poles as a single arc joining $z=\alpha$ to $z=\beta$ (or $\rho=-1$ to $\rho=1$) holds as long as $\mu_p'(z)dz$ remains positive on the cut. A critical situation occurs when μ_p' happens to vanish right on the cut, i.e., if dF/dz vanishes at a point where the real part of F vanishes too.



The locus of (x, At) with $B_1 = \pi - 2ix$, $B_2 = -2iAt$, where this happens is called the (first) caustic in [8]. We then have $\rho_1 = R^{-1/2}e^{i\theta}$, $\rho_2 = -R^{1/2}e^{i\theta}$, with real R and θ. For a trial value of At, we look for R and θ such that $(R+1/R)/2 = 2At/\sin 2\theta\ -1$ (from $2x = \log R + (1/R - R)\sin^2\theta$). Knowing ρ_1 and ρ_2, one looks for the zero of the analytic function dF/dz, or $dF/d\rho$. This yields the equation $\mu_p' = 0$ in (24) as

$$L_1 - L_2 = \left(1 + \frac{1}{\rho_1\rho_2}\right)(\rho_2 - \rho_1)\frac{\rho - \rho^{-1}}{2}. \tag{26}$$

One then manages to have the real part of $F = 0$ as well.

Some values:

x	At	$\sin\theta$	ρ_2	$R = -\rho_2/\rho_1$	ρ	F
0.001	0.500973	0.6990	-0.749 -0.733 i	1.098	-1.27846 -0.24352 i	6.282 i
0.010	0.509711	0.6781	-0.854 -0.788 i	1.350	-1.52506 -0.51989 i	6.261 i
0.100	0.596697	0.5838	-1.324 -0.952 i	2.660	-2.06555 -1.23283 i	5.922 i
0.500	1.009193	0.3636	-2.919 -1.139 i	9.817	-2.98988 -2.26747 i	4.371 i
1.000	1.672677	0.2167	-5.311 -1.179 i	29.600	-3.78112 -2.72755 i	2.923 i
2.000	4.344519	0.0798	-14.669 -1.174 i	216.553	-4.47798 -2.79719 i	1.175 i
3.000	11.684073	0.0294	-39.858 -1.171 i	1590.003	-4.58262 -2.77939 i	0.439 i

We see that $\theta \to \pi/4$ when $x \to 0$, and that $\theta \to 0$ when $x \to \infty$, but many features are still unexplained...

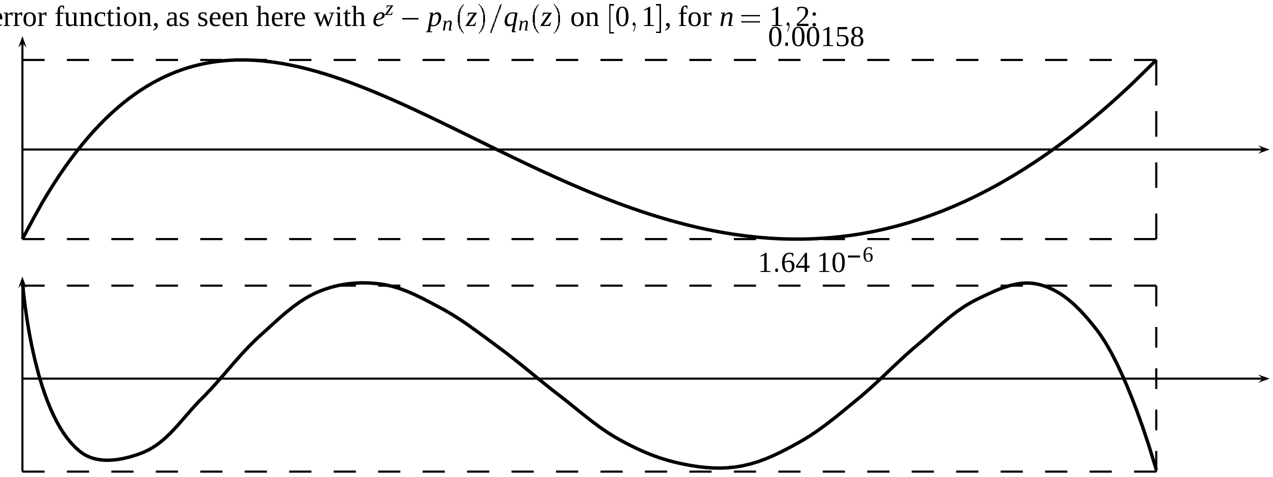
Here is a tentative explanation of the behaviour for large x: as it seems that $|\rho_1| << |\rho| << |\rho_2|$, the logarithms are approximated by $L_1 \approx \rho_1(\rho^{-1} - \rho) - \log\rho$, $L_2 \approx (\rho - \rho^{-1})/\rho_2 + \log\rho$, the equation (26) becomes $\dfrac{\log\rho}{\rho - \rho^{-1}} \approx i\xi$, with $\xi = \theta\sqrt{R}/2$. Also, $F/\theta \approx (\rho - \rho^{-1})[i(\rho + \rho^{-1})/2 - \xi^{-1} - 2\xi]$ must be pure imaginary, making a second equation for ξ and ρ, whence fixed solutions. And

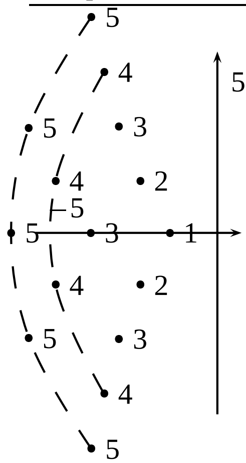
$$\frac{At}{e^x} \approx \frac{\theta R/4}{\sqrt{R}\,\exp(-\theta^2 R/2)} = \frac{\xi}{2}\exp(2\xi^2).$$

```
Script V1.1 session started Thu Feb 17 11:24:23 2000

C:\calc\pari>gp
                   GP/PARI CALCULATOR Version 2.0.12 (alpha)
                           Copyright (C) 1989-1998 by
          C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

? \r expcaus2
...
    rho=-4.5988543935324601646O - 2.7759982804064249263l*I

 F/theta= -1.14496588753536343500 E-40 + 14.98295430453600041 58*I
? xi                            0.585318492448534646977
? (xi/2)*exp(2*xi*xi)           0.580682668039111487078
? quit

C:\calc\pari>exit        Script completed Thu Feb 17 11:29:16 2000
```

5. Best rational approximation to $e^{-(An+B)z}$ on a real interval

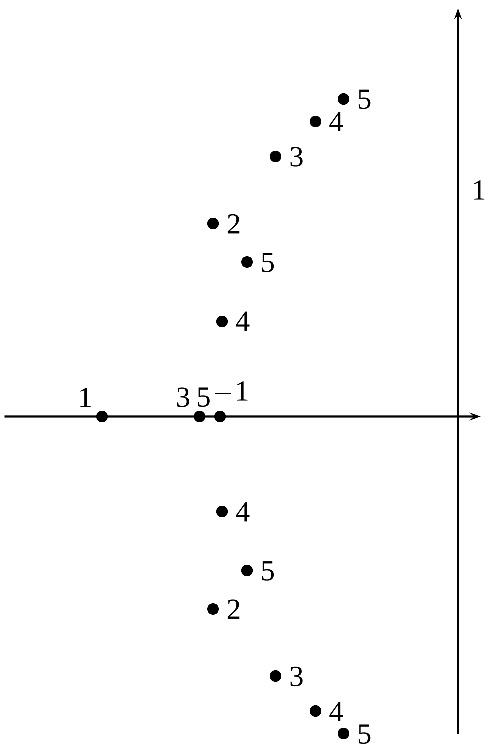
Best rational approximation to $\exp(-z)$ on a given real interval, say $[0,c]$ has a strict equioscillating error function, as seen here with $e^z - p_n(z)/q_n(z)$ on $[0,1]$, for $n = 1, 2$:





For varying degrees, we have a now familiar scaling effect best seen through the poles:

Sets of poles expand and tend to follow the Padé poles; errors decrease factorially fast with n(here, the error is about $\dfrac{e^{-1/2}n!(n-1)!}{4^{2n+1}(2n+1)!(2n-1)!}$. For an accurate asymptotic picture, see Braess' proof of Meinardus' conjecture).



We find a stable picture if we look at the poles of the best approximants of degree n to $\exp(-nz)$. Moreover, the norms E_n of the errors tend to decrease in an exponential way with n:

$n=1$	2	3	4	5	6
1.58 E-3	3.197 E-5	5.921 E-7	1.068 E-8	1.91 E-10	3.383 E-12

The ratio of two successive errors seems to tend towards a limit of about $1/60$. The exact value, as it will be shown later (in (36), p. 18), is $\rho = 1/57.0699681\cdots$. Could we have $E_n \sim C\rho^n$, and what is the value of C? I can't wait: here are the products $E_n\rho^{-n}$:

$n=1$	2	3	4	5	6
0.090	0.104	0.110	0.113	0.116	0.117

Hmmm, what could it be? The numbers follow the approximate formula $0.125 - 0.05/(n+1/2)$. The limit 0.125 is reasonably close to an estimate which will be given in § 5.2.

Ah, an obscure insight (hindsight?) coming from long and painful experiments with the $'1/9'$ problem tells me to try $\exp(-(n+1/2)z)$ instead of $\exp(-nz)$, and to multiply the errors by $\rho^{-n+1/2}$:

$n=0$	1	2	3	4	5	6
0.197	4.161 E-3	7.610 E-5	1.356 E-6	2.406 E-8	4.244 E-10	7.470 E-12
1.488	1.794	1.872	1.904	1.928	1.941	1.950

Aha! Now, the limit seems to be 2. This phenomenon will also be explained in § 5.2.

5.1. **Root asymptotics.**

We expect the poles to tend to be ultimately distributed on a fixed arc F with a limit distribution $d\mu_p$, and the interpolation points on $E=[0,c]$ with a limit distribution $d\mu_i$, so that the complex potential

$$\mathcal{V}(z) = \mathcal{V}_p(z) - \mathcal{V}_i(z) := \int_F \log\frac{1}{z-t}\,d\mu_p(t) - \int_E \log\frac{1}{z-t}\,d\mu_i(t) \tag{27}$$

satisfies (9) with $\varphi(z) = \exp(-Az)$:

$$(\mathcal{V}_+(z) + \mathcal{V}_-(z))/2 + \frac{Az}{2} = \text{ a real constant } = \sigma \text{ on } F, \tag{28}$$

and equioscillation on E:

$$(\mathcal{V}_+ + \mathcal{V}_-)/2 = \text{ another real constant } = \rho \text{ on } E, \tag{29}$$

$$\int_E d\mu_i(t) = \int_F d\mu_p(t) = 1 \tag{30}$$

(charges on E and F), equivalent to $\mathcal{V}$ bounded at ∞, actually, $\mathcal{V}'(z) \sim$ constant z^{-2} for large z, and $\displaystyle\int_C \frac{\partial V(t)}{\partial n}|dt| = -2\pi$ on any contour containing F but not E, or also, that the imaginary part of $\mathcal{V}$ increases by π on $[0,c]$.

From (10), $\mathcal{V}' + A/2$ takes opposite values on the two sides of F, and also

$$\mathcal{V}'(z) = -\int_{E\bigcup F} \frac{d\mu(t)}{z-t} = -⨍_{E\bigcup F} \frac{d\mu(t)}{z-t} \pm \pi i\mu'(z) \tag{31}$$

when z tends to a point of E or F, and where $⨍$ is the Cauchy principal value. We therefore have

$$⨍_{E\bigcup F} \frac{d\mu(t)}{z-t} = ⨍_F \frac{d\mu_p(t)}{z-t} - \int_E \frac{d\mu_i(t)}{z-t} = \frac{A}{2},\ z \in F, \tag{32}$$

which is an integral equation for the distribution μ_p, to be considered with (29) as another equation for μ_p and μ_i...

Now, there are various ways to go further, and to conclude with more or less neat expressions. There may be wrong turns, which may however yield a useful piece of information.

We turn to a classical way to deal with the Sokhotskyi-Plemelj formulas (31)-(32) in the $z-$plane, by considering $\sqrt{z(z-c)(z-a)(z-b)}\ \mathcal{V}'(z)$ which is meromorphic outside F, even holomorphic, as the product remains bounded near 0 and c. Best combination is

$$\sqrt{\frac{z(z-c)}{(z-a)(z-b)}}\ \mathcal{V}'(z) = \frac{A}{2\pi i}\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\ \frac{dt}{z-t}, \tag{33}$$

where one not only got rid of unwanted constants, but, as $\mathcal{V}'(z)$ is only $O(z^{-2})$ at ∞, leaves

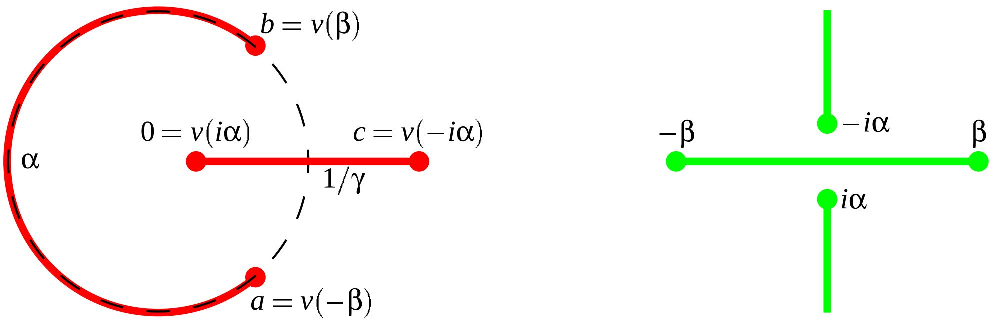
$$\int_a^b \sqrt{\frac{t(t-c)}{(t-a)(t-b)}}\ dt = 0 \tag{34}$$

as a bonus!! (34) gives one equation for a and b, knowing c (and another equation will be worked later on, from the unit charge condition (30)). For instance, when $c = 0$, we have indeed a vanishing integral of an odd function if $a = -b$, but, as we know (or suspect) that a and b are complex conjugates, we see that a and b must be opposite pure imaginary numbers, as they are indeed in the Padé case. To work (34) a bit further, we see that it is a complete elliptic integral of the third kind (complete because one integrates on a arc joining two branchpoints; of the third kind because the incomplete integral behaves like a logarithm somewhere [near ∞, the square root is $1+(a+b-c)/(2t)+\cdots$]).

A convenient transformation sending the four branchpoints 0, c, a, and b on and from a symmetric set is

$$t = \frac{\alpha + iv}{1+i\gamma v}.$$

So, $v = i\alpha$ is mapped on $t = 0$, one must have, for $v = -i\alpha$, $\dfrac{2\alpha}{1+\gamma\alpha} = c$, and $\dfrac{\alpha \pm i\beta}{1 \pm i\gamma\beta} = a, b$. As neither a nor b is known, we may as well take α and β, keeping in mind that $\gamma = \dfrac{2}{c} - \dfrac{1}{\alpha}$ (for given α, a and b are on a circle of diametral points α and $1/\gamma$).



The $z-$ plane and the $v-$plane.

complete elliptic integrals of first and second kind: $\mathsf{K} = (\alpha^2+\beta^2)^{1/2}\int_0^\beta \frac{dv}{[(v^2+\alpha^2)(\beta^2-v^2)]^{1/2}}$,
$\mathsf{E} = \alpha^2(\alpha^2+\beta^2)^{1/2}\int_0^\beta \frac{1}{v^2+\alpha^2}\frac{dv}{[(v^2+\alpha^2)(\beta^2-v^2)]^{1/2}} = (\alpha^2+\beta^2)^{-1/2}\int_0^\beta \left[\frac{v^2+\alpha^2}{\beta^2-v^2}\right]^{1/2} dv.$

$$Ac = \frac{\pi^2}{\left(\dfrac{1}{\alpha\gamma}-\alpha\gamma\right)\mathsf{E}(\mathsf{K}-\mathsf{E})} \tag{35}$$

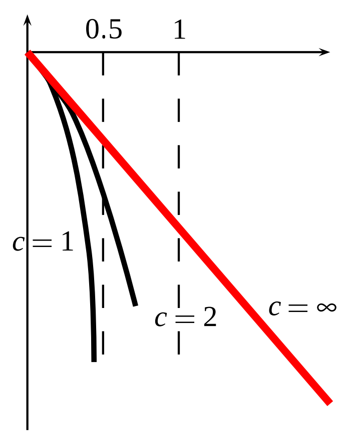
which, together with (34), gives α, γ, K, etc., from Ac (no wonder that everything depends essentially on the product Ac: remember that we approximate $\exp(-nAz)$ on $0 \leqslant z \leqslant c$, equivalent up to a scaling to the approximation of $\exp(-nAct)$ on $[0,1]$.

Check: when $c \to \infty$, $\alpha\gamma = -1+2\alpha/c \to -1$, $1/(\alpha\gamma)-\alpha\gamma \sim -4\alpha/c$, and we should check that $-4\alpha\mathsf{E}(\mathsf{K}-\mathsf{E}) \to \pi^2$. Yes: $|a|^2 = ab = \dfrac{\alpha^2+\beta^2}{1+\gamma^2\beta^2} \to \alpha^2$, and we know that $\mathsf{E} \to \mathsf{K}/2$ when $c \to \infty$, so that the limit of (35) is $|a|\mathsf{K}^2 = \pi^2$, confirmed by $|a| = \pi/\omega$ of [5, § 3.4, eq. (34)].

Now, (34) becomes $\displaystyle\int_{-\beta}^{\beta} \frac{1}{(1+i\gamma v)^2}\sqrt{\frac{\alpha^2+v^2}{\beta^2-v^2}}\,dv = 0.$

Some points of the locus:

$a/c = \overline{b}/c$	α/c	β/c	γc
$0.023671-0.0280i$	-0.0390	0.0809	27.614
$0.061784-0.0757i$	-0.1159	0.2206	10.628
$0.091190-0.1152i$	-0.1910	0.3385	7.2343
$0.114713-0.1488i$	-0.2647	0.4412	5.7773
$0.134055-0.1781i$	-0.3371	0.5327	4.9659
$0.150306-0.2040i$	-0.4084	0.6156	4.4482
$0.204616-0.3022i$	-0.7512	0.9493	3.3310
$0.241306-0.3830i$	-1.1399	1.2531	2.8772
$0.358549-0.8401i$	-6.3630	3.7085	2.1571
$0.406333-1.3084i$	-20.529	7.8098	2.0487
$0.425044-1.6475i$	-39.115	11.848	2.0255
$0.439587-2.0532i$	-73.476	17.878	2.0136
$0.445607-2.2839i$	-100.04	21.890	2.0099



For each c, the locus of a is a curve with vertical asymptote of abscissa $c/2$, and of tangent at the origin matching the $c = \infty$ locus, given by $\arg a = -0.860274\ldots$ (see [5, end of § 3.2]).

Finally, the error decreases like ρ^n, with

$$\log\frac{1}{\rho} = \pi\frac{\alpha\gamma(\mathsf{K}-\mathsf{E})(\mathsf{K}'-\mathsf{E}')-\mathsf{E}\mathsf{E}'}{(\alpha\gamma-1)\mathsf{E}(\mathsf{K}-\mathsf{E})} \tag{36}$$

5.2. **Strong asymptotics .**

Consider rational approximants to functions $f^n g$, and suppose that the Hermite-Walsh error formula can already be written as

$$f^n(z)g(z)-\frac{p_n(z)}{q_n(z)} \sim e^{\mathcal{W}_n(z)}\frac{1}{2\pi i}\int_C f^n(t)g(t)e^{-\mathcal{W}_n(t)}\frac{dt}{z-t},$$

where $\mathcal{W}_n$ is a "smoothed" approximation of the discrete potential created by the poles and the interpolation points. The function $\exp(\mathcal{W}_n)$ (corresponding to Nuttall's χ_1 and/or χ_2 [4]) has branch points, even if f and g are entire. What is this function? The influence of f is overwhelming in the determination of the branchpoints and other main features when n is large. So, we solve first with f, and find the active part $F\subset C$ and the main behaviour $(\exp(\mathcal{W}_n))^{1/n}\to\exp(2\mathcal{V})$ (root asymptotics, also called zero order asymptotics by Nuttall).

Aptekarev [1] established in some cases a more accurate picture $\mathcal{W}_n = 2n\mathcal{V}+\widetilde{\mathcal{V}}+o(1)$ (strong asymptotics, also called first order asymptotics by Nuttall). I give here a probably very sloppy account of Aptekarev's wonderful results (to be available soon):

> *Also sprache Aptekarev*: $\widetilde{\mathcal{V}}$ is (multivalued) analytic outside $E\bigcup F$, with a period $2\pi i$ about F, and $-2\pi i$ about E, corresponding to a positive unit charge on F, and a negative unit charge on E, with $\widetilde{\mathcal{V}}_+ + \widetilde{\mathcal{V}}_-$ constant on E, $\widetilde{\mathcal{V}}(z)_+ + \widetilde{\mathcal{V}}(z)_- + 2\log g(z) =$ another constant on F, and finally $\widetilde{\mathcal{V}}(z) =$ const. $+o(1)$ when $z\to\infty$ (if E and F are bounded).
>
> Moreover, the error norm is $E_n \sim 2\rho^n\tilde{\rho}$, where $2\log\tilde{\rho} = \mathrm{Re}\,\{(\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z))_E - [\widetilde{\mathcal{V}}_+(z)+\widetilde{\mathcal{V}}_-(z)+2\log g(z)]_F\}$.

This means also that $\widetilde{\mathcal{V}}'$ is analytic outside E and F, taking opposite values on the two sides of E, and with $\widetilde{\mathcal{V}}'+g'(z)/g(z)$ taking opposite values on the two sides of F.

Important special case: if $g=\sqrt{f}$, the conditions on $\widetilde{\mathcal{V}}$ are exactly the conditions (29)-(30) which we already saw for $\mathcal{V}$ itself! So, $\widetilde{\mathcal{V}}=\mathcal{V}$, and $\tilde{\rho}=\sqrt{\rho}$ in this case.

Remark: the real part of $\widetilde{\mathcal{V}}+\log g$ need not, and will normally not be a constant on F. However, the cut on which the boundary conditions for $\widetilde{\mathcal{V}}$ are set may be modified (keeping the endpoints as the endpoints of F), and one may dream to find the locus $\widetilde{F}$ where $\widetilde{\mathcal{V}}+\log g$ has a constant real part. The use and even the existence of $\widetilde{F}$ seem questionable (Aptekarev). It may be wiser and more useful to look for a locus F_n where the whole complex potential $\mathcal{V}_n = 2n\mathcal{V}+\widetilde{\mathcal{V}}+n\log f+\log g$ has a constant real part, as this locus may be a fair approximation to the set of poles for a given value of n (Nuttall).

Application to best approximation to $\exp(-(nA+B)z)$ on $[0,c]$:

$E_n\sim 2\rho^n\rho_B$, where $2\log\rho_B = \mathrm{Re}\,\{(\mathcal{V}_{B,+}+(z)+\mathcal{V}_{B,-}(z))_E - [\mathcal{V}_{B,+}(z)+\mathcal{V}_{B,-}(z)+2Bz]_F\}$, $\mathcal{V}_B' = \widetilde{\mathcal{V}}'$ being analytic outside $E\bigcup F$, taking opposite values on the two sides of $E=[0,c]$, $\mathcal{V}_B'(z)+B$ taking opposite values on the two sides of F, or any arc of endpoints a and b, and corresponding to a positive unit charge on F, and a negative unit charge on E, and finally $\mathcal{V}_B'(z) =$ const. $z^{-2}+\cdots$ when $z\to\infty$.

The problem is solved by $\mathcal{V}_{A/2}=\mathcal{V}$ if $B=A/2$.

And if $B=0$? Then, $\mathcal{V}_0''$ is the simple algebraic function $\mathcal{V}_0''(z)=\dfrac{\text{constant}}{\sqrt{z(z-c)(z-a)(z-b)}}$ associated to the potential of a plain (and plane) condenser $(E,\widetilde{F})$, although we do not need to know what $\widetilde{F}$ is. The capacity is $2\mathsf{K}/(\pi\mathsf{K}')$, and

$\rho_0=\exp\left(-\dfrac{\pi}{2}\dfrac{\mathsf{K}'}{\mathsf{K}}\right).$

And for any B,

$$\mathcal{V}_B=\frac{2B}{A}\mathcal{V}+\left(1-\frac{2B}{A}\right)\mathcal{V}_0 \tag{37}$$

does the trick, see Meinguet [7] for such relations. So,

$$\rho_B=\rho^{B/A}\rho_0^{(1-2B/A)}.$$

and we just have to get $\rho_0=\exp(-1/C)$, where C is the plain condenser capacity of $(E,\widetilde{F})$.

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