Prices versus quantities: stock pollution control with repeated choice of the instrument [1]

by

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Abstract. One examines strategies of pollution control through choices between taxes and tradable permits, supposed to be decided at several time periods $t_1, t_2, \ldots, t_I$. At each of these time periods, the choice depends on the solution of a dynamic programming problem involving the expectations of the polluting factor of production function $z(t)$ and the pollutant stock function $S(t)$. According to coefficients of these two functions in the dynamic problem, it is shown under a broad functional setting that permits are decided for a while, followed by decisions of taxes for all the remaining periods. Finite $(I < \infty)$, as well as infinite $(I = \infty)$ horizon is considered.

Introduction

Taxes and quotas of tradable pollution permits are two important and intensively studied economic instruments for pollution control. These paper is concerned with the comparison of taxes and permits when the regulatory authority and the polluters have asymmetric information about abatement costs, and when environmental damages are due to a pollutant that accumulates. At each period of time, the regulator chooses the type and the level of the instrument that maximises a welfare functional depending on the expected flow of production minus damage costs, taking in account the expected answer of the polluters to the chosen environmental policy.

We depart from the literature concerned by this subject (a.o. Hoel and Karp [2,3], Newell and Pizer [4], Pizer [5]) by allowing the possibility of switching between instruments at each period, by considering finite as well as infinite horizon frameworks, and by considering a more general formulation of the regulator's objective and of the pollutant accumulation.

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We do not restrict ourselves to the infinite horizon case studied in the above mentioned literature where the choice of the instrument appears to be constant through time, and we show in the finite horizon case that permits are decided for a while, followed by decisions of taxes for all the remaining periods.

1. The model

The model has both continuous and discrete time features. t is the continuous time variable $(0 \leq t \leq H)$, which is divided in periods of length h, which are indexed by the discrete time index i $(i \in \{1, 2, ..., I\}$, with $h = H/I)$. The regulator chooses the instrument and its level at the beginning of each period i, and keeps this level constant for the whole period.

To this end, one must estimate the reaction of the polluting firms:

1.1. The polluting firm reaction.

At each step i, the representative polluting firm is assumed to maximise a quadratic profit functional involving

$$a_i z(t) - \frac{b}{2} z^2(t) - \tau_i z(t), \qquad [i-1]h \leqslant t \leqslant ih, \tag{1}$$

in which $az - bz^2/2$ is the firm's production function, where z is the polluting factor of production (for example, emissions linked to energy), a_i is the realization at period i of a, a discrete i.i.d. random process with mean μ and variance σ^2, and b is a positive constant. τ_i is either the level of the tax chosen by the regulator, or the observed price of permits at period i. In problem (1), it is assumed that the representative firm does not take account of the future, because it is too small to influence the aggregate stock of pollutant, and thus has no power to influence the future environmental policy. The firm is thus unable to manipulate the future decisions of the regulator, so that its profit maximisation problem is solved at any time by

$$z(t) = z_i := \frac{a_i - \tau_i}{b}, \ [i-1]h \leq t \leq ih \tag{2}$$

If the chosen instrument at step i is a tax, the behaviour of the firm is described by (2). If the chosen instrument are permits, then whatever the random shock a_i, their price will adjust so that the demand of permits by polluters equals the quantity of permits supplied by the regulator, $\overline{x}_i$. In that case,

$$z(t) = \overline{x}_i, \ [i-1]h \leq t \leq ih \tag{3}$$

Remark that the values of τ_i or $\overline{x}_i$ will be determined by the regulator as solutions of an optimization problem. And the decision to order tax or permits is also a part of the solution.

Let ξ be a binary decision variable determining the type of the instrument (tax or permits), with $\xi = 0$ if permits are chosen and $\xi = 1$ if a tax is chosen. Let $x = E\{z\}$ be the expectation of z. Given (2) and (3), x can be taken as the decision variable determining the level of the instrument (the level of the tax or the quantity of permits), whether this instrument is a tax or a quota.

We summarize both (2) and (3) as

$$z(t) = x_i + \xi_i \left[\frac{a_i - \mu}{b} \right] \tag{4}$$

where the unknowns for the ith period are ξ_i and x_i. If $\xi_i = 0$ (permits), x_i must be understood as $\overline{x}_i$; if $\xi_i = 1$ (tax), x_i means $\frac{\mu - \tau_i}{b}$, i.e., τ_i will be recovered as $\mu - bx_i$.

1.2. The stock of pollutant.

We suppose that the evolution of the stock of pollutant $S(t)$ is given in $t \in [(i-1)h, ih]$ by

$$S(t) = \alpha(t - [i-1]h)S_{i-1} + \beta(t - [i-1]h)z_i, \tag{5}$$

where S_{i-1} is the value $S([i-1]h)$, and z_i is the value seen above of the piecewise constant function $z(t)$ on $([i-1]h, ih)$. α and β are continuous functions, and $\alpha(0) = 1$, $\beta(0) = 0$.

A typical instance of (5) results when $S(t)$ is a solution of the differential equation

$$\frac{dS(t)}{dt} = z(t) - \delta S(t), \ S(0) = S_0 \text{ given} \tag{6}$$

where δ is the rate of decay of the stock of pollution ($\delta > 0$) and where the number of firms has been normalized to 1. As $z(t)$ is the constant z_i on $([i-1]h, ih)$, we solve immediately (6) as

$$S(t) = \frac{z_i}{\delta} + \left[S_{i-1} - \frac{z_i}{\delta} \right] \exp(-\delta(t - [i-1]h)),$$

whence

$$\alpha(t) = \exp(-\delta t) , \qquad \beta(t) = \frac{1 - \exp(-\delta t)}{\delta}. \tag{7}$$

1.3. The regulator's decision.

The stock of pollutant $S(t)$ results in damages to society that are equal to $\pi S^2(t)/2$, where π is a positive parameter.

At the beginning of each step i (i.e. when $t = [i-1]h$), that is *before* observing the current and future random shocks a_j ($j = i, ..., I$), but knowing however their mean and variance, the regulator chooses the type and the level of the instrument that maximise the expected flow of payoffs defined as the difference between production and damages $\pi S^2/2$, subject to the behaviour of the polluters (described by (2) or (3)) and to the stock equation (6). At step i, the regulator solves the following dynamic programming problem :

$$V_i(S_{i-1}) = \max_{\xi_i, x_i} E \left\{ \mathcal{F}_i \left(a(t)z(t) - \frac{b}{2}z^2(t) - \frac{\pi}{2}S^2(t) \right) \right\}, \tag{8}$$

subject to (5), and to (4) giving $z(t)$ in the ith period, with $V_{I+1}(S_H) = 0$ and where $S_j := S(jh)$, and where we only require $\mathcal{F}_i$ to be a linear positive functional involving functions on $[i-1]h < t < H$, and submitted to the condition (10) below.

A typical example of what we have in mind is an integral

$$\mathcal{F}_i(f) = \int_{[i-1]h}^{H} f(t) \exp(-r(t - [i-1]h)) \, dt, \tag{9}$$

where r is the exogeneous (positive) discount rate, but other forms have been considered (see examples later on).

We also suppose that the functionals $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are related by

$$\mathcal{F}_i(f) = \mathcal{F}(f(t + [i-1]h)) + \varepsilon \mathcal{F}_{i+1}(f), \tag{10}$$

where $\mathcal{F}$ involves functions defined on $t \in [0, h]$.

Remark that $\mathcal{F}_i(f) = \sum_{j=i}^{I} \varepsilon^{j-i} \mathcal{F}(f(t + [j-1]h))$.

In our example (9), $\mathcal{F}(f) = \int_0^h f(t)\, e^{-rt}\, dt$, and $\varepsilon = e^{-rh}$. We then have $\mathcal{F}(1) = \dfrac{1 - e^{-rh}}{r}$, from (7), $\mathcal{F}(\alpha) = \dfrac{1 - e^{-[r+\delta]h}}{r + \delta}$, $\mathcal{F}(\beta) = \dfrac{1 - e^{-rh}}{r\delta} - \dfrac{1 - e^{-[r+\delta]h}}{[r + \delta]\delta}$, etc. Using such expressions in the subsequent discussions should soon be an absolute nightmare. That's why we shall keep the $\mathcal{F}$ notation in further calculations as far as possible. Our theory is therefore able to cope with several formulas. As a second example, Hoel and Karp use in [2] an elementary integration formula $\mathcal{F}(f) = \dfrac{1 - e^{-rh}}{r}\, f(0)$, therefore, their $\mathcal{F}_i(f)$ is $\dfrac{1 - e^{-rh}}{r} \sum_{j=i}^{I} e^{-r[j-i]h} f([j - 1]h)$.

2. Optimal solution.

Theorem. *For any set of functionals $\mathcal{F}_i$ satisfying (10), the problems (8) are solved with values $\xi_i = 0$ (permits) or $\xi_i = 1$ (tax) determined whether the ratio π/b is larger or smaller than a computable value η_i. Moreover, $\eta_1 < \eta_2 < \cdots < \eta_I$.*
If $H \to \infty$ (infinite horizon), $\eta_1, \eta_2, \ldots$ tend towards a same limit η^*.
This means that,

    (1) if $\pi/b < \eta_1$, taxes will be chosen for all periods,
    (2) if $\pi/b > \eta_I$, quotas will be chosen for all periods,
    (3) if $\eta_1 < \pi/b < \eta_I$, quotas will be chosen while $\pi/b > \eta_i$, and taxes later on.

When the horizon is infinite, we have taxes or permits for all the periods, depending only on the ratio π/b, whether it is smaller or larger than η^*.

Proof.

    (1) We first look at how the decision has to be taken with respect to the *last* period $i = I$. We perform the calculation of (8), using (5), and knowing that a and x (and therefore z) are constants on $(t_{I-1}, t_I) = (H - h, H)$:

$$\mathcal{F}_I \left(a_I z_I - \frac{b}{2} z_I^2 - \frac{\pi}{2}[\alpha(t - [I-1]h)S_{I-1} + \beta(t - [I-1]h)z_I]^2 \right)$$

$$= \mathcal{F} \left(a_I z_I - \frac{b}{2} z_I^2 - \frac{\pi}{2}[\alpha(t)S_{I-1} + \beta(t)z_I]^2 \right) \qquad \text{from (10)}$$

$$= \left(a_I z_I - \frac{b}{2} z_I^2 \right) \mathcal{F}(1) - \frac{\pi}{2}[S_{I-1}^2 \mathcal{F}(\alpha^2) + 2 z_I S_{I-1} \mathcal{F}(\alpha\beta) + z_I^2 \mathcal{F}(\beta^2)]$$

which shows already that the result will be a quadratic polynomial in S_{I-1}.
We now compute the mathematical expectation in (8), knowing that a_I is a value of a random variable of mean μ and variance σ^2, and, from (4), $E\{z_I\} = x_I$, $E\{a_I z_I\} = E\{a_I x_I + \xi_I(a_I^2 - \mu a_I)/b\} = \mu x_I + \xi_I \sigma^2/b$, and $E\{z_I^2\} = E\{x_I^2 + 2\xi_I x_I(a_I - \mu)/b + \xi_I(a_I - \mu)^2/b^2\} = x_I^2 + \xi_I \sigma^2/b^2$. So,

$$E\left\{ \mathcal{F}_I \left(a_I z_I - \frac{b}{2} z_I^2 - \frac{\pi}{2} S^2(t) \right) \right\} = \left(\mu x_I + \xi_I \frac{\sigma^2}{b} - \frac{b}{2} x_I^2 - \frac{\xi_I b}{2} \frac{\sigma^2}{b^2} \right) \mathcal{F}(1)$$

$$- \frac{\pi}{2} \left[S_{I-1}^2 \mathcal{F}(\alpha^2) + 2 x_I S_{I-1} \mathcal{F}(\alpha\beta) + \left[x_I^2 + \xi_I \frac{\sigma^2}{b^2} \right] \mathcal{F}(\beta^2) \right]$$

$$= \xi_I \frac{\sigma^2}{2b} \mathcal{F}(1) - \frac{\pi}{2} S_{I-1}^2 \mathcal{F}(\alpha^2) - \xi_I \frac{\pi}{2} \frac{\sigma^2}{b^2} \mathcal{F}(\beta^2) + [\mu \mathcal{F}(1) - \pi S_{I-1} \mathcal{F}(\alpha\beta)] x_I - [b \mathcal{F}(1) + \pi \mathcal{F}(\beta^2)] \frac{x_I^2}{2}$$

Maximizing with respect to x_I, i.e., by putting

$$x_I = \frac{\mu \mathcal{F}(1) - \pi S_{I-1} \mathcal{F}(\alpha \beta)}{b \mathcal{F}(1) + \pi \mathcal{F}(\beta^2)}, \tag{11}$$

one finds the required quadratic polynomial in S_{I-1}

$$\xi_I \frac{\sigma^2}{2b} \left[\mathcal{F}(1) - \frac{\pi}{b} \mathcal{F}(\beta^2) \right] - \frac{\pi}{2} S_{I-1}^2 \mathcal{F}(\alpha^2) + \frac{[\mu \mathcal{F}(1) - \pi S_{I-1} \mathcal{F}(\alpha \beta)]^2}{2b[\mathcal{F}(1) + \frac{\pi}{b} \mathcal{F}(\beta^2)]}. \tag{12}$$

The complete solution of (8) at the final period is

$$V_I(S_{I-1}) = -\frac{\pi}{2} S_{I-1}^2 \mathcal{F}(\alpha^2) + \frac{[\mu \mathcal{F}(1) - \pi S_{I-1} \mathcal{F}(\alpha \beta)]^2}{2b[\mathcal{F}(1) + \frac{\pi}{b} \mathcal{F}(\beta^2)]} + \xi_I \frac{\sigma^2}{2b} \left[\mathcal{F}(1) - \frac{\pi}{b} \mathcal{F}(\beta^2) \right]. \tag{13}$$

We emphasize the quadratic character of the function V_I by writing (13) as

$$V_I(S_{I-1}) = \rho_0^{(I)} + \rho_1^{(I)} S_{I-1} + \frac{\rho_2^{(I)} \pi}{2} S_{I-1}^2, \tag{14}$$

(borrowing a notation of [2]).

Remark that $\rho_2^{(I)} = \pi \dfrac{\mathcal{F}^2(\alpha \beta) - \mathcal{F}(\alpha^2) \mathcal{F}(\beta^2) - [b/\pi] \mathcal{F}(1) \mathcal{F}(\alpha^2)}{b \mathcal{F}(1) + \pi \mathcal{F}(\beta^2)}$ is *negative*: as $\mathcal{F}$ is a positive linear functional, $\mathcal{F}(\alpha \beta)$ is a scalar product.

The sign of the coefficient of ξ_I in (13) tells if tax or permit will be decided for the last period, i.e., whether

$$\eta_I = \frac{\mathcal{F}(1)}{\mathcal{F}(\beta^2)} \lessgtr \frac{\pi}{b}. \tag{15}$$

In the simple case where $\mathcal{F}(f) = f(h)$ and $\beta(t) = t$, the result is $1/h^2$. If $\mathcal{F}(f) = \int_0^h f(t)\, dt$, $\eta_I = 3/h^2$. And if $\mathcal{F}(f) = f(0)$, $\eta_I = \infty$ (always tax during the last period).

(2) We come now to the general problem (8) for an intermediate period $([i-1]h, ih)$.
We will show by induction that a quadratic expression similar to (14), i.e.,

$$V_{i+1}(S_i) = \rho_0^{(i+1)} + \rho_1^{(i+1)} S_i + \frac{\rho_2^{(i+1)} \pi}{2} S_i^2, \tag{16}$$

holds at the $i + 1^{\text{th}}$ period. Therefore, using (10),

$$\mathcal{F}_i \left(a(t) z(t) - \frac{b}{2} z^2(t) - \frac{\pi}{2} S^2(t) \right) = \mathcal{F} \left(a_i z_i - \frac{b}{2} z_i^2 - \frac{\pi}{2} S^2(t + [i-1]h) \right) + \varepsilon V_{i+1}(S_i),$$

as all the parameters have been settled in $\mathcal{F}_{i+1}$.

Assuming (16), we proceed as above in looking for the maximum of $E\{\mathcal{F}_i\}$.

$$E\{\mathcal{F}_i\} = \left(\xi_i \frac{\sigma^2}{2b} + \mu x_i - \frac{b}{2}x_i^2\right)\mathcal{F}(1) - \frac{\pi}{2}[S_{i-1}^2\mathcal{F}(\alpha^2) + 2x_i S_{i-1}\mathcal{F}(\alpha\beta) + \frac{\xi_i\sigma^2}{b^2}\mathcal{F}(\beta^2) + x_i^2\mathcal{F}(\beta^2)]$$

$$+ \varepsilon\rho_0^{(i+1)} + \varepsilon\rho_1^{(i+1)}[\alpha(h)S_{i-1} + \beta(h)x_i] + \frac{\varepsilon\rho_2^{(i+1)}\pi}{2}\left[[\alpha(h)S_{i-1} + \beta(h)x_i]^2 + \beta^2(h)\xi_i\frac{\sigma^2}{b^2}\right]$$

$$= \frac{\xi_i\sigma^2}{2b}\mathcal{F}(1) - \frac{\pi}{2}S_{i-1}^2\mathcal{F}(\alpha^2) - \frac{\pi}{2}\frac{\xi_i\sigma^2}{b^2}\mathcal{F}(\beta^2)$$

$$+ \varepsilon\rho_0^{(i+1)} + \varepsilon\rho_1^{(i+1)}\alpha(h)S_{i-1} + \varepsilon\frac{\rho_2^{(i+1)}}{2}\pi\alpha^2(h)S_{i-1}^2 + \varepsilon\xi_i\frac{\rho_2^{(i+1)}\pi\beta^2(h)\sigma^2}{2b^2}$$

$$+[\mu\mathcal{F}(1) - \pi S_{i-1}\mathcal{F}(\alpha\beta) + \varepsilon\rho_1^{(i+1)}\beta(h) + \varepsilon\rho_2^{(i+1)}\pi\alpha(h)\beta(h)S_{i-1}]x_i - [b\mathcal{F}(1) + \pi\mathcal{F}(\beta^2) - \varepsilon\rho_2^{(i+1)}\pi\beta^2(h)]x_i^2/2,$$

Maximum is reached at

$$x_i = \frac{\mu\mathcal{F}(1) - \pi S_{i-1}\mathcal{F}(\alpha\beta) + \varepsilon\rho_1^{(i+1)}\beta(h) + \varepsilon\rho_2^{(i+1)}\pi\alpha(h)\beta(h)S_{i-1}}{b\mathcal{F}(1) + \pi\mathcal{F}(\beta^2) - \varepsilon\rho_2^{(i+1)}\pi\beta^2(h)}, \tag{17}$$

resulting in

$$\frac{\xi_i\sigma^2}{2b}\mathcal{F}(1) - \frac{\pi}{2}S_{i-1}^2\mathcal{F}(\alpha^2) - \frac{\pi}{2}\frac{\sigma^2}{b^2}\mathcal{F}(\beta^2)$$

$$+ \varepsilon\rho_0^{(i+1)} + \varepsilon\rho_1^{(i+1)}\alpha(h)S_{i-1} + \varepsilon\frac{\rho_2^{(i+1)}}{2}\pi\alpha^2(h)S_{i-1}^2 + \varepsilon\xi_i\frac{\rho_2^{(i+1)}\pi\beta^2(h)\sigma^2}{2b^2}$$

$$+ \frac{[\mu\mathcal{F}(1) - \pi S_{i-1}\mathcal{F}(\alpha\beta) + \varepsilon\rho_1^{(i+1)}\beta(h) + \varepsilon\rho_2^{(i+1)}\pi\alpha(h)\beta(h)S_{i-1}]^2}{2[b\mathcal{F}(1) + \pi\mathcal{F}(\beta^2) - \varepsilon\rho_2^{(i+1)}\pi\beta^2(h)]},$$

We confirm therefore the inductive step from (16)

$$V_i(S_{i-1}) = \rho_0^{(i)} + \rho_1^{(i)}S_{i-1} + \frac{\rho_2^{(i)}\pi}{2}S_{i-1}^2, \tag{18}$$

where we only look at the $\xi_i\sigma^2$ terms in $\rho_0^{(i)} = \xi_i\frac{\sigma^2}{2b}R_i(\pi/b) + \cdots$:

$$R_i\left(\frac{\pi}{b}\right) = \mathcal{F}(1) - \frac{\pi}{b}\mathcal{F}(\beta^2) + \varepsilon\frac{\pi}{b}\rho_2^{(i+1)}\beta^2(h), \tag{19}$$

so that we also will have to care for $\rho_2^{(i)}$:

$$\rho_2^{(i)} = -\mathcal{F}(\alpha^2) + \varepsilon\rho_2^{(i+1)}\alpha^2(h) + \pi\frac{[-\mathcal{F}(\alpha\beta) + \varepsilon\rho_2^{(i+1)}\alpha(h)\beta(h)]^2}{b\mathcal{F}(1) + \pi\mathcal{F}(\beta^2) - \varepsilon\rho_2^{(i+1)}\pi\beta^2(h)}, \tag{20}$$

which shows that the ρ_2's, and therefore the R_i's in (19), are rational functions of π/b.

(3) We proceed now with a close inspection of (20) in order to discuss the positive root η_i of $R_i(\pi/b) = 0$, so to determine the value of the ratio π/b which will trigger tax or permits during the ith step:

(a) We work (20), showing that $\rho_2^{(i)}$ is actually a rational function of first degree of $\rho_2^{(i+1)}$:

$$\rho_2^{(i)} = \frac{\frac{\pi}{b}[\varepsilon\rho_2^{i+1}\mathcal{F}(\gamma^2) + \mathcal{F}^2(\alpha\beta) - \mathcal{F}(\alpha^2)\mathcal{F}(\beta^2)] + \varepsilon\alpha^2(h)\mathcal{F}(1)\rho_2^{(i+1)} - \mathcal{F}(1)\mathcal{F}(\alpha^2)}{\mathcal{F}(1) + \frac{\pi}{b}[\mathcal{F}(\beta^2) - \varepsilon\beta^2(h)\rho_2^{(i+1)}]}, \tag{21}$$
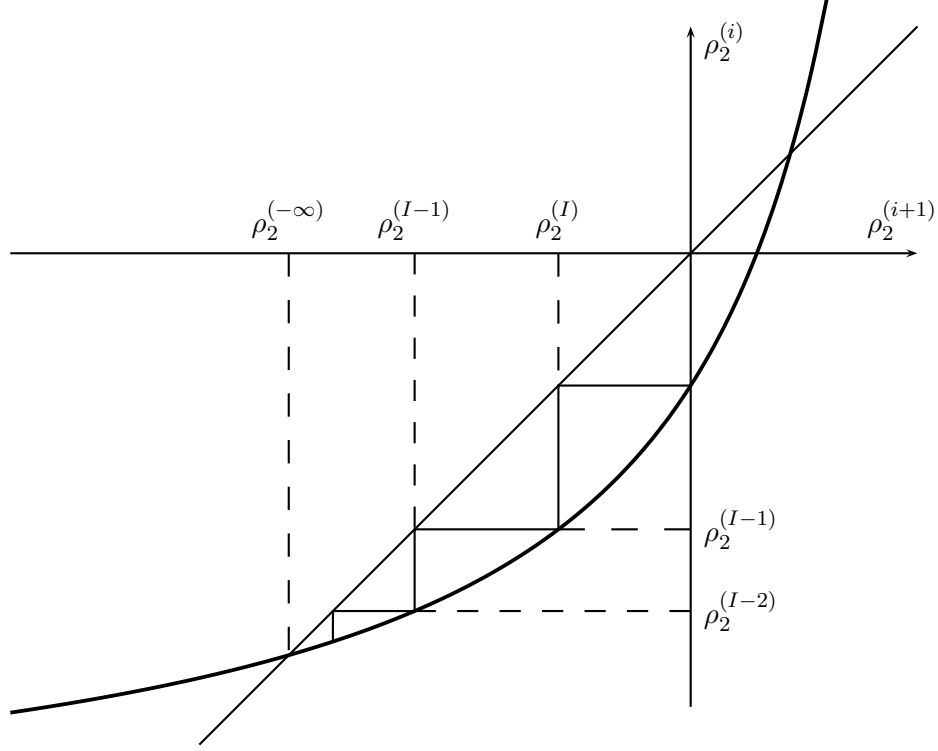
FIGURE 1. Successive values of $\rho_2^{(i)}$.

where the function γ is $\gamma(t) := \beta(h)\alpha(t) - \alpha(h)\beta(t)$. We already see that all the $\rho_2^{(i)}$'s are negative when $i \leqslant I$, as $\rho_2^{(I+1)} = 0$, and as the numerator of (21) is negative $(\mathcal{F}^2(\alpha\beta) \leqslant \mathcal{F}(\alpha^2)\mathcal{F}(\beta^2)$, and $\mathcal{F}(1)\mathcal{F}(\alpha^2) > 0)$.

(b) We now proceed with the simple fraction expansion of (21):

$$\rho_2^{(i)} = -\frac{\mathcal{F}(\gamma^2)}{\beta^2(h)} - \frac{b}{\pi}\frac{\alpha^2(h)}{\beta^2(h)}\mathcal{F}(1) + \frac{b}{\pi\beta^2(h)}\frac{\left[\alpha(h)\mathcal{F}(1) - \frac{\pi}{b}\mathcal{F}(\beta\gamma)\right]^2}{\mathcal{F}(1) + \frac{\pi}{b}[\mathcal{F}(\beta^2) - \varepsilon\beta^2(h)\rho_2^{(i+1)}]}, \qquad (22)$$

showing (see fig.1) that $\rho_2^{(i)}$ is an nondecreasing function of $\rho_2^{(i+1)}$. Therefore, the sequence $0 = \rho_2^{(I+1)} \geqslant \rho_2^{(I)} \geqslant \rho_2^{(I-1)} \geqslant \cdots$ is a nonincreasing nonpositive one, converging through an elementary iteration process towards the negative root $\rho_2^{(-\infty)}$ of

$$\frac{\pi}{b}\varepsilon\beta^2(h)\rho_2^2 + \left[\frac{\pi}{b}[\varepsilon\mathcal{F}(\gamma^2) - \mathcal{F}(\beta^2)] - [1 - \varepsilon\alpha^2(h)]\mathcal{F}(1)\right]\rho_2 - \frac{\pi}{b}[\mathcal{F}(\alpha^2)\mathcal{F}(\beta^2) - \mathcal{F}^2(\alpha\beta)] - \mathcal{F}(1)\mathcal{F}(\alpha^2) = 0. \qquad (23)$$
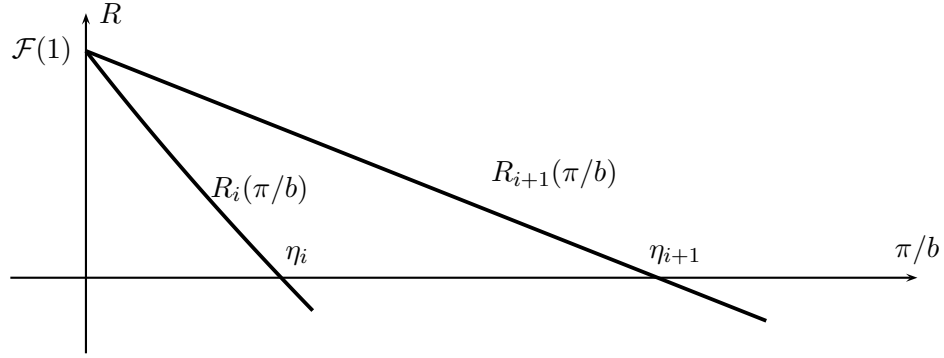
FIGURE 2. Zeros of two successive R functions.

If needed, we can get a more quantitative description of the sequence of the $\rho_2^{(i)}$'s. Indeed, with (22) written as $\rho_2^{(i)} = -A + \dfrac{B^2}{C - \rho_2^{(i+1)}}$,

$$\rho_2^{(i)} + A = \cfrac{B^2}{C + A - \cfrac{B^2}{C + A - \cfrac{B^2}{C + A - \cfrac{B^2}{C + A}}}}$$
$$\ddots$$
$$\cfrac{}{C + A - \dfrac{B^2}{C}}$$

Such continued fractions with constant elements are easily evaluated through powers of the roots of $r^2 - (C + A)r + B^2 = 0$ [1, § 123, p. 501], say, r_1 and r_2. Then, the relevant combination is soon found to be

$$\rho_2^{(i)} + A = B^2 \frac{r_1^{I-i+1} - r_2^{I-i+1} - A[r_1^{I-i} - r_2^{I-i}]}{r_1^{I-i+2} - r_2^{I-i+2} - A[r_1^{I-i+1} - r_2^{I-i+1}] = C[r_1^{I-i+1} - r_2^{I-i+1}] - B^2[r_1^{I-i} - r_2^{I-i}]}$$

whence $\rho_2^{(i)} = \dfrac{(B^2 - AC)[r_1^{I-i+1} - r_2^{I-i+1}]}{C[r_1^{I-i+1} - r_2^{I-i+1}] - B^2[r_1^{I-i} - r_2^{I-i}]}$

As r_1 and r_2 are positive and unequal[4], $\rho_2^{(i)}$ tends exponentially fast towards $\rho_2^{(-\infty)}$ when $i \to -\infty$.

(c) We now return to (21) in order to show that each $(\pi/b)\rho_2^{(i)}(\pi/b)$ is a nonincreasing function of π/b: indeed, it is true for $\rho_2^{(I+1)} \equiv 0$; if it is true for $(\pi/b)\rho_2^{(i+1)}(\pi/b)$, the denominator of (21) is a positive nondecreasing function, and the numerator times π/b is a nonpositive nonincreasing function.

Therefore, from (19), $R_i(\pi/b)$ is a nonincreasing function of π/b too, which must vanish for one positive value η_i of π/b (as $R_i(0) = \mathcal{F}(1) > 0$, and $R_i(\infty) = -\infty$ if $i < I$).

Finally, as $R_i < R_{i+1}$, $\eta_i < \eta_{i+1}$, (see fig.2), which ends the proof of the theorem.

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[4]One finds $[\sqrt{r_1} \pm \sqrt{r_2}]^2 = A + C \pm 2B = \dfrac{b\mathcal{F}(1)}{\pi\beta^2(h)}[\varepsilon^{-1/2} \pm \alpha(h)]^2 + \dfrac{1}{\beta^2(h)}[\varepsilon^{-1/2}\beta \mp \gamma]^2 > 0.$

3. Examples.

With the integral form of $\mathcal{F}$ from (9) above, and $\delta = 0.005, r = 0.03$ (found in [2]) in (7), we are able to compute some values of $R_i(\pi/b)$ (actually, $R_i/\mathcal{F}(1)$) when $h = 10$. Recall that $R_i < 0$ means that permits will hold for the period $(i-1)h < t < ih$, and that tax will be chosen if $R_i > 0$.

π/b	$R_{-\infty}$	R_{I-3}	R_{I-2}	R_{I-1}	R_I
0	1	1	1	1	1
$\eta^* = 0.0102390619$	0	0.00175409676	0.0150992067	0.122913662	0.695404832
$\eta_{I-3} = 0.0102631672$	-0.00174275376	0	0.0133000540	0.121058490	0.694687740
$\eta_{I-2} = 0.0104418439$	-0.0146311280	-0.0129701024	0	0.107330730	0.689372401
$\eta_{I-1} = 0.0118571462$	-0.115019270	-0.113880153	-0.103281030	0	0.647269499
$\eta_I = 0.0336153131$	-1.47231842	-1.47231520	-1.47201966	-1.44479355	0
0.0600374213	-2.95549322	-2.95549322	-2.95549322	-2.95549322	-0.786014045

The strange phenomenon of the last row occurs at $\pi/b = \alpha(h)\mathcal{F}(1)/\mathcal{F}(\beta\gamma)$ where all the R_i's have the same value, from R_{I-1} downwards.

We now look at values of η_i, i.e., the ratio π/b separating permits from tax, for various values of h. The value η^* corresponds to infinite horizon. We also show $\eta^*(HK)$ as calculated by Hoel and Karp [2, formula (15) p.377].

h	$\eta^*(HK)$	η^*	η_{I-3}	η_{I-2}	η_{I-1}	η_I
100	0.00199	0.000839	0.000839	0.000839	0.000839	0.000991
10	0.00897	0.010239	0.010263	0.010441	0.011857	0.033615
1	0.53543	0.645472	0.655971	0.696356	0.886452	3.033980
0.1	50.3504	61.00348	62.21340	66.55009	86.00395	300.3377

For an empirical illustration of pollution control in the framework of climate change, Hoel and Karp [2] give two estimates of π/b which are 0.0000137 and 0.00002. For all the values of h, r, and δ considered here, the tables above show that these estimates are largely lower than η^*, indicating that the regulator will choose the tax in all cases.

It is interesting to look at analytic formulas for small h, as done by Hoel and Karp [2]. Then, $\alpha(t) \approx 1$ and $\beta(t) \approx t$, so, $\mathcal{F}(1)$ and $\mathcal{F}(\alpha^2) \approx h$, $\mathcal{F}(\beta^2) \approx h^3/3$, etc.

from (20), $\rho_2^{(i)} \approx -h + \rho_2^{(i+1)} + \pi\dfrac{\left[-\dfrac{h^2}{2} + h\rho_2^{(i+1)}\right]^2}{bh + \pi\dfrac{h^3}{3} - \pi h^2\rho_2^{(i+1)}}$: $\rho_2^{(I+1)} = 0$, $\rho_2^{(I)} \approx -h\dfrac{1 + \dfrac{\pi}{b}\dfrac{h^2}{12}}{1 + \dfrac{\pi}{b}\dfrac{h^2}{3}}, \ldots$ con-

verging towards $\rho_2^{(-\infty)} \approx -\sqrt{\dfrac{b}{\pi} + \dfrac{h^2}{12}}$;

$R_I(\pi/b) \approx h[1 - (h^2/3)(\pi/b)]$, $R_{I-1}(\pi/b) \approx h\dfrac{1 - h^2(\pi/b) - (7h^4/36)(\pi/b)^2}{1 + (h^2/3)(\pi/b)}, \ldots$

$R_{-\infty}(\pi/b) \approx h[1 - (h^2/3)(\pi/b) - h\sqrt{\pi/b + (h^2/12)(\pi/b)^2}]$;

$\eta_I \approx 3/h^2$, $\eta_{I-1} \approx 6/(7h^2), \ldots$ $\eta^* \approx 6[5 - 2\sqrt{6}]/h^2 \approx 0.6/h^2$.

With simpler formulas, such as in [2], $\mathcal{F}(f) = \mathcal{F}(1)f(0)$, then $\mathcal{F}(\beta) = \mathcal{F}(\beta\gamma) = 0$, $\mathcal{F}(\gamma^2) = \mathcal{F}(1)\beta^2(h)$, $\rho_2^{(I)} = -\mathcal{F}(1)$, $R_I(\pi/b) = \mathcal{F}(1)$, $R_{I-1}(\pi/b) = \mathcal{F}(1)[1 - \varepsilon\beta^2(h)\pi/b], \ldots$

$$\rho_2^{(-\infty)} = \mathcal{F}(1)\dfrac{1 - \varepsilon\alpha^2(h) - \varepsilon\beta^2(h)\pi/b - \sqrt{[1 - \varepsilon\alpha^2(h) - \varepsilon\beta^2(h)\pi/b]^2 + 4\varepsilon\beta^2(h)\pi/b}}{2\varepsilon\beta^2(h)\pi/b}$$

$\eta_I = \infty$, $\eta_{I-1} = 1/(\varepsilon\beta^2(h)), \ldots, \eta^* = \eta_{-\infty} = [2 - \varepsilon\alpha^2(h)]/[2\varepsilon\beta^2(h)]$, in agreement with [2, formula (15) p.377], with $\varepsilon = \exp(-rh)$, $\alpha(h) = \exp(-\delta h)$, and $\beta(h) = h$.

4. References

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