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An estimate in the spirit of Poincaré's inequality

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Abstract. We show that if $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded Lipschitz domain and $(\rho_n) \subset L^1(\mathbb{R}^N)$ is a sequence of nonnegative radial functions weakly converging to δ_0, then

$$\int_\Omega |f - f_\Omega|^p \leq C \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n \left(|x - y|\right)\, dx\, dy$$

for all $f \in L^p(\Omega)$ and $n \geq n_0$, where f_Ω denotes the average of f on Ω. The above estimate was suggested by some recent work of Bourgain, Brezis and Mironescu [2]. As $n \to \infty$ we recover Poincaré's inequality. The case $N = 1$ requires an additional assumption on (ρ_n). We also extend a compactness result of Bourgain, Brezis and Mironescu.

1. Introduction and main results

Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with Lipschitz boundary and let $1 \leq p < \infty$. It is a well-known fact that there exists a constant $A_0 = A_0(p, \Omega) > 0$ such that the following form of Poincaré's inequality holds :

$$\int_\Omega |f - f_\Omega|^p \leq A_0 \int_\Omega |Df|^p \quad \forall f \in W^{1,p}(\Omega), \tag{1}$$

where $f_\Omega := \frac{1}{|\Omega|} \int_\Omega f$.

On the other hand, let $(\rho_n) \subset L^1(\mathbb{R}^N)$ be a sequence of **radial** functions satisfying

$$\begin{cases} \rho_n \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \rho_n = 1 \quad \forall n \geq 1, \\ \displaystyle\lim_{n \to \infty} \int_{|h| > \delta} \rho_n(h)\, dh = 0 \quad \forall \delta > 0. \end{cases} \tag{2}$$

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In this case, we have the following pointwise limit (see [2], see also [6] for a simpler proof)

$$\lim_{n\to\infty} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy = K_{p,N}\int_\Omega |Df|^p \tag{3}$$

for every $f \in W^{1,p}(\Omega)$, where $K_{p,N} = \fint_{S^{N-1}} |e_1\cdot\sigma|^p\,d\mathcal{H}^{N-1}$.

Motivated by this, we show the following estimate related to (1) :

Theorem 1.1. *Assume $N \geq 2$. Let $(\rho_n) \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying (2). Given $\delta > 0$, there exists $n_0 \geq 1$ sufficiently large such that*

$$\int_\Omega |f - f_\Omega|^p \leq \left(\frac{A_0}{K_{p,N}} + \delta\right)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \tag{4}$$

for every $f \in L^p(\Omega)$ and $n \geq n_0$.

The choice of $n_0 \geq 1$ depends not only on $\delta > 0$, but also on p, Ω and on the sequence $(\rho_n)_{n\geq 1}$. Special cases of this inequality have been used in the study of the Ginzburg-Landau model (see [3,4]; see also Corollaries 2.1–2.4 below).

We first point out that (4) is stronger than (1), in the sense that the right-hand side of (4) can be always estimated by $\int_\Omega |Df|^p$. In fact, given $f \in W^{1,p}(\Omega)$, we first extend f to $\mathbb{R}^N$ so that $f \in W^{1,p}(\mathbb{R}^N)$. It is then easy to see that (see e.g. [2, Theorem 1])

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \leq \int_{\mathbb{R}^N} |Df|^p \leq C\int_\Omega |Df|^p. \tag{5}$$

If $N = 1$, then one can construct examples of sequences $(\rho_n) \subset L^1(\mathbb{R})$ for which (4) fails (see [2, Counterexample 1]). In this case, we need to impose an additional condition on (ρ_n); see Theorem 1.3 below.

Theorem 1.1 can be deduced from the following compactness result :

Theorem 1.2. *Assume $N \geq 2$. Let $(\rho_n) \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying (2). If $(f_n) \subset L^p(\Omega)$ is a bounded sequence such that*

$$\int_\Omega\int_\Omega \frac{|f_n(x)-f_n(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \leq B \quad \forall n \geq 1, \tag{6}$$

then (f_n) is relatively compact in $L^p(\Omega)$.
Assume that $f_{n_j} \to f$ in $L^p(\Omega)$. Then

(a) $f \in W^{1,p}(\Omega)$ if $1 < p < \infty$;
(b) $f \in BV(\Omega)$ if $p = 1$.

In both cases, we have $\int_\Omega |Df|^p \leq \dfrac{B}{K_{p,N}}$, where B is given by (6).

This result was already known under the additional assumption that ρ_n is radially nondecreasing for every $n \geq 1$ (see [2, Theorem 4]).

We now consider the case $N = 1$.

Given $\rho_n \in L^1(\mathbb{R})$, we shall assume that ρ_n is defined for every $x \in \mathbb{R}$ in the following way

$$\rho_n(x) = \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{2r} \displaystyle\int_{x-r}^{x+r} \rho_n & \text{if } x \text{ is a Lebesgue point of } \rho_n, \\ +\infty & \text{otherwise.} \end{cases}$$

Given $\theta_0 \in (0, 1)$ we define

$$\rho_{n,\theta_0}(x) := \inf_{\theta_0 \leq \theta \leq 1} \rho_n(\theta x) \quad \forall x \in \mathbb{R}.$$

By construction,

$$\rho_{n,\theta_0}(x) \leq \rho_n(\theta x) \quad \forall x \in \mathbb{R} \quad \forall \theta \in [\theta_0, 1]. \tag{7}$$

We then have the following result :

Theorem 1.3. *Let $(\rho_n) \subset L^1(\mathbb{R})$ be a sequence of functions satisfying* (2). *Assume there exist $\theta_0 \in (0, 1)$ and $\alpha_0 > 0$ such that*

$$\int_{\mathbb{R}} \rho_{n,\theta_0} \geq \alpha_0 > 0 \quad \forall n \geq 1. \tag{8}$$

If $(f_n) \subset L^p(0, 1)$ is a bounded sequence such that

$$\int_0^1 \int_0^1 \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(x - y)\, dx\, dy \leq B \quad \forall n \geq 1, \tag{9}$$

then (f_n) is relatively compact in $L^p(0, 1)$.
Moreover, all the other statements of Theorems 1.1 and 1.2 are also valid. In particular, inequality (4) *holds with $\Omega = (0, 1)$.*

Most of the results in this paper were announced in [9].

2. Some examples

We now state some inequalities coming from Theorems 1.1 and 1.3. We denote by $Q = (0, 1)^N$ the N-dimensional unit cube. In all cases, condition (2) is satisfied for $N \geq 1$; it is also easy to see that (8) holds when $N = 1$.

For every $N \geq 1$ we then have the following corollaries :

Corollary 2.1 (Bourgain-Brezis-Mironescu [3]).

$$\int_Q |f - f_Q|^p \leq C_{s_0}(1 - s)p \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}}\, dx\, dy \quad \forall f \in L^p(Q),$$

for every $0 < s_0 < s < 1$.

This inequality takes into account the correction factor $(1-s)^{1/p}$ we should put in front of the Gagliardo seminorm $|f|_{W^{s,p}}$ as $s \uparrow 1$. In [3], the authors study related estimates arising from the Sobolev imbedding $L^q \hookrightarrow W^{s,p}$ for the critical exponent $\frac{1}{q} = \frac{1}{p} - \frac{s}{N}$; see also [7] for a more elementary approach.

Corollary 2.2 (Bourgain-Brezis-Mironescu [4]).

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^p} \frac{dx\,dy}{(|x-y| + \varepsilon)^N}$$

for every $f \in L^p(Q)$ *and* $0 < \varepsilon < \varepsilon_0$.

A stronger form of this inequality is the following

Corollary 2.3.

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \underset{|x-y|>\varepsilon}{\int_Q \int_Q} \frac{|f(x) - f(y)|^p}{|x-y|^{N+p}} \, dx\,dy \quad \forall f \in L^p(Q),$$

for every $0 < \varepsilon < \varepsilon_0 \ll 1$.

We have been informed by H. Brezis that Bourgain and Brezis [1] have proved that

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{(|x-y| + \varepsilon)^{N+p}} \, dx\,dy \quad \forall f \in L^p(Q),$$

for every $0 < \varepsilon < \varepsilon_0$, using a Paley-Littlewood decomposition of f. Note that this estimate can be deduced instead from the corollary above.

Here is another example :

Corollary 2.4.

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{N+p}{\varepsilon^{N+p}} \underset{|x-y|<\varepsilon}{\int_Q \int_Q} |f(x) - f(y)|^p \, dx\,dy \quad \forall f \in L^p(Q),$$

for every $0 < \varepsilon < \varepsilon_0$.

Concerning the behavior of the constants in these inequalities, let A_0 denote the best constant in (1). Then in Corollary 2.1 the constant C_{s_0} can be chosen so that

$$C_{s_0} \to \frac{A_0}{K_{p,N} |S^{N-1}|} \quad \text{as } s_0 \uparrow 1.$$

Similarly, in Corollaries 2.2–2.4 we have C_{ε_0} converging to the same limit as $\varepsilon_0 \downarrow 0$.

Applying Theorem 1.1 to $p = 1$ and $f = \chi_E$, where $E \subset Q$ is any measurable set, we get (see also [3] for related results) :

Corollary 2.5. *Let $N \geq 2$. Given a sequence of radial functions $(\rho_n) \subset L^1(\mathbb{R}^N)$ satisfying (2), then for any $C > A_0/K_{1,N}$ there exists $n_0 \geq 1$ such that*

$$|E||Q\backslash E| \leq C \int_E \int_{Q\backslash E} \frac{\rho_n(|x-y|)}{|x-y|}\, dx\, dy \quad \forall E \subset Q \text{ measurable} \quad \forall n \geq n_0.$$

3. Estimates in dimension $N = 1$

Given any $g \in L^p(\mathbb{R})$, let $G_p : [0,\infty) \to [0,\infty)$ be the (continuous) function defined by

$$G_p(t) = \int_{\mathbb{R}} |g(x+t) - g(x)|^p\, dx \quad \forall t \geq 0. \tag{10}$$

We start with the following

Lemma 3.1. *Given $0 < s < t$, let $k \in \mathbb{N}$ and $\theta \in [0,1)$ be such that $\frac{t}{s} = k + \theta$. Then there exists $C_p > 0$ such that for every $g \in L^p(\mathbb{R})$ we have*

$$\frac{G_p(t)}{t^p} \leq C_p \left\{ \frac{G_p(s)}{s^p} + \frac{G_p(\theta s)}{t^p} \right\}. \tag{11}$$

Proof. Note that

$$\begin{aligned}
|g(x+t) - g(x)|^p &= |g(x+ks+\theta s) - g(x)|^p \\
&\leq 2^{p-1}\Big\{ |g(x+ks) - g(x)|^p + \\
&\qquad\qquad + |g(x+ks+\theta s) - g(x+ks)|^p \Big\} \\
&\leq 2^{p-1}k^{p-1} \sum_{j=0}^{k-1} |g(x+js+s) - g(x+js)|^p + \\
&\qquad\qquad + 2^{p-1}|g(x+ks+\theta s) - g(x+ks)|^p.
\end{aligned}$$

Integrating with respect to $x \in \mathbb{R}$ and changing variables we get

$$G_p(t) \leq 2^{p-1}k^p G_p(s) + 2^{p-1}G_p(\theta s).$$

Recall that $k \leq \frac{t}{s}$. We then conclude that (11) holds with $C_p = 2^{p-1}$.

Another estimate we shall need is given by the lemma below :

Lemma 3.2. *Let $r > 0$. There exists a constant $C_p > 0$ so that the following holds : for every $g \in L^p(0,2r)$ such that $g = 0$ a.e. in $(r,2r)$ we have*

$$\int_0^r |g|^p \leq C_p r^p \int_0^r \frac{|g(x+t) - g(x)|^p}{t^p}\, dx \quad \forall t \in (0,r). \tag{12}$$

Proof. By a scaling argument, it suffices to prove the lemma for $r = 1$. We now extend $g \in L^p(0,2)$ to the entire half-line so that $g = 0$ a.e. in $(1,\infty)$. Given $0 < t < 1$, let $k \geq 1$ be the first integer satisfying $kt \geq 1$. In particular, for $x \in (0,1)$ we have $x + kt > 1$, thus

$$|g(x)|^p = |g(x+kt) - g(x)|^p \leq k^{p-1} \sum_{j=0}^{k-1} |g(x+jt+t) - g(x+jt)|^p.$$

Integrating this inequality with respect to x we get

$$\begin{aligned}\int_0^1 |g|^p &\leq k^{p-1} \sum_{j=0}^{k-1} \int_0^\infty |g(x+jt+t) - g(x+jt)|^p \, dx \\ &= k^p \int_0^\infty |g(x+t) - g(x)|^p \, dx = k^p \int_0^1 |g(x+t) - g(x)|^p \, dx.\end{aligned}$$

Note however that $k \leq \dfrac{2}{t}$. The lemma now follows by taking $C = 2^p$.

4. Compactness in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $N \geq 2$

Given $f \in L^p(\mathbb{R}^N)$, we consider $F_p : \mathbb{R}^N \to [0,\infty)$ defined by

$$F_p(h) = \int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx \quad \forall h \in \mathbb{R}^N.$$

This function is continuous and satisfies

$$F_p(h_1 + h_2) \leq 2^{p-1}\big[F_p(h_1) + F_p(h_2)\big] \quad \forall h_1, h_2 \in \mathbb{R}^N.$$

We have the following

Lemma 4.1. *Assume $N \geq 2$. Then there exists $C_p > 0$ such that*

$$\int_{S^{N-1}} \frac{F_p(tv)}{t^p} \, d\sigma(v) \leq C_p \int_{S^{N-1}} \frac{F_p(sv)}{s^p} \, d\sigma(v) \quad \textit{for every } 0 < s < t. \tag{13}$$

Proof. Let $0 < s < t < \infty$. Given $v \in S^{N-1}$ and $w \in (\mathbb{R}v)^\perp$, we apply the one dimensional estimate in Lemma 3.1 to the function

$$g(\tau) = f(w + \tau v) \quad \text{for a.e. } \tau \geq 0.$$

Integrating the resulting expression with respect to $w \in (\mathbb{R}v)^\perp$, it follows that for every $v \in S^{N-1}$ we have

$$\frac{F_p(tv)}{t^p} \leq C_p \left\{ \frac{F_p(sv)}{s^p} + \frac{F_p(\theta sv)}{t^p} \right\} \tag{14}$$

for some $\theta \in [0,1)$ (depending on s and t). We now split the proof into two cases :

Case 1. N is even.

Let $O \in O(N)$ be an orthogonal transformation such that $\langle Ow, w\rangle = 0$ for every $w \in \mathbb{R}^N$ (this is possible since N is even). We then consider

$$O_1 w := \frac{\theta}{2} w + \sqrt{1 - \frac{\theta^2}{4}} Ow,$$
$$O_2 w := \frac{\theta}{2} w - \sqrt{1 - \frac{\theta^2}{4}} Ow.$$

Note that $O_1, O_2 \in O(N)$ and

$$\theta w = O_1 w + O_2 w \quad \forall w \in \mathbb{R}^N,$$

thus

$$F_p(\theta s v) \le 2^{p-1}\Big\{F_p(s\, O_1 v) + F_p(s\, O_2 v)\Big\}.$$

Inserting this inequality into (14) we get

$$\frac{F_p(tv)}{t^p} \le C_p \frac{F_p(sv) + F_p(s\, O_1 v) + F_p(s\, O_2 v)}{s^p}.$$

Integrating with respect to $v \in S^{N-1}$ we obtain (13).

Case 2. N is odd.

Let $v \in S^{N-1}$. We denote by S_v^{N-2} the $(N-2)$-sphere orthogonal to v :

$$S_v^{N-2} := S^{N-1} \cap (\mathbb{R}v)^{\perp}.$$

Reasoning as in the previous case, we see that

$$\int_{S_v^{N-2}} \frac{F_p(tw)}{t^p}\, d\mathcal{H}^{N-2} \le C_p \int_{S_v^{N-2}} \frac{F_p(sw)}{s^p}\, d\mathcal{H}^{N-2}. \tag{15}$$

On the other hand, on S^{N-1} we consider the measure μ defined as

$$\mu(A) = \int_{S^{N-1}} \mathcal{H}^{N-2}\big(A \cap S_v^{N-2}\big)\, d\sigma(v) \quad \text{for every Borel set } A \subset S^{N-1}.$$

Note that μ is invariant under orthogonal transformations, i.e. $\mu(OA) = \mu(A)$ for every $O \in O(N)$, and $\mu(S^{N-1}) = |S^{N-2}|$. It then follows that

$$\mu = |S^{N-2}|\, \mathcal{H}^{N-1}\lfloor_{S^{N-1}}.$$

We now integrate (15) with respect to $v \in S^{N-1}$. Using the observation above we get (13).

The lemma above implies the following compactness result :

Proposition 4.2. *Assume $N \ge 2$. Let $(f_n) \subset L^p(\mathbb{R}^N)$ be a bounded sequence of functions such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n\left(|x-y|\right) dx\, dy \le B \quad \forall n \ge 1. \tag{16}$$

Then (f_n) is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. Fix $t_0 > 0$. Let $n_0 \geq 1$ be such that

$$\int_{B_{t_0}} \rho_n \geq \frac{1}{2} \quad \forall n \geq n_0.$$

We first prove the following

Claim. There exists a constant $C = C(p, N, B) > 0$ such that

$$\int_{S^{N-1}} F_{n,p}(tv)\, d\sigma(v) \leq C t_0^p \tag{17}$$

for every $0 < t < t_0$ and every $n \geq n_0$. ($F_{n,p}$ denotes the function F_p associated to f_n).

In fact, let $s, \tau > 0$ be such that $0 < s < t_0 \leq \tau$. It follows from the previous lemma that

$$\int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p}\, d\sigma(v) \leq C_p \int_{S^{N-1}} \frac{F_{n,p}(sv)}{s^p}\, d\sigma(v).$$

We now multiply both sides by $s^{N-1}\rho_n(s)$ and then integrate the resulting expression with respect to s running from 0 to t_0. We get

$$\begin{aligned}
\frac{1}{2|S^{N-1}|} \int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p}\, d\sigma(v) &\leq \int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p}\, d\sigma(v) \int_0^{t_0} \rho_n(s) s^{N-1}\, ds \\
&\leq C \int_0^{t_0} \int_{S^{N-1}} \frac{F_{n,p}(sv)}{s^p} \rho_n(s) s^{N-1}\, d\sigma(v)\, ds \\
&\leq C \int_{\mathbb{R}^N} \frac{F_{n,p}(h)}{|h|^p} \rho_n(|h|)\, dh.
\end{aligned}$$

Note that the last term is precisely the double integral in the left-hand side of (16). We then conclude that

$$\int_{S^{N-1}} F_{n,p}(\tau v)\, d\sigma(v) \leq C\tau^p \quad \forall \tau \geq t_0 \quad \forall n \geq n_0.$$

We now let $0 < t < t_0$. Using the above estimate with $\tau = t_0$ and $\tau = t + t_0$ we get

$$\begin{aligned}
&\int_{S^{N-1}} F_{n,p}(tv)\, d\sigma(v) \leq \\
&\qquad \leq 2^{p-1} \left\{ \int_{S^{N-1}} F_{n,p}(t_0 v)\, d\sigma + \int_{S^{N-1}} F_{n,p}((t + t_0)v)\, d\sigma \right\} \\
&\qquad \leq 2^{p-1} C \big[t_0^p + (t + t_0)^p \big] \leq C t_0^p
\end{aligned}$$

for every $n \geq n_0$. This proves the claim.

Once we reach at this point we can proceed as in [2].

We first set $\Phi_\delta := \dfrac{1}{|B_\delta|}\chi_{B_\delta}$. For any $0 < \delta < t_0$, it follows from the previous estimate that

$$\begin{aligned}\int_{\mathbb{R}^N} |\Phi_\delta * f_n(x) - f_n(x)|^p\,dx &= \int_{\mathbb{R}^N} \left| \fint_{B_\delta} \big[f_n(x+h) - f_n(x)\big]\,dh \right|^p dx \\ &\le \int_{\mathbb{R}^N} \fint_{B_\delta} |f_n(x+h) - f_n(x)|^p\,dh\,dx \\ &= \frac{1}{|B_\delta|} \int_0^\delta \int_{S^{N-1}} F_{n,p}(tv)\,d\sigma(v)\,t^{N-1}\,dt \\ &\le \frac{Ct_0^p}{|B_\delta|} \int_0^\delta t^{N-1}\,dt \le Ct_0^p.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^N} |\Phi_\delta * f_n(x) - f_n(x)|^p\,dx \le Ct_0^p \quad \forall n \ge n_0 \quad \forall \delta \in (0, t_0). \tag{18}$$

We now conclude the proof by applying a variant of the Fréchet-Kolmogorov Theorem. In fact, since (f_n) is bounded in $L^p(\mathbb{R}^N)$, then for every $\delta > 0$ fixed the sequence $(\Phi_\delta * f_n)$ is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ (see [5, Corollary IV.27]), hence it is totally bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Using (18), it follows that (f_n) is also totally bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, which implies that (f_n) is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

5. An L^p-estimate near the boundary of Ω

In this section we shall prove the following

Lemma 5.1. *Assume $N \ge 2$. Then there exist constants $r_0 > 0$ (depending on Ω and on the sequence $(\rho_n)_{n\ge1}$) and $C_1, C_2 > 0$ (depending on p, Ω and N) so that the following holds : given $0 < r < r_0$ we can find $n_0 \ge 1$ such that*

$$\int_\Omega |f|^p \le C_1 \int_{\Omega_r} |f|^p + C_2 r^p \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|)\,dx\,dy \tag{19}$$

for every $f \in L^p(\Omega)$ and $n \ge n_0$.

Proof. Let $x_0 \in \partial\Omega$. Without loss of generality, we may assume that $x_0 = 0$. Take $r_0 > 0$ sufficiently small such that (up to a rotation of $\partial\Omega$) the set $\partial\Omega \cap B_{4r_0}$ is the graph of a Lipschitz function γ. For simplicity, we can also assume that γ has Lipschitz constant at most $1/2$.
Given $0 < r < r_0$, we consider the graph of γ :

$$\Gamma_r := \Big\{ x = (x', \gamma(x')) \in \mathbb{R}^N : x' \in B'_r \Big\}.$$

Let Λ be the upper half cone

$$\Lambda := \Big\{ x = (x', x_N) \in \mathbb{R}^N : |x'| \le x_N \Big\}.$$

We also define

$$\Omega_r := \Big\{ x \in \Omega : d(x, \partial\Omega) > r \Big\}.$$

Because of the upper bound on the Lipschitz constant of γ, we have

$$\Omega \cap B_{r/2} \subset \Gamma_r + (\Lambda \cap B_r) \subset \Omega \cap B_{3r} \tag{20}$$

for every $0 < r < r_0$. We first prove the following

Claim. There exists $n_0 \ge 1$ depending on $r \in (0, r_0)$ such that if $f \in L^p(\Omega)$ and $f = 0$ a.e. in Ω_r, then

$$\int_{\Omega \cap B_{r/2}} |f|^p \le C r^p \int_{\Omega \cap B_{4r}} \int_{\Omega \cap B_{4r}} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n \left(|x - y|\right) \, dx \, dy \tag{21}$$

for every $n \ge n_0$.

In fact, given $\xi \in \Gamma_r$ and $v \in \Lambda \cap S^{N-1}$, we consider the function

$$g(t) = f(\xi + tv) \quad \text{for a.e. } t \in (0, 2r).$$

Applying Lemma 3.2 to g we get

$$\int_0^r |f(\xi + sv)|^p \, ds \le C r^p \int_0^r \frac{|f(\xi + sv + tv) - f(\xi + sv)|^p}{t^p} \, ds$$

for every $0 < t < r$.

Recall that $\xi = (x', \gamma(x'))$ for some $x' \in B_r' \subset \mathbb{R}^{N-1}$. We first integrate the above estimate with respect to $x' \in B_r'$ and then we perform the change of coordinates

$$y = (x', \gamma(x')) + sv$$

with respect to the variables x' and s. Using (20) we then find

$$\begin{aligned} \int_{\Omega \cap B_{r/2}} |f|^p &\le C r^p \int_{\Gamma_r + (\Lambda \cap B_r)} \frac{|f(y + tv) - f(y)|^p}{t^p} \, dy \\ &\le C r^p \int_{\Omega \cap B_{3r}} \frac{|f(y + tv) - f(y)|^p}{t^p} \, dy. \end{aligned} \tag{22}$$

Take $n_0 \ge 1$ sufficiently large so that

$$\int_{B_r} \rho_n \ge \frac{1}{2} \quad \forall n \ge n_0.$$

Since each ρ_n is a radial function, there exists $c > 0$ such that

$$\int_{\Lambda \cap B_r} \rho_n \ge c \quad \forall n \ge n_0.$$

We now multiply (22) by $\rho_n(t)t^{N-1}$. Integrating the resulting expression with respect to $t \in (0,r)$ and $v \in \Lambda \cap S^{N-1}$ we get

$$
\begin{aligned}
c\int_{\Omega\cap B_{r/2}} |f|^p &\le Cr^p \int_{\Omega\cap B_{3r}}\int_{\Lambda\cap B_r} \frac{|f(y+h)-f(y)|^p}{|h|^p}\rho_n(|h|)\,dh\,dy\\
&\le Cr^p \int_{\Omega\cap B_{4r}}\int_{\Omega\cap B_{4r}} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy.
\end{aligned}
$$

This completes the proof of the claim.

Using a standard covering argument, it follows from the claim above that there exists $n_0 \ge 1$ depending on $r \in (0,r_0)$ such that if $f \in L^p(\Omega)$ and $f = 0$ a.e. in Ω_r, then

$$
\int_{\Omega\setminus\Omega_{r/4}} |f|^p \le Cr^p \int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \tag{23}
$$

for every $n \ge n_0$, where the constant $C > 0$ is independent of f, r and n.

We now take $f \in L^p(\Omega)$ arbitrary. In other words, we do not impose any restriction on the set $\operatorname{supp} f$.
Let $\zeta \in C^\infty(\Omega)$ be such that $\zeta \equiv 0$ on Ω_r, $\zeta \equiv 1$ on $\Omega\setminus\Omega_{r/2}$, $0 \le \zeta \le 1$ on Ω and $|\nabla\zeta| \le C/r$ on Ω. Applying (23) to the function ζf we get

$$
\begin{aligned}
\int_{\Omega\setminus\Omega_{r/4}} |f|^p &\le Cr^p \int_{\Omega}\int_{\Omega} \frac{|\zeta(x)f(x)-\zeta(y)f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy\\
&\le 2^{p-1}Cr^p \biggl\{ \int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy +\\
&\qquad + \int_{\Omega}\int_{\Omega} |f(x)|^p \frac{|\zeta(x)-\zeta(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \biggr\}.
\end{aligned}
$$

We now estimate the second double-integral in the right-hand side. Since $\zeta(x) = \zeta(y) = 1$ for every $x, y \in \Omega\setminus\Omega_{r/2}$ we have

$$
\int_{\Omega}\int_{\Omega} |f(x)|^p \frac{|\zeta(x)-\zeta(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy = \iint\limits_{\substack{x\in\Omega\setminus\Omega_{r/4}\\ y\in\Omega_{r/2}}} + \iint\limits_{\substack{x\in\Omega_{r/4}\\ y\in\Omega}} .
$$

Note that $d\bigl(\Omega\setminus\Omega_{r/4}, \Omega_{r/2}\bigr) = r/4$, thus

$$
\iint\limits_{\substack{x\in\Omega\setminus\Omega_{r/4}\\ y\in\Omega_{r/2}}} \le \frac{C}{r^p}\int_{|h|>\frac{r}{4}} \rho_n \cdot \int_{\Omega} |f|^p \quad \text{and} \quad \iint\limits_{\substack{x\in\Omega_{r/4}\\ y\in\Omega}} \le \frac{C}{r^p}\int_{\Omega_{r/4}} |f|^p.
$$

We then conclude that

$$\begin{aligned}\int_\Omega |f|^p &= \int_{\Omega_{r/4}} |f|^p + \int_{\Omega\setminus\Omega_{r/4}} |f|^p \\ &\leq C\int_{\Omega_{r/4}} |f|^p + Cr^p \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy+ \\ &\qquad + C\int_{|h|>\frac{r}{4}} \rho_n \cdot \int_\Omega |f|^p.\end{aligned}$$

Taking $n_0 \geq 1$ large enough so that

$$\int_{|h|>\frac{r}{4}} \rho_n \leq \frac{1}{2C} \quad \forall n \geq n_0,$$

we see that (19) holds.

6. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.2. Given $l \geq 1$, we fix $\varphi_l \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on $\Omega_{1/l}$. It is easy to see that the sequence $(\varphi_l f_n)_{n\geq 1}$ satisfies the assumptions of Proposition 4.2. In particular, (f_n) is relatively compact in $L^p(\Omega_l)$. Applying a standard diagonalization argument, we can extract a subsequence (f_{n_j}) such that $f_{n_j} \to f$ in $L^p_{\mathrm{loc}}(\Omega)$. Since the original sequence is bounded, $f \in L^p(\Omega)$.

Claim. $f \in BV(\Omega)$ if $p=1$ and $f \in W^{1,p}(\Omega)$ if $1<p<\infty$; moreover,

$$\int_\Omega |Df|^p \leq \frac{B}{K_{p,N}}. \tag{24}$$

Let $\varphi \in C_0^\infty(B_1)$ be such that $\varphi \geq 0$ and $\int \varphi = 1$. Given $\delta > 0$, we define

$$\varphi_\delta(x) := \frac{1}{\delta^N}\varphi\left(\frac{x}{\delta}\right) \quad \forall x \in \mathbb{R}^N.$$

It follows from Jensen's inequality and estimate (6) that

$$\int_{\Omega_\delta}\int_{\Omega_\delta} \frac{|\varphi_\delta * f_n(x) - \varphi_\delta * f_n(y)|^p}{|x-y|^p}\rho_n\left(|x-y|\right)\,dx\,dy \leq B \quad \forall n \geq 1. \tag{25}$$

We now observe that for each $\delta > 0$ fixed, the subsequence $(\varphi_\delta * f_{n_j})_{j\geq 1}$ converges to $\varphi_\delta * f$ in $C^2(\overline{\Omega}_\delta)$. Taking $n_j \to \infty$ in (25) we get (see e.g. [8, Remark 7])

$$K_{p,N}\int_{\Omega_\delta} \left|D(\varphi_\delta * f)\right|^p \leq B \quad \forall \delta > 0.$$

The claim now follows by taking $\delta \to 0$.

We are left to prove that $f_{n_j} \to f$ in $L^p(\Omega)$.

In order to show this, we apply (19) with f replaced by $f_{n_j} - f$. Using (5) and (6) we get

$$\int_\Omega |f_{n_j} - f|^p \leq C_1 \int_{\Omega_r} |f_{n_j} - f|^p + C_2 r^p 2^{p-1} \left(B + C \int_\Omega |Df|^p \right)$$

for every $n_j \geq n_0(r)$. For $r > 0$ fixed we let $j \to \infty$. It follows that

$$\limsup_{j\to\infty} \int_\Omega |f_{n_j} - f|^p \leq C_2 r^p 2^{p-1} \left(B + C \int_\Omega |Df|^p \right).$$

Taking $r \to 0$, we conclude that $f_{n_j} \to f$ in $L^p(\Omega)$.

As a corollary to Theorem 1.2 we have

Proof of Theorem 1.1. Let $A_0 > 0$ be the best constant of the inequality (1). Assume by contradiction that there exists $C > A_0/K_{p,N}$ for which (4) fails for every $n \geq n_0$. This means there exists a sequence (f_n) in $L^p(\Omega)$ verifying the following properties :

$$\int_\Omega |f_n|^p = 1 \quad \text{and} \quad \int_\Omega f_n = 0, \tag{26}$$

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n\left(|x-y|\right)\, dx\, dy < \frac{1}{C}. \tag{27}$$

Note that (f_n) satisfies the assumptions of Theorem 1.2. We can then extract a convergent subsequence $f_{n_j} \to f$ in $L^p(\Omega)$. In particular, it follows from (26) that

$$\int_\Omega |f|^p = 1 \quad \text{and} \quad \int_\Omega f = 0.$$

On the other hand, from (27) we have

$$\int_\Omega |Df|^p \leq \frac{1}{K_{p,N} C}.$$

These two facts imply that $1 \leq \dfrac{A_0}{K_{p,N} C}$, a contradiction.

7. Proof of Theorem 1.3

We first observe that after replacing the sequence ρ_n by $\frac{\rho_n(t)+\rho_n(-t)}{2}$, we can always assume that each ρ_n is an even function. Note that (9) still holds with the same constant B.

To prove the theorem we shall follow the sames steps as before. We start with a compactness lemma :

Lemma 7.1. *Assume there exists $\theta_0 \in (0,1)$ and $\alpha_0 > 0$ such that* (8) *holds. If $(f_n) \subset L^p(\mathbb{R})$ is a bounded sequence of functions such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n\,(x-y)\, dx\, dy \le B \quad \forall n \ge 1, \tag{28}$$

then (f_n) is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R})$.

Proof. Let $\ell_0 \ge 1$ be a fixed integer. We first prove the following

Claim. Estimate (11) still holds with θ replaced by

$$\tilde{\theta} := 1 - \frac{\theta}{\ell_0} = 1 - \frac{1}{\ell_0}\left(\frac{t}{s} - k\right)$$

(with the constant C_p also depending on ℓ_0).

Indeed, it suffices to notice that

$$G_p(\theta s) \le \ell_0^p\, G_p\Big(\frac{\theta s}{\ell_0}\Big) \le 2^{p-1}\ell_0 \Big\{ G_p(s) + G_p\Big(s - \frac{\theta s}{\ell_0}\Big)\Big\}.$$

Inserting this inequality into (11), the claim follows.

Given $\theta_0 \in (0,1)$, we take $\ell_0 \ge 2$ sufficiently large so that $1/\ell_0 < 1 - \theta_0$; in particular, we have $\theta_0 < \tilde{\theta} \le 1$.

We now fix $t_0 > 0$. Take $n_0 \ge 1$ sufficiently large so that

$$\int_0^{t_0} \rho_{n,\theta_0} \ge \frac{\alpha_0}{4} \quad \forall n \ge n_0.$$

We know from our claim that

$$\frac{F_{n,p}(\tau)}{\tau^p} \le C \left\{ \frac{F_{n,p}(s)}{s^p} + \frac{F_{n,p}(\tilde{\theta} s)}{\tau^p} \right\}$$

for every $0 < s < t_0 \le \tau$. We multiply both sides of this inequality by ρ_{n,θ_0}. Using (7) and integrating the resulting expression from 0 to t_0 we get

$$\frac{\alpha_0}{4}\frac{F_{n,p}(\tau)}{\tau^p} \le C\left\{ \int_0^\infty \frac{F_{n,p}(s)}{s^p}\rho_n(s)\, ds + \frac{1}{\tau^p}\int_0^{t_0} F_{n,p}(\tilde{\theta}s)\rho_n(\tilde{\theta}s)\, ds \right\} \tag{29}$$

for every $\tau \ge t_0$ and $n \ge n_0$. We now estimate the second integral in the right-hand side of this inequality. We first observe that

$$\frac{1}{\tau^p}\int_0^{t_0} F_{n,p}(\tilde{\theta}s)\rho_n(\tilde{\theta}s)\, ds \le \int_0^\tau \frac{F_{n,p}(\tilde{\theta}s)}{(\tilde{\theta}s)^p}\rho_n(\tilde{\theta}s)\, ds =: I.$$

We then make the change of variables $h = \tilde{\theta}s$ (note that $\tilde{\theta}$ is a function of s for fixed τ). Recall that, by definition,

$$\tilde{\theta}s = \left(\frac{k}{\ell_0} + 1\right)s + \frac{\tau}{\ell_0} \quad \text{for } k \le \frac{\tau}{s} < k+1.$$

Thus,

$$
\begin{aligned}
I &= \sum_{k=1}^{\infty} \int_{\frac{\tau}{k+1}}^{\frac{\tau}{k}} \frac{F_{n,p}(\tilde{\theta} s)}{(\tilde{\theta} s)^p} \rho_n(\tilde{\theta} s)\, ds \\
&= \sum_{k=1}^{\infty} \int_{(1-\frac{1}{\ell_0})\frac{\tau}{k+1}}^{\frac{\tau}{k}} \frac{F_{n,p}(h)}{h^p} \rho_n(h)\, \frac{dh}{\frac{k}{\ell_0}+1} \le C \int_0^{\infty} \frac{F_{n,p}(h)}{h^p} \rho_n(h)\, dh.
\end{aligned} \tag{30}
$$

This last inequality comes from the fact that $\frac{1}{k_0}$ belongs to at most Ck_0 intervals of the form

$$\Big((1-\frac{1}{\ell_0})\frac{1}{k+1}; \frac{1}{k}\Big) \quad \text{for } k \ge 1.$$

Inserting (30) into (29) and using (28) we conclude that

$$\frac{F_{n,p}(\tau)}{\tau^p} \le \frac{C}{\alpha_0} \int_0^{\infty} \frac{F_{n,p}(s)}{s^p} \rho_n(s)\, ds \le \frac{C}{\alpha_0} B$$

for every $\tau \ge t_0$ and $n \ge n_0$. Proceeding as in the proof of (17), this implies that

$$\int_{\mathbb{R}} |f_n(x+t) - f_n(x)|^p\, dx \le C t_0^p \quad \forall t \in (0, t_0) \quad \forall n \ge n_0.$$

In other words, the sequence (f_n) is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R})$ (see [5, Theorem IV.25]).

The analogue of Lemma 5.1 is the following

Lemma 7.2. *There exist $r_0 > 0$ (depending on $(\rho_n)_{n\ge1}$) and constants $C_1, C_2 > 0$ (depending on p) so that the following holds : given $0 < r < r_0$ we can find $n_0 \ge 1$ such that*

$$\int_0^1 |f|^p \le C_1 \int_r^{1-r} |f|^p + C_2 r^p \int_0^1 \int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n(x-y)\, dx\, dy \tag{31}$$

for every $f \in L^p(0,1)$ and $n \ge n_0$.

Proof. We proceed exactly as in the proof of Lemma 5.1. Actually, this case is even simpler since the claim is essentially contained in Lemma 3.2. Note in particular that condition (8) is not needed.

Theorem 1.3 can now be proved as in the previous section.

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