# On the distributions of the form $\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ 

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#### Abstract

We present some properties of the distributions $T$ of the form $\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$, with $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$, which arise in the study of the 3-d Ginzburg-Landau problem; see Bourgain, Brezis and Mironescu (C. R. Acad. Sci. Paris, Ser. I 331 (2000) 119-124). We show that there always exists an irreducible representation of $T$. We also extend a result of D. Smets (C. R. Acad. Sci. Paris, Ser. I 334 (2002) 371-374) which says that $T$ is a measure iff $T$ can be written as a finite sum of dipoles.


## 1 Introduction

Given a complete metric space $(X, d)$ and points $\left(p_{i}\right),\left(n_{i}\right) \subset X$ such that $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$, we consider the following linear functional in $[\operatorname{Lip}(X)]^{*}$ :

$$
\begin{equation*}
T:=\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \tag{1}
\end{equation*}
$$

given by

$$
\begin{equation*}
\langle T, \zeta\rangle=\sum_{i}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right] \quad \forall \zeta \in \operatorname{Lip}(X) . \tag{2}
\end{equation*}
$$

Note that $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$ implies that $T$ is well-defined and continuous in $\operatorname{Lip}(X)$.

In this paper, we present some properties satisfied by $T$. Our proofs rely on the existence of irreducible representations of $T$, a notion which we introduce below; see Definition 7.

[^0]In applications, $T$ describes the location and the topological degree of singularities of maps $u$ defined on $X$ with values into a sphere $S^{k}$. Assume for instance that $X=\mathbb{R}^{3}$ and $k=2$. We then consider

$$
H^{1}\left(\mathbb{R}^{3} ; S^{2}\right)=\left\{u: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}: \int_{\mathbb{R}^{3}}|\nabla u|^{2}<\infty \text { and }|u|=1 \text { a.e. in } \mathbb{R}^{3}\right\} .
$$

Note that for any $u \in H^{1}\left(\mathbb{R}^{3} ; S^{2}\right)$ we have

$$
D(u):=\left(u \cdot u_{y} \wedge u_{z}, u \cdot u_{z} \wedge u_{x}, u \cdot u_{x} \wedge u_{y}\right) \in L^{1}\left(\mathbb{R}^{3} ; \mathbb{R}^{3}\right)
$$

In particular, the distribution div $D(u)$ is well-defined in $\mathbb{R}^{3}$; moreover, one can show there exist sequences of points $\left(r_{i}\right),\left(q_{i}\right) \subset \mathbb{R}^{3}$ such that (see [4])

$$
\begin{gather*}
\sum_{i=1}^{\infty} d\left(r_{i}, q_{i}\right)<\frac{1}{8 \pi} \int_{\mathbb{R}^{3}}|\nabla u|^{2},  \tag{3}\\
\operatorname{div} D(u)=4 \pi \sum_{i=1}^{\infty}\left(\delta_{r_{i}}-\delta_{q_{i}}\right) \quad \text { in } \mathcal{D}^{\prime}\left(\mathbb{R}^{3}\right) . \tag{4}
\end{gather*}
$$

Another example, but now arising from the Ginzburg-Landau model in 3-d, is when we take $X=S^{2}$ and $k=1$. In this case, we consider maps $u$ which belong to $H^{1 / 2}\left(S^{2} ; S^{1}\right)$. The way we define $T(u)$ in this setting, however, is much more involved. We refer the reader to $[1,2]$ for details.

Given a finite number of points (not necessarily distinct) $p_{1}, \ldots, p_{k}, n_{1}, \ldots, n_{k}$ in $X$, the length of the minimal connection between these points is given by (see [5])

$$
\begin{equation*}
L:=\min _{\sigma \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right), \tag{5}
\end{equation*}
$$

where $S_{k}$ denotes the group of permutations of $\{1, \ldots, k\}$. It can be shown that the number $L$ satisfies (see [5]; see also [3] for an elementary proof)

$$
\begin{equation*}
L=\sup _{\left.|\zeta|\right|_{\text {Lip }} \leq 1} \sum_{i=1}^{k}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right] \tag{6}
\end{equation*}
$$

where $|\zeta|_{\text {Lip }}$ denotes the best Lipschitz constant of $\zeta$. Moreover, the supremum in (6) is achieved.

More generally, consider two sequences $\left(p_{i}\right),\left(n_{i}\right) \subset X$ such that

$$
\begin{equation*}
\sum_{i} d\left(p_{i}, n_{i}\right)<\infty \tag{7}
\end{equation*}
$$

(By abuse of notation, we allow sequences indexed on a finite subset of $\mathbb{N}$, which includes the previous case).

Motivated by (6), we define the length of the minimal connection between these points as

$$
\begin{equation*}
\|T\|:=\sup _{\substack{\zeta \in \operatorname{Lip}(X) \\|\zeta| \text { Lip } \leq 1}}\langle T, \zeta\rangle=\sup _{|\zeta|_{\text {Lip }} \leq 1} \sum_{i}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right] \tag{8}
\end{equation*}
$$

where $T$ is the linear functional given by (1). We point out that the supremum is still achieved in this case; see Proposition 18. In Section 2 we compare this number with some alternative definitions.

Let

$$
\mathcal{Z}:=\left\{\begin{array}{l|l}
T \in[\operatorname{Lip}(X)]^{*} & \begin{array}{l}
T \text { can be written in the form (1) for some } \\
\left(p_{i}\right),\left(n_{i}\right) \subset X \text { such that } \sum_{i} d\left(p_{i}, n_{i}\right)<\infty
\end{array} \tag{9}
\end{array}\right\} .
$$

Note that if $T \in \mathcal{Z}$ then $-T \in \mathcal{Z}$, and $T_{1}+T_{2} \in \mathcal{Z}$ whenever $T_{1}, T_{2} \in \mathcal{Z}$. As we shall see in the Appendix, $\mathcal{Z}$ is a complete metric space with respect to the distance induced by $\|\cdot\|$.

We also introduce the notion of support of $T$ :
Definition 1 Let $\left(\omega_{i}\right)_{i \in I}$ be the family of all open subsets of $X$ such that, for each $i \in I$, the following holds: if $\zeta \in[\operatorname{Lip}(X)]^{*}$ and $\zeta \equiv 0$ on $X \backslash \omega_{i}$, then $\langle T, \zeta\rangle=0$. We set $\operatorname{supp} T:=X \backslash \bigcup_{i \in I} \omega_{i}$.

Clearly, $\operatorname{supp} T \subset \overline{\bigcup_{i}\left\{p_{i}\right\} \cup \bigcup_{i}\left\{n_{i}\right\}}$, although the strictly inequality can actually occur; see, however, Theorem 11 below. Note that there are several possible representations of $T$ as a sum of the form (1). Moreover, such representations need not be equivalent modulo a permutation of points. In fact, if $\left(q_{i}\right)$ is a sequence rapidly converging to $p$ in $X$ (in the sense that $\sum_{i} d\left(q_{i}, q_{i+1}\right)<\infty$ ), then we can write $\delta_{p}-\delta_{n}=\sum_{i=1}^{\infty}\left(\delta_{q_{i+1}}-\delta_{q_{i}}\right)$ in $[\operatorname{Lip}(X)]^{*}$, where $n:=q_{1}$.

The next proposition is the analogue of (5) in our more general setting (see [2, Lemma 12'] and also Proposition 18 below)

Proposition 2 For any $T \in \mathcal{Z}$ we have

$$
\begin{equation*}
\|T\|=\inf _{\substack{\left(\tilde{p}_{i}\right) \\\left(n_{i}\right)}}\left\{\sum_{i} d\left(\tilde{p}_{i}, \tilde{n}_{i}\right): T=\sum_{i}\left(\delta_{\tilde{p}_{i}}-\delta_{\tilde{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}\right\} \tag{10}
\end{equation*}
$$

In contrast with the case of a finite number of points, the infimum above need not be achieved in general; see Example 5 below. Here is a case where it is still attained:


Fig. 1. Dipoles $\delta_{p_{i}}-\delta_{n_{i}}$ in Example 5
Theorem 3 If $\mathcal{H}^{1}(\operatorname{supp} T)=0$, then the infimum in (10) is attained. In other words, there exist $\left(\tilde{p}_{i}\right),\left(\tilde{n}_{i}\right)$ in $X$ such that

$$
\begin{equation*}
\|T\|=\sum_{i} d\left(\tilde{p}_{i}, \tilde{n}_{i}\right) \quad \text { and } \quad T=\sum_{i}\left(\delta_{\tilde{p}_{i}}-\delta_{\tilde{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{11}
\end{equation*}
$$

Above, $\mathcal{H}^{1}$ denotes the 1-dimensional Hausdorff measure. In particular, if the set $\overline{\bigcup_{i}\left\{p_{i}\right\} \cup \bigcup_{i}\left\{n_{i}\right\}}$ is countable, then Theorem 3 holds.

In any case, it is always possible to decompose $T$ in terms of simpler functionals, taking into account the length of its minimal connection. But let us first introduce a definition:

Definition $4 T \in \mathcal{Z}$ is said to be regular in $X$ if there exist $\left(\tilde{p}_{i}\right),\left(\tilde{n}_{i}\right) \subset X$ such that

$$
\|T\|=\sum_{i} d\left(\tilde{p}_{i}, \tilde{n}_{i}\right) \quad \text { and } \quad T=\sum_{i}\left(\delta_{\tilde{p}_{i}}-\delta_{\tilde{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} .
$$

$T \in \mathcal{Z}$ is singular in $X$ if whenever $T=T_{1}+T_{2},\|T\|=\left\|T_{1}\right\|+\left\|T_{2}\right\|$ and $T_{1}$ is regular, then $T_{1}=0$.

Here is an example of $T \in \mathcal{Z}$ which is singular:
Example 5 Let $X=[0,1]$ and $C_{\alpha} \subset[0,1]$ be a Cantor-type set with Lebesgue measure $\alpha \in(0,1)$. We denote by $\left(J_{k}\right)_{k \geq 1}, J_{k}=\left(n_{k}, p_{k}\right)$, the sequence of disjoint open intervals which are removed from $[0,1]$ in the construction of $C_{\alpha}$. We then take $p_{0}=0$ and $n_{0}=1$. In Section 6 we show that $T=\sum_{i \geq 0}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ is singular and $\|T\|=\alpha$. For descriptive purposes we can think of representing each dipole $\delta_{p_{i}}-\delta_{n_{i}}$ as an arrow pointing from $n_{i}$ to $p_{i}$. In Figure 1 we represent $T$ geometrically according to this convention.

We have the following
Theorem 6 For any $T \in \mathcal{Z}$ there exist $T_{\text {reg }}, T_{\text {sing }} \in \mathcal{Z}$ such that $T_{\text {reg }}$ is regular, $T_{\text {sing }}$ is singular,

$$
\begin{equation*}
T=T_{\text {reg }}+T_{\text {sing }} \quad \text { and } \quad\|T\|=\left\|T_{\text {reg }}\right\|+\left\|T_{\text {sing }}\right\| . \tag{12}
\end{equation*}
$$

Moreover, there exists $\left(T_{j}\right) \subset \mathcal{Z}$ such that

$$
\begin{equation*}
T_{\text {sing }}=\sum_{j} T_{j}, \quad\left\|T_{\text {sing }}\right\|=\sum_{j}\left\|T_{j}\right\| \quad \text { and } \quad\left\|T_{j}\right\|=\mathcal{H}^{1}\left(\operatorname{supp} T_{j}\right) \quad \forall j . \tag{13}
\end{equation*}
$$

In addition, each set $\operatorname{supp} T_{j}$ is homeomorphic to the Cantor set in $\mathbb{R}$.
The decomposition of $T$ in terms of a regular and a singular part, as in (12), need not be unique; see Example 63.

We point out that Theorem 3 is a special case of the above. In fact, it follows from the proof of Theorem 6 that $T_{\text {reg }}, T_{\text {sing }}$ and $\left(T_{j}\right)$ can be chosen so that

$$
\operatorname{supp} T=\operatorname{supp} T_{\text {reg }} \cup \operatorname{supp} T_{\text {sing }} \quad \text { and } \quad \bigcup_{j} \operatorname{supp} T_{j} \subset \operatorname{supp} T_{\text {sing }} .
$$

Therefore, if $\mathcal{H}^{1}(\operatorname{supp} T)=0$, then $\left\|T_{j}\right\|=\mathcal{H}^{1}\left(\operatorname{supp} T_{j}\right)=0$ for each $j$. We conclude that $T_{\text {sing }}=\sum_{j} T_{j}=0$ in $[\operatorname{Lip}(X)]^{*}$ and so $T=T_{\text {reg }}$ is regular in $X$.

A natural question regarding $T \in \mathcal{Z}$ is whether it has a "simplest" representation in the following sense:

Definition 7 The representation $\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ is reducible if there exist $\mathbb{N}_{1} \subset$ $\mathbb{N}_{2} \subset \mathbb{N}$, $\operatorname{card} \mathbb{N}_{1}<\operatorname{card} \mathbb{N}_{2}$, and points $r_{i}, q_{i} \in X, i \in \mathbb{N}_{1}$, such that

$$
\begin{equation*}
\sum_{i \in \mathbb{N}_{2}}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=\sum_{i \in \mathbb{N}_{1}}\left(\delta_{r_{i}}-\delta_{q_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{14}
\end{equation*}
$$

$\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ will be called irreducible if it is not reducible.
The next result states that one can always find an irreducible representation of $T$ :

Theorem 8 Any linear functional $T \in \mathcal{Z}$ has an irreducible representation. More precisely, there exist sequences $\left(\hat{p}_{i}\right),\left(\hat{n}_{i}\right)$ in $X$, satisfying (7), such that

$$
\begin{equation*}
T=\sum_{i}\left(\delta_{\hat{p}_{i}}-\delta_{\hat{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}, \tag{15}
\end{equation*}
$$

and so that this representation is irreducible.
Our proof of Theorem 8 relies on the notion of maximal paths; see Section 5 . This approach requires the following lemma:

Lemma 9 If

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=\left(\delta_{r_{1}}-\delta_{q_{1}}\right)+\left(\delta_{r_{2}}-\delta_{q_{2}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{16}
\end{equation*}
$$

for some $r_{1}, q_{1}, r_{2}, q_{2} \in X$, then there exists $\tilde{\mathbb{N}} \subset \mathbb{N}$ such that

$$
\begin{equation*}
\sum_{i \in \tilde{\mathbb{N}}}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { equals } \quad\left(\delta_{r_{1}}-\delta_{q_{1}}\right) \quad \text { or } \quad\left(\delta_{r_{1}}-\delta_{q_{2}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{17}
\end{equation*}
$$

A simple consequence of this is the corollary below which simplifies our notion of irreducible representations (see also Proposition 51):

Corollary $10 \sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ is reducible if, and only if, one of the following conditions hold:
(a) $p_{i}=n_{j}$ for some $i, j \geq 1$;
(b) there exists an infinite set $\tilde{\mathbb{N}} \subset \mathbb{N}$ such that

$$
\begin{equation*}
\sum_{i \in \tilde{\mathbb{N}}}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=\delta_{r}-\delta_{q} \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{18}
\end{equation*}
$$

for some $r, q \in X$.
If $T$ can be written as a finite sum of dipoles of the form $\delta_{p}-\delta_{n}$, then the irreducible representation of $T$ is unique (modulo a permutation of the points). This need not be the case in general. Assume, for example, that $X=[0,1]$, and let $\left(p_{i}\right),\left(n_{i}\right)$ be two sequences converging to 0 such that $p_{i}>n_{i}>p_{i+1}$ for every $i \geq 1$. Then

$$
\begin{aligned}
& \sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \\
& \left(\delta_{p_{1}}-\delta_{0}\right)+\sum_{i=1}^{\infty}\left(\delta_{p_{i+1}}-\delta_{n_{i}}\right) \\
& \left(\delta_{p_{1}}-\delta_{0}\right)+\left(\delta_{p_{2}}-\delta_{0}\right)+\sum_{i=1}^{\infty}\left(\delta_{p_{i+2}}-\delta_{n_{i}}\right), \quad \cdots
\end{aligned}
$$

are all irreducible representations of the same operator in $[\operatorname{Lip}[0,1]]^{*}$.
However, we have the following
Theorem 11 Assume (15) is an irreducible representation of $T$. Then

$$
\begin{equation*}
\operatorname{supp} T=\overline{\bigcup_{i}\left\{\hat{p}_{i}\right\} \cup \bigcup_{i}\left\{\hat{n}_{i}\right\}} \tag{19}
\end{equation*}
$$

In particular, if $\zeta \in \operatorname{Lip}(X)$ and $\zeta \equiv 0$ on $\operatorname{supp} T$, then $\langle T, \zeta\rangle=0$.
A simple consequence of Theorem 11 is the corollary below:
Corollary 12 Let $T \in \mathcal{Z}$. If $\operatorname{supp} T$ is finite, then there exist finitely many
points $\hat{p}_{1}, \ldots, \hat{p}_{k_{0}}, \hat{n}_{1}, \ldots, \hat{n}_{k_{0}} \in X$ such that

$$
\begin{equation*}
T=\sum_{i=1}^{k_{0}}\left(\delta_{\hat{p}_{i}}-\delta_{\hat{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{20}
\end{equation*}
$$

Another result in this direction is the theorem below which completely solves an open problem raised by H. Brezis. We denote by $\operatorname{BLip}(X)$ the subspace of bounded Lipschitz functions:

Theorem 13 Let $T \in \mathcal{Z}$. Assume that

$$
\begin{equation*}
|\langle T, \zeta\rangle| \leq C\|\zeta\|_{\infty} \quad \forall \zeta \in \operatorname{BLip}(X) \tag{21}
\end{equation*}
$$

for some $C>0$. Then, there exist points $a_{1}, \ldots, a_{k}$ and integers $d_{1}, \ldots, d_{k}$, $\sum_{i} d_{i}=0$, such that

$$
\begin{equation*}
T=\sum_{i=1}^{k} d_{i} \delta_{a_{i}} \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{22}
\end{equation*}
$$

We point out that (20) is equivalent to saying that (21) holds (since $\sum_{i} d_{i}=0$ ). Theorem 13 has been proved by Smets [8] (using the Riesz Representation Theorem) under the additional assumption that $X$ is locally compact. Our proof instead makes use of the existence of irreducible representations of $T$, which only requires $X$ to be complete. Very simple examples show that Theorem 13 is no longer true without this assumption on $X$.

We now present the notion of indecomposable functionals taken from Federer [6].

Given $T \in \mathcal{Z}$, we define

$$
\begin{equation*}
m(T):=\sup _{\|\zeta\|_{\infty}=1}\langle T, \zeta\rangle \quad \forall \zeta \in \operatorname{BLip}(X) \tag{23}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathcal{I}:=\{T \in \mathcal{Z}: m(T)<\infty\} . \tag{24}
\end{equation*}
$$

It follows from Theorem 13 that $T \in \mathcal{I}$ if, and only if, $T$ can be written in terms of finitely many dipoles. In fact, we have $m(T)=2 k_{0}$, where $k_{0} \geq 0$ is the smallest integer such that (20) holds. Moreover,

$$
\begin{equation*}
m\left(T_{1}+T_{2}\right) \leq m\left(T_{1}\right)+m\left(T_{2}\right) \quad \forall T_{1}, T_{2} \in \mathcal{I} . \tag{25}
\end{equation*}
$$

We now consider $\mathcal{I}$ equipped with the norm

$$
\begin{equation*}
N(T):=\|T\|+m(T) \quad \forall T \in \mathcal{I} . \tag{26}
\end{equation*}
$$

As in Federer [6, $\S 4.2 .25]$, we say that $T \in \mathcal{I}$ is indecomposable if there exists no $S \in \mathcal{I}$ with

$$
\begin{equation*}
S \neq 0 \neq T-S \quad \text { and } \quad N(T)=N(S)+N(T-S) \tag{27}
\end{equation*}
$$

It is then easy to see that $T \in \mathcal{I}$ is indecomposable if, and only if, there exist $r, q \in X$ such that $T=\left(\delta_{r}-\delta_{q}\right)$ in $[\operatorname{Lip}(X)]^{*}$. Thus, every element in $\mathcal{I}$ can be written as a finite sum of indecomposable parts, which coincides with a minimal connection of $T$. Note however that this notion is restricted to the subspace $\mathcal{I} \varsubsetneqq \mathcal{Z}$.

Most of the results in this paper were announced in [7].

## 2 Alternative definitions of minimal connections

Throughout this paper, we shall always consider two sequences of points $\left(p_{i}\right)$ and $\left(n_{i}\right)$ in $X$ such that $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$.

Let $T:=\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ in $[\operatorname{Lip}(X)]^{*}$. There are several alternative ways of defining the length of the minimal connection between $\left(p_{i}\right)$ and $\left(n_{i}\right)$ :

## Definition 14

$$
\begin{equation*}
L_{1}:=\inf _{\substack{\sigma: \mathbb{N} \rightarrow \mathbb{N} \\ \text { bijection }}} \sum_{i=1}^{\infty} d\left(p_{i}, n_{\sigma(i)}\right) . \tag{28}
\end{equation*}
$$

Definition 15

$$
\begin{equation*}
L_{2}:=\lim _{k \rightarrow \infty} \min _{\sigma \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right) . \tag{29}
\end{equation*}
$$

Definition 16

$$
\begin{equation*}
L_{3}:=\inf _{\left(\tilde{n}_{i}\right)}\left\{\sum_{i=1}^{\infty} d\left(p_{i}, \tilde{n}_{i}\right): T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{\tilde{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}\right\} . \tag{30}
\end{equation*}
$$

Definition 17

$$
\begin{equation*}
L_{4}:=\inf _{\substack{\left(\tilde{p}_{i}\right) \\\left(\tilde{n}_{i}\right)}}\left\{\sum_{i=1}^{\infty} d\left(\tilde{p}_{i}, \tilde{n}_{i}\right): T=\sum_{i=1}^{\infty}\left(\delta_{\tilde{p}_{i}}-\delta_{\tilde{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}\right\} . \tag{31}
\end{equation*}
$$

Clearly, we have

$$
\begin{equation*}
L_{1} \leq L_{2} \quad \text { and } \quad\|T\| \leq L_{4} \leq L_{3} \tag{32}
\end{equation*}
$$

Using (6) and $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$ we can actually prove the following (see also [2])

## Proposition 18

$$
\begin{equation*}
L_{1} \leq L_{2}=L_{3}=L_{4}=\|T\| . \tag{33}
\end{equation*}
$$

Moreover, the supremum in (8) is achieved.

## PROOF.

Step 1. $L_{3} \leq L_{2}$.
Given $k \geq 1$ and $\sigma \in S_{k}$, we extend $\sigma$ to $\mathbb{N}$ so that $\sigma(i)=i$ for every $i>k$. In particular, $T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{\sigma(i)}}\right)$ in $[\operatorname{Lip}(X)]^{*}$. By definition, we have

$$
\begin{equation*}
L_{3} \leq \sum_{i=1}^{\infty} d\left(p_{i}, n_{\sigma(i)}\right)=\sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right)+\sum_{i>k} d\left(p_{i}, n_{i}\right) \tag{34}
\end{equation*}
$$

Since $\sigma \in S_{k}$ is arbitrary, we conclude that

$$
\begin{equation*}
L_{3} \leq \min _{\sigma \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right)+\sum_{i>k} d\left(p_{i}, n_{i}\right) . \tag{35}
\end{equation*}
$$

Letting $k \rightarrow \infty$, we get $L_{3} \leq L_{2}$.
Step 2. $L_{2} \leq\|T\|$.
Given $\varepsilon>0$, we fix $k \geq 1$ large enough so that $\sum_{i>k} d\left(p_{i}, n_{i}\right)<\varepsilon$. Let $\sigma \in S_{k}$ and $\zeta \in \operatorname{Lip}(X),|\zeta|_{\text {lip }} \leq 1$, be such that

$$
\begin{equation*}
\sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right)=\sum_{i=1}^{k}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right] . \tag{36}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
L_{2}-\varepsilon \leq \sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right) \leq \sum_{i=1}^{\infty}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right]+\varepsilon \leq\|T\|+\varepsilon . \tag{37}
\end{equation*}
$$

Since $\varepsilon>0$ is arbitrary, we must have $L_{2} \leq\|T\|$.
In view of (32), (33) follows the two previous steps.
Step 3. The supremum in (8) is attained.
For each $k \geq 1$, let $\zeta_{k} \in \operatorname{Lip}(X),\left|\zeta_{k}\right|_{\operatorname{Lip}} \leq 1$, be such that

$$
\sum_{i=1}^{k}\left[\zeta_{k}\left(p_{i}\right)-\zeta_{k}\left(n_{i}\right)\right]=\min _{\sigma \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right) .
$$

For the sake of normalization, we may assume that $\zeta_{k}\left(x_{0}\right)=0$ for some fixed $x_{0} \in X$. In particular, for each $i \geq 1$ the sequences $\left(\zeta_{k}\left(x_{i}\right)\right)_{k}$, where $x_{i}=p_{i}$ or $n_{i}$, are bounded. Passing to a subsequence if necessary, we may assume that all the limits

$$
\tilde{\zeta}\left(x_{i}\right):=\lim _{k \rightarrow \infty} \zeta_{k}\left(x_{i}\right), \quad x_{i}=p_{i}, n_{i}
$$

exist. It is easy to see that $\tilde{\zeta}$, defined on $A:=\bigcup_{i}\left\{p_{i}\right\} \cup \bigcup_{i}\left\{n_{i}\right\}$, satisfies

$$
L=\sum_{i}\left[\tilde{\zeta}\left(p_{i}\right)-\tilde{\zeta}\left(n_{i}\right)\right]
$$

(we use here that $\sum_{i} d\left(p_{i}, n_{i}\right)<\infty$ ).
On the other hand, since $|\tilde{\zeta}|_{\text {Lip }(A)} \leq 1$ we can extend $\tilde{\zeta}$ to $X$ without increasing its Lipschitz constant (take for instance $\zeta(x):=\inf _{a \in A}\{\tilde{\zeta}(a)+d(x, a)\}$ ). We conclude the supremum in (8) is achieved.

Remark 19 The strict inequality $L_{1}<\|T\|$ may actually occur in (33). In fact, take $\left(a_{i}\right)_{i \in \mathbb{Z}}$ such that $\sum_{i} d\left(a_{i}, a_{i+1}\right)<\infty$. In particular, both limits

$$
r:=\lim _{i \rightarrow+\infty} a_{i} \text { and } q:=\lim _{i \rightarrow-\infty} a_{i}
$$

exist (since $X$ is complete). Thus,

$$
T=\sum_{i=-\infty}^{+\infty}\left(\delta_{a_{i+1}}-\delta_{a_{i}}\right)=\delta_{r}-\delta_{q} \quad \text { in }[\operatorname{Lip}(X)]^{*} .
$$

Note that $\|T\|=d(r, q)$ but $L_{1}=0$.
Remark 20 The infimum in (28) need not be achieved in general. Consider the sequence of points $\left(p_{i}\right)_{i \geq 1}$ and $\left(n_{i}\right)_{i \geq 1}$ given in Example 5. We claim that $L_{1}=0$, even though $p_{i} \neq n_{j} \forall i, j$. In fact, given $\varepsilon>0$ we can find $i_{1}, j_{1} \in \mathbb{N}$, $i_{1}, j_{1} \neq 1$, such that $\left|p_{1}-n_{i_{1}}\right|+\left|n_{1}-p_{j_{1}}\right|<\frac{\varepsilon}{2}$. We set $\sigma(1):=i_{1}$ and $\sigma\left(j_{1}\right):=1$. Proceeding by induction, at each step $k>1$ we can extend this bijection $\sigma$ so that

$$
\sigma:\{1, \ldots, k\} \cup\left\{j_{1}, \ldots, j_{k}\right\} \longrightarrow\left\{i_{1}, \ldots, i_{k}\right\} \cup\{1, \ldots, k\}
$$

satisfies

$$
\sum_{l=1}^{k}\left|p_{l}-n_{\sigma(l)}\right|<\varepsilon \quad \forall k \geq 1 .
$$

At the end, we conclude that $L_{1} \leq \varepsilon$. Since $\varepsilon>0$ was arbitrary, the claim follows.

## 3 Cycles

As we have already mentioned in Example 5, we can think of identifying each dipole $\delta_{p_{i}}-\delta_{n_{i}}$ with an arrow pointing from $n_{i}$ to $p_{i}$. In order to make a clear distinction between all the dipoles, we shall usually indicate each $\delta_{p_{i}}-\delta_{n_{i}}$ by its index $i$. This way we will be able to distinguish equal dipoles arising from different indices.

Our strategy to deal with the linear functional $T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ will be to equip the set of arrows $i$ with a suitable order relation. The motivation of this approach comes from elementary concepts in Geometry, as it will soon become clear.

We start with the following:
Definition $21 A$ chain $(\Lambda, \leq)$ is a set of indices $\Lambda \subset \mathbb{N}$ equipped with a partial order relation $\leq$.

In general, we shall call $\Lambda$ itself a chain, $\leq$ being implicitly understood. The order $\leq$ induces an orientation in the set of dipoles $\left(\delta_{p_{i}}-\delta_{n_{i}}\right)_{i \in \Lambda}$.

We shall usually be interested in the order relation $\leq$ modulo cyclic permutations of the elements in $\Lambda$. In order to make this precise, we start with an auxiliary notion:

Definition $22 A$ subchain $\Lambda_{1} \subset \Lambda$ (equipped with the order relation induced from $\Lambda$ ) is called a segment if whenever $\lambda_{1} \leq \lambda \leq \lambda_{2}$ in $\Lambda$ and $\lambda_{1}, \lambda_{2} \in \Lambda_{1}$, then $\lambda \in \Lambda_{1}$.

We now introduce the notion of a cycle:
Definition 23 Given two chains $\Lambda, \tilde{\Lambda}$, we write $\Lambda \sim \tilde{\Lambda}$ if
(i) $\Lambda=\tilde{\Lambda}$ (as sets);
(ii) there exist two disjoint segments $\Lambda_{1}, \Lambda_{2} \subset \Lambda$ such that $\Lambda=\Lambda_{1} \cup \Lambda_{2}$ and the inclusions $\Lambda_{1}, \Lambda_{2} \subset \tilde{\Lambda}$ are order preserving.

It is easy to see that $\sim$ defines an equivalence relation in the class of all chains. The equivalence class $[\Lambda]$ of $\Lambda$ induced by $\sim$ will be called $a$ cycle.

Assume $\Lambda$ is the finite chain containing $\lambda_{1} \leq \cdots \leq \lambda_{k}$, which we denote as $\left(\lambda_{1} \cdots \lambda_{k}\right)$. In this special case, $[\Lambda]$ will be the union of all cyclic permutations of $\Lambda$, namely

$$
[\Lambda]=\left\{\left(\lambda_{1} \cdots \lambda_{k}\right),\left(\lambda_{2} \cdots \lambda_{k} \lambda_{1}\right), \ldots,\left(\lambda_{k} \lambda_{1} \cdots \lambda_{k-1}\right)\right\} .
$$

Since any representative of $[\Lambda]$ ( $\Lambda$ now being finite or infinite) contains the same set of indices, we can actually think of $[\Lambda]$ as being the set of indices $i \in \Lambda$ itself. Moreover, [ $\Lambda$ ] has a well-defined orientation, induced by the order of any of its representatives $\tilde{\Lambda} \in[\Lambda]$.

We now define

$$
\begin{align*}
T_{[\Lambda]} & :=\sum_{\lambda \in \Lambda}\left(\delta_{p_{\lambda}}-\delta_{n_{\lambda}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}, \\
\ell_{[\Lambda]} & :=\sum_{\lambda \in \Lambda} d\left(p_{\lambda}, n_{\lambda}\right),  \tag{38}\\
L_{[\Lambda]} & :=\left\|T_{[\Lambda]}\right\| .
\end{align*}
$$

We call $\ell_{[\Lambda]}$ the length of $[\Lambda]$.
Given $\varepsilon>0$, an $\varepsilon$-chain $\Lambda_{\varepsilon}=\left(\lambda_{1} \cdots \lambda_{k}\right)$ is a finite subchain of $\Lambda$ such that if $i \in \Lambda$ and $i \in\left\{1, \ldots,\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right\}$, then $i \in \Lambda_{\varepsilon}$. Note that if $\Lambda$ is infinite, then it has an infinite number of $\varepsilon$-chains (for an $\varepsilon>0$ fixed), since one can always add to $\Lambda_{\varepsilon}$ indices in $\Lambda$ outside $\left\{1, \ldots,\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right\}$.

The co-length of $\left[\Lambda_{\varepsilon}\right]$ is the number

$$
\begin{equation*}
\ell_{\left[\Lambda_{\varepsilon}\right]}^{*}:=d\left(p_{\lambda_{1}}, n_{\lambda_{2}}\right)+\cdots+d\left(p_{\lambda_{k-1}}, n_{\lambda_{k}}\right)+d\left(p_{\lambda_{k}}, n_{\lambda_{1}}\right) . \tag{39}
\end{equation*}
$$

It measures the total jump from one dipole to the next one as we travel along $\left[\Lambda_{\varepsilon}\right]$.

Lemma 24 If $\Lambda_{\varepsilon_{1}} \subset \Lambda_{\varepsilon_{2}}$, then

$$
\begin{equation*}
\ell_{\left[A_{\varepsilon_{1}}\right]}+\ell_{\left[s_{\varepsilon_{1}}\right]}^{*} \leq \ell_{\left[A_{\varepsilon_{2}}\right]}+\ell_{\left[\Lambda_{\varepsilon_{2}}\right]}^{*} . \tag{40}
\end{equation*}
$$

PROOF. It suffices to check (40) when $\Lambda_{\varepsilon_{2}}$ differs from $\Lambda_{\varepsilon_{1}}$ by exactly one index and then argue by induction. In order to add an index $i_{2}$ between $i_{1}$ and $i_{3}$, we just need to apply the triangle inequality to get

$$
d\left(p_{i_{1}}, n_{i_{3}}\right) \leq d\left(p_{i_{1}}, n_{i_{2}}\right)+d\left(p_{i_{2}}, n_{i_{2}}\right)+d\left(p_{i_{2}}, n_{i_{3}}\right) .
$$

Notice that the second term in the right-hand side enters in the definition of the length $\ell_{\left[\Lambda_{\varepsilon_{2}}\right]}$, while the other two appear in the definition of the co-length $\ell_{\left[\Lambda_{\varepsilon_{2}}\right]}^{*}$. This proves the lemma.

A simple consequence of (40) is the equality below:

## Proposition 25

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}:=\lim _{\varepsilon \downarrow 0}\left(\inf _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*}\right)=\lim _{\varepsilon \downarrow 0}\left(\sup _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*}\right), \tag{41}
\end{equation*}
$$

where both the infimum and the supremum are taken over the class of all $\varepsilon$ chains of $\Lambda$.

We define the common number $\ell_{[\Lambda]}^{*}$ in (41) to be the co-length of $[\Lambda]$.

PROOF. We denote by $\ell^{*}$ the limit in the right-hand side of (41) (note that it is well-defined, but may be infinite). Given $m<\ell^{*}$, let $\tilde{\Lambda}_{\varepsilon}$ be an $\varepsilon$-chain of $\Lambda$ such that $m<\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}^{*}$.

We now take a sequence of $\varepsilon_{j}$-chains $\Lambda_{\varepsilon_{j}}$, where $\varepsilon_{j} \downarrow 0$, such that

$$
\lim _{j \rightarrow \infty} \ell_{\left[\Lambda_{\varepsilon_{j}}\right]}^{*}=\lim _{\varepsilon \downarrow 0}\left(\inf _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*}\right) .
$$

Since $\tilde{\Lambda}_{\varepsilon}$ is finite, there exists $j_{0} \geq 1$ sufficiently large so that $\Lambda_{\varepsilon_{j}} \supset \tilde{\Lambda}_{\varepsilon}$ for every $j \geq j_{0}$.

Applying (40) we get

$$
m<\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}^{*} \leq \ell_{\left[\Lambda_{\varepsilon_{j}}\right]}^{*}+\left(\ell_{\left[\Lambda_{\varepsilon_{j}}\right]}-\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}\right) \quad \forall j \geq j_{0} .
$$

Taking $j \rightarrow \infty$ and then $\varepsilon \downarrow 0$, we conclude that

$$
m \leq \lim _{\varepsilon \downarrow 0}\left(\inf _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*}\right),
$$

from which (41) follows.

Combining Lemma 24 and Proposition 25, we get
Corollary 26 Given a chain $\Lambda$, for any subchain $\tilde{\Lambda} \subset \Lambda$ we have

$$
\begin{equation*}
\ell_{[\tilde{\Lambda}]}+\ell_{[\tilde{\Lambda}]}^{*} \leq \ell_{[\Lambda]}+\ell_{[\Lambda]}^{*} . \tag{42}
\end{equation*}
$$

Corollary 27 Assume $\Lambda$ is a chain. If $\Lambda_{1} \subset \Lambda_{2} \subset \cdots \subset \Lambda$ is an increasing sequence of subchains such that $\Lambda=\bigcup_{k} \Lambda_{k}$, then

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}=\lim _{k \rightarrow \infty} \ell_{\left[\Lambda_{k}\right]}^{*} . \tag{43}
\end{equation*}
$$

Note that for every $\Lambda_{\varepsilon}$ we have

$$
\begin{equation*}
L_{\left[\Lambda_{\varepsilon}\right]} \leq \min \left\{\ell_{\left[\Lambda_{\varepsilon}\right]}, \ell_{\left[\Lambda_{\varepsilon}\right]}^{*}\right\}, \tag{44}
\end{equation*}
$$

since both $\ell_{\left[\Lambda_{\varepsilon}\right]}$ and $\ell_{\left[\Lambda_{\varepsilon}\right]}^{*}$ correspond to special choices of permutations in (5).
Taking $\varepsilon \downarrow 0$, we conclude that

$$
\begin{equation*}
L_{[\Lambda]} \leq \min \left\{\ell_{[\Lambda]}, \ell_{[\Lambda]}^{*}\right\} . \tag{45}
\end{equation*}
$$

There are three cases of interest when the equality holds in the estimate above:
Definition 28 Assume $[\Lambda]$ is a cycle.
(a) $[\Lambda]$ is a minimal cycle if $L_{[\Lambda]}=\ell_{[\Lambda]}$;
(b) $[\Lambda]$ is a co-minimal cycle if $L_{[\Lambda]}=\ell_{[\Lambda]}^{*}$;
(c) $[\Lambda]$ is a loop if $\ell_{[\Lambda]}^{*}=0$ (this is a special case of (b)); in particular,

$$
\begin{equation*}
T_{[\Lambda]}=0 \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{46}
\end{equation*}
$$

Here are some examples:
Example 29 Assume $\Lambda=(12 \cdots k)$, that is to say, consider the dipoles $\delta_{p_{1}}-\delta_{n_{1}}, \ldots, \delta_{p_{k}}-\delta_{n_{k}}$, oriented in this order. We have:
(i) If $L_{[\Lambda]}=\ell_{[\Lambda]}$, then the pairs $\left[p_{1}, n_{1}\right], \ldots,\left[p_{k}, n_{k}\right]$ form a minimal connection.
(ii) If $L_{[\Lambda]}=\ell_{[\Lambda]}^{*}$, then a minimal connection is given by $\left[p_{1}, n_{2}\right], \ldots,\left[p_{k-1}, n_{k}\right]$, $\left[p_{k}, n_{1}\right]$.
(iii) More generally, let $\sigma \in S_{k}$ be a permutation which minimises (5). Recall that $\sigma$ can be written as a composition of disjoint cycles (in the algebraic sense), say $\sigma_{1}, \ldots, \sigma_{j}$. Note, however, that each $\sigma_{l}$ induces in a natural way a cycle $\left[\Lambda_{l}\right]$ (in the sense of Definition 23). For instance, if

$$
\sigma_{1}: 1 \mapsto i_{1} \mapsto \cdots \mapsto i_{\alpha} \mapsto 1,
$$

then $\Lambda_{1}=\left(1 i_{1} \cdots i_{\alpha}\right)$. This way, we can write $\{1, \ldots, k\}=\Lambda_{1} \cup \cdots \cup \Lambda_{j}$ so that

$$
\begin{equation*}
L_{[\Lambda]}=\sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right)=\sum_{l=1}^{j} \sum_{i \in \Lambda_{l}} d\left(p_{i}, n_{\sigma(i)}\right)=\sum_{l=1}^{j} \ell_{\left[\Lambda_{l}\right]}^{*} . \tag{47}
\end{equation*}
$$

Figure 2 shows such a decomposition with $k=6, \Lambda_{1}=\left(\begin{array}{ll}1 & 4\end{array}\right), \Lambda_{2}=\left(\begin{array}{ll}3 & 5\end{array}\right)$ and $\Lambda_{3}=(6)$. In Proposition 31 we extend this construction to the case of an infinite number of points.

Example 30 Let $X=[0,1]$ and $p_{i}, n_{i} \in[0,1]$ be as in Example 5. We consider $\Lambda_{0}=\mathbb{N} \cup\{0\}$ oriented clockwise with respect to Figure 1. Using the equality $L_{2}=\|T\|$ in Proposition 18, it is easy to see that

$$
L_{\left[\Lambda_{0}\right]}=\alpha=\ell_{\left[\Lambda_{0}\right]}^{*},
$$

where $\alpha$ is the Lebesgue measure of $C_{\alpha}$. In other words, $\left[\Lambda_{0}\right]$ is a co-minimal cycle.

$\left[\Lambda_{1}\right]$

[ $\Lambda_{2}$ ]

$\left[\Lambda_{3}\right]$

Fig. 2. Decomposition of $[\Lambda]$ in terms of three co-minimal cycles $\left[\Lambda_{1}\right],\left[\Lambda_{2}\right]$ and $\left[\Lambda_{3}\right]$ as in Example 29

Note that if we consider the cycle $\left[\Lambda_{0}\right]_{\text {anti }}$ oriented in the opposite direction (i.e. counterclockwise with respect to Figure 1), then

$$
\ell_{\left[\Lambda_{0}\right]_{\text {anti }}^{*}}^{*}=\ell_{\left[\Lambda_{0}\right]}+\ell_{\left[\Lambda_{0}\right]}^{*}=2 .
$$

The proposition below extends (47) to the case of infinitely many points:
Proposition 31 Let

$$
\begin{equation*}
T:=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{48}
\end{equation*}
$$

There exists a sequence of disjoint co-minimal cycles $\left[\Lambda_{j}\right]$ such that $\mathbb{N}=\cup_{j} \Lambda_{j}$ and

$$
\begin{equation*}
\|T\|=\sum_{j} \ell_{\left[\Lambda_{j}\right]}^{*} . \tag{49}
\end{equation*}
$$

PROOF. For each $k \geq 1$, let $\sigma \in S_{k}$ be such that

$$
\begin{equation*}
\sum_{i=1}^{k} d\left(p_{i}, n_{\sigma(i)}\right)=\min _{\tilde{\sigma} \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\tilde{\sigma}(i)}\right) . \tag{50}
\end{equation*}
$$

It follows from Example 29 (iii), that we can write $\{1, \ldots, k\}=\Lambda_{1} \cup \cdots \cup \Lambda_{j}$ in terms of disjoint chains (this decomposition actually depends on $k$ ) so that (47) holds. For $i>k$, we let $\Lambda_{i}=(i)$.

We now relabel $\Lambda_{1}, \ldots, \Lambda_{j}, \Lambda_{k+1}, \ldots$ as

$$
\Lambda_{1, k}, \Lambda_{2, k}, \Lambda_{3, k}, \ldots
$$

so that the smallest integer in $\Lambda_{j_{1}, k}$, is less than the smallest integer in $\Lambda_{j_{2}, k}$ whenever $j_{1}<j_{2}$.

By construction, $1 \in \Lambda_{1, k}$ for every $k \geq 1$.

Let $\alpha_{k}$ be the smallest integer in $\Lambda_{1, k}$ greater than 1 . If $\alpha_{k} \rightarrow \infty$ as $k \rightarrow \infty$, then we set $\Lambda_{1}:=(1)$. Otherwise, $\left(\alpha_{k}\right)$ has a convergent subsequence $\alpha_{k_{l}} \rightarrow a_{1}$; since $\alpha_{k_{l}}$ is an integer, we actually have $\alpha_{k_{l}}=a_{1}$ for all $l$ sufficiently large.

Let $\beta_{l}$ be the smallest integer in $\Lambda_{1, k_{l}}$ greater than $a_{1}$. If $\beta_{l} \rightarrow \infty$, then we set $\Lambda_{1}:=\left(1 a_{1}\right)$. Otherwise, passing to a further subsequence if necessary, we may assume that $\beta_{l}=b_{1}$, for all $l$ large enough; moreover, we can also assume that one of the following inclusions is order preserving:

$$
\left(1 a_{1} b_{1}\right) \subset \Lambda_{1, k_{l}} \quad \forall l \text { large } \quad \text { or } \quad\left(1 b_{1} a_{1}\right) \subset \Lambda_{1, k_{l}} \quad \forall l \text { large. }
$$

Using a standard diagonalization argument, we can construct a subsequence $\left(k_{l}\right)$ (not necessarily the same as the one above) and a chain $\Lambda_{1}$, containing 1, such that the following holds:
(a) given an $\varepsilon$-chain $\Lambda_{1, \varepsilon} \subset \Lambda_{1}$, we can find $N=N\left(\Lambda_{1, \varepsilon}\right) \geq 1$ sufficiently large so that $\Lambda_{1, \varepsilon} \subset \Lambda_{1, k_{l}}$ for every $l \geq N$, and this inclusion is order preserving.

We now repeat the same construction as above with $\Lambda_{2, k_{l}}$ and so on (the only difference here is that we should start with the smallest integer in the set $\mathbb{N} \backslash \Lambda_{1}$, which necessarily belongs to $\Lambda_{2, k_{l}}$ for $l$ sufficiently large). This way we can construct disjoint chains $\Lambda_{2}, \Lambda_{3}, \ldots$ and a universal subsequence ( $k_{l}$ ) (here we apply once again a diagonalization argument) so that
(b) $\mathbb{N}=\bigcup_{j \geq 1} \Lambda_{j}$;
(c) property (a) holds for every $\Lambda_{j}$, after replacing $\Lambda_{1}$ by $\Lambda_{j}$.

By (b), we have

$$
\begin{equation*}
T=\sum_{j} T_{\left[\Lambda_{j}\right]} \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{51}
\end{equation*}
$$

Moreover, (c) implies that

$$
\begin{equation*}
\ell_{\left[\Lambda_{j}\right]}=\lim _{l \rightarrow \infty} \ell_{\left[\Lambda_{j, k_{l}}\right]} \quad \forall j . \tag{52}
\end{equation*}
$$

On the other hand, it follows from (c) and (40) that

$$
\begin{equation*}
\ell_{\left[\Lambda_{j, \varepsilon}\right]}+\ell_{\left[\Lambda_{j, \varepsilon}\right]}^{*} \leq \ell_{\left[\Lambda_{\left.j, k_{l}\right]}\right]}+\ell_{\left[\Lambda_{\left.j, k_{l}\right]}\right]}^{*} \quad \forall l \geq N . \tag{53}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\ell_{\left[\Lambda_{j}\right]}^{*} \leq \liminf _{l \rightarrow \infty} \ell_{\left[\Lambda_{j, k_{l}}\right]}^{*} . \tag{54}
\end{equation*}
$$

We now rewrite (50) as

$$
\begin{equation*}
\sum_{j} \ell_{\left[\Lambda_{j, k}\right]}^{*}=\min _{\tilde{\sigma} \in S_{k}} \sum_{i=1}^{k} d\left(p_{i}, n_{\tilde{\sigma}(i)}\right)+\sum_{i>k} d\left(p_{i}, n_{i}\right) \tag{55}
\end{equation*}
$$

Applying Proposition 18 we obtain

$$
\begin{equation*}
\lim _{l \rightarrow \infty} \sum_{j} \ell_{\left[\Lambda_{j, k}\right]}^{*}=\|T\| . \tag{56}
\end{equation*}
$$

Combining (51), (54) and (56) we get

$$
\|T\| \leq \sum_{j}\left\|T_{\left[\Lambda_{j}\right]}\right\| \leq \sum_{j} \ell_{\left[\Lambda_{j}\right]}^{*} \leq \sum_{j} \liminf _{l \rightarrow \infty} \ell_{\left[\Lambda_{j, k_{l}}\right]}^{*} \leq \lim _{l \rightarrow \infty} \sum_{j} \ell_{\left[\Lambda_{j, k_{l}}\right]}^{*}=\|T\| .
$$

Therefore, we must have equality everywhere. In particular,

$$
\begin{equation*}
\|T\|=\sum_{j}\left\|T_{\left[\Lambda_{j}\right]}\right\| \quad \text { and } \quad\left\|T_{\left[\Lambda_{j}\right]}\right\|=\ell_{\left[\Lambda_{j}\right]}^{*} \quad \forall j, \tag{57}
\end{equation*}
$$

which is precisely (49).

We now present some properties of $[\Lambda]$ when $\ell_{[\Lambda]}^{*}<\infty$. Let us first introduce some notation

Definition 32 Let $\Lambda$ be a chain. Given a family of points $\left(x_{\lambda}\right)_{\lambda \in \Lambda}$, we say that the limit

$$
a:=\lim _{\lambda \in \Lambda \uparrow} x_{\lambda}
$$

exists if given $\varepsilon>0$ there exists $\lambda_{0} \in \Lambda$ such that $d\left(x_{\lambda}, a\right)<\varepsilon \quad \forall \lambda \geq \lambda_{0}$.
The limit $a:=\lim _{\lambda \in \Lambda \downarrow} x_{\lambda}$ is defined similarly, after replacing $\lambda \geq \lambda_{0}$ by $\lambda \leq \lambda_{0}$ in the above.

Proposition 33 If $\ell_{[\Lambda]}^{*}<\infty$, then the following limits exist

$$
\begin{equation*}
\lim _{\lambda \in \Lambda \uparrow} p_{\lambda}, \quad \lim _{\lambda \in \Lambda \downarrow} p_{\lambda}, \quad \lim _{\lambda \in \Lambda \uparrow} n_{\lambda} \quad \text { and } \quad \lim _{\lambda \in \Lambda \downarrow} n_{\lambda} . \tag{58}
\end{equation*}
$$

PROOF. It suffices to show the first limit exists (since all the others can be derived from this case). Moreover, because $\Lambda$ is countable, we only need to show that for every increasing sequence $\left(\lambda_{j}\right)_{j \geq 1}$ in $\Lambda,\left(p_{\lambda_{j}}\right)$ converges.

We have

$$
\begin{equation*}
\sum_{j=1}^{\infty} d\left(p_{\lambda_{j}}, p_{\lambda_{j+1}}\right) \leq \sum_{j=1}^{\infty}\left\{d\left(p_{\lambda_{j}}, n_{\lambda_{j}}\right)+d\left(n_{\lambda_{j}}, p_{\lambda_{j+1}}\right)\right\} \leq \ell_{[\Lambda]}+\ell_{[\Lambda]}^{*} . \tag{59}
\end{equation*}
$$

Therefore, $\left(p_{\lambda_{j}}\right)$ is Cauchy, so it converges.
Corollary 34 If $\ell_{[\Lambda]}^{*}<\infty$, then $\bigcup_{\lambda \in \Lambda}\left\{p_{\lambda}\right\}$ and $\bigcup_{\lambda \in \Lambda}\left\{n_{\lambda}\right\}$ are relatively compact in $X$. In particular, $\operatorname{supp} T_{[\Lambda]}$ is compact.

Remark 35 Let $\Lambda_{1}$ and $\Lambda_{2}$ be two disjoint chains such that $\ell_{\left[\Lambda_{1}\right]}^{*}+\ell_{\left[\Lambda_{2}\right]}^{*}<\infty$. We take $\Lambda:=\Lambda_{1} \cup \Lambda_{2}$ with the order induced from each $\Lambda_{i}$ and such that $\Lambda_{1} \leq \Lambda_{2}$. In view of Proposition 33 we can define

$$
r_{i}:=\lim _{\lambda \in \Lambda_{i} \uparrow} p_{\lambda} \quad \text { and } \quad q_{i}:=\lim _{\lambda \in \Lambda_{i} \downarrow} n_{\lambda} \quad \text { for } i=1,2 .
$$

Clearly, we have

$$
\begin{align*}
L_{[\Lambda]} & \leq L_{\left[\Lambda_{1}\right]}+L_{\left[\Lambda_{2}\right]}, \\
\ell_{[\Lambda]} & =\ell_{\left[\Lambda_{1}\right]}+\ell_{\left[\Lambda_{2}\right]},  \tag{60}\\
\ell_{[\Lambda]}^{*} & =\left(\ell_{\left[\Lambda_{1}\right]}^{*}-d\left(r_{1}, q_{1}\right)\right)+d\left(r_{1}, q_{2}\right)+\left(\ell_{\left[\Lambda_{2}\right]}^{*}-d\left(r_{2}, q_{2}\right)\right)+d\left(r_{2}, q_{1}\right) .
\end{align*}
$$

In particular,

$$
\begin{equation*}
\ell_{[\Lambda]}^{*} \geq\left(\ell_{\left[\Lambda_{1}\right]}^{*}-d\left(r_{1}, q_{1}\right)\right)+\left(\ell_{\left[\Lambda_{2}\right]}^{*}-d\left(r_{2}, q_{2}\right)\right) . \tag{61}
\end{equation*}
$$

## 4 Simple cycles

Throughout this section, we shall assume that $[\Lambda]$ is a nonempty cycle such that $\ell_{[\Lambda]}^{*}<\infty$. Recall that

$$
\begin{equation*}
T_{[\Lambda]}=\sum_{\lambda \in \Lambda}\left(\delta_{p_{\lambda}}-\delta_{n_{\lambda}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{62}
\end{equation*}
$$

We define the gap of $[\Lambda]$ to be the number given by

$$
\begin{equation*}
\operatorname{gap}[\Lambda]:=\sup _{\tilde{\Lambda} \in[\Lambda]} d\left(\lim _{\lambda \in \tilde{\Lambda} \uparrow} p_{\lambda}, \lim _{\lambda \in \tilde{\Lambda} \downarrow} n_{\lambda}\right) . \tag{63}
\end{equation*}
$$

Roughly speaking, gap $[\Lambda]$ measures the jump of $[\Lambda]$ accross two adjacent dipoles, while the co-length $\ell_{[\Lambda]}^{*}$ measures the total jump along [ $\Lambda$ ]. We point out that, since $\ell_{[\Lambda]}^{*}<\infty$, the supremum in (63) is actually achieved.

Example 36 Assume $\Lambda$ is finite, say $\Lambda=(1 \cdots k)$. In this case, we have

$$
\operatorname{gap}[\Lambda]=\max \left\{d\left(p_{1}, n_{2}\right), \ldots, d\left(p_{k-1}, n_{k}\right), d\left(p_{k}, n_{1}\right)\right\} .
$$

In particular, if gap $[\Lambda]=0$, then $T_{[\Lambda]}=0$ in $[\operatorname{Lip}(X)]^{*}$. This need not be the case in general. In fact, in Example 30 we have gap $\left[\Lambda_{0}\right]=0$, even though $L_{\left[\Lambda_{0}\right]}=\alpha>0$.

We now consider the following
Definition $37[\Lambda]$ is a closed cycle if gap $[\Lambda]=0$.
For example, we have
Lemma 38 If $[\Lambda]$ is a co-minimal cycle and $T_{[\Lambda]}$ is singular in $X$, then $[\Lambda]$ is a closed cycle.

PROOF. Let $[\Lambda]$ be a co-minimal cycle such that gap $[\Lambda]>0$. Without loss of generality, we may assume that the supremum in (63) is achieved by $\Lambda$ itself:

$$
\begin{equation*}
d\left(n_{0}, p_{0}\right)>0, \quad \text { where } n_{0}:=\lim _{\lambda \in \Lambda \uparrow} p_{\lambda} \text { and } p_{0}:=\lim _{\lambda \in \Lambda \downarrow} n_{\lambda} . \tag{64}
\end{equation*}
$$

We define the chain $\Lambda_{0}:=\Lambda \cup\{0\}$ oriented in such a way that 0 is the largest element in $\Lambda_{0}$. Applying Remark 35 with $\Lambda_{1}:=\Lambda, \Lambda_{2}:=\{0\}, r_{1}=q_{2}:=n_{0}$ and $q_{1}=r_{2}:=p_{0}$, we get

$$
\begin{equation*}
\left\|T_{\left[\Lambda_{0}\right]}\right\| \leq \ell_{\left[\Lambda_{0}\right]}^{*}=\ell_{[\Lambda]}^{*}-d\left(n_{0}, p_{0}\right)=\left\|T_{[\Lambda]}\right\|-d\left(n_{0}, p_{0}\right) \leq\left\|T_{\left[\Lambda_{0}\right]}\right\| \tag{65}
\end{equation*}
$$

(we use the triangle inequality to obtain the last estimate). Thus,

$$
\begin{equation*}
T_{[\Lambda]}=\left(\delta_{n_{0}}-\delta_{p_{0}}\right)+T_{\left[\Lambda_{0}\right]} \quad \text { and } \quad\left\|T_{[\Lambda]}\right\|=d\left(n_{0}, p_{0}\right)+\left\|T_{\left[\Lambda_{0}\right]}\right\| . \tag{66}
\end{equation*}
$$

We conclude that $T_{[\Lambda]}$ is not singular.

In order to introduce the notion of simple cycles, we shall need an auxiliary
Definition 39 A subchain $\Lambda_{1} \subset \Lambda$ is a segment of $[\Lambda]$ if $\Lambda_{1}$ is a segment of some representative $\tilde{\Lambda} \in[\Lambda]$ (see Definition 22). Equivalently, $\Lambda_{1} \subset \Lambda$ is a segment of $[\Lambda]$ if either $\Lambda_{1}$ or $\Lambda \backslash \Lambda_{1}$ is a segment of $\Lambda$.

A simple cycle will be defined as follows:
Definition $40[\Lambda]$ is a simple cycle if
(i) $[\Lambda]$ is a closed cycle;
(ii) if $\Lambda_{1}$ is a segment of $[\Lambda]$ such that $\left[\Lambda_{1}\right]$ is a closed cycle, then $\Lambda_{1}=\Lambda$.

Since gap $[\Lambda]=0$, condition (ii) in the definition above is equivalent to saying that
(ii') if $\Lambda_{1} \varsubsetneqq \Lambda$ is a segment, then $\lim _{\lambda \in \Lambda_{1} \uparrow} p_{\lambda} \neq \lim _{\lambda \in \Lambda_{1} \downarrow} n_{\lambda}$.
Note that $\left[\Lambda_{0}\right]$ given by Example 30 is a simple cycle.
The orientation of a simple cycle $[\Lambda]$ is compatible with the topology induced by $X$ on the set $\cup_{\lambda \in \Lambda}\left\{p_{\lambda}, n_{\lambda}\right\}$ in the following sense:

Lemma 41 Assume $[\Lambda]$ is a simple cycle. Given
(i) a sequence $\left(\lambda_{k}\right)_{k \geq 1}$ in $\Lambda$ such that either $p_{\lambda_{k}} \rightarrow p_{\lambda_{0}}$ or $n_{\lambda_{k}} \rightarrow p_{\lambda_{0}}$,
(ii) two indices $\mu_{1}, \mu_{2} \in \Lambda$ such that $\mu_{1}<\lambda_{0}<\mu_{2}$ with respect to some representative $\tilde{\Lambda} \in[\Lambda]$,
then there exists $k_{0} \geq 1$ such that

$$
\begin{equation*}
\mu_{1}<\lambda_{k}<\mu_{2} \quad \text { in } \tilde{\Lambda} \quad \forall k \geq k_{0} . \tag{67}
\end{equation*}
$$

PROOF. Assume by contradiction there exist $p_{\lambda_{k}} \rightarrow p_{\lambda_{0}}$ and $\mu_{1}<\lambda_{0}<\mu_{2}$ in $\Lambda$ such that (67) does not hold. (The case when $n_{\lambda_{k}} \rightarrow p_{\lambda_{0}}$ can be dealt with in a similar way).

Replacing $\Lambda$ by another representative of $[\Lambda]$, we can assume that

$$
\begin{equation*}
\mu_{1}<\lambda_{0}<\mu_{2} \leq \lambda_{k} \quad \text { in } \Lambda \quad \forall k \geq k_{0} . \tag{68}
\end{equation*}
$$

Moreover, passing to a subsequence if necessary, we can also assume that $\left(\lambda_{k}\right)_{k \geq 1}$ is either nondecreasing or nonincreasing in $\Lambda$. We consider each one of these possibilities separately:

Case 1. $\left(\lambda_{k}\right)_{k \geq 1}$ is nondecreasing in $\Lambda$.
Let

$$
\begin{equation*}
\Lambda_{1}:=\bigcup_{k=1}^{\infty}\left(\lambda_{0}<\lambda \leq \lambda_{k}\right) . \tag{69}
\end{equation*}
$$

Note that $\Lambda_{1}$ is a segment of $\Lambda$. We claim that gap $\left[\Lambda_{1}\right]=0$. In order to see this, it suffices to show that

$$
\begin{equation*}
\lim _{\lambda \in \Lambda_{1} \downarrow} n_{\lambda}=p_{\lambda_{0}}=\lim _{\lambda \in \Lambda_{1} \uparrow} p_{\lambda} . \tag{70}
\end{equation*}
$$

The first equality holds because gap $[\Lambda]=0$, while the second one follows from $p_{\lambda_{k}} \rightarrow p_{\lambda_{0}}$. Therefore, we have constructed a closed segment $\left[\Lambda_{1}\right]$ strictly contained in $[\Lambda]$, which is a contradiction.

Case 2. $\left(\lambda_{k}\right)_{k \geq 1}$ is nonincreasing in $\Lambda$.

In this case, we take

$$
\begin{equation*}
\Lambda_{1}:=\bigcap_{k=1}^{\infty}\left(\lambda_{0}<\lambda \leq \lambda_{k}\right) . \tag{71}
\end{equation*}
$$

In order to get a contradiction, it suffices to show that the second equality in (70) holds, and then argue as before.

If $\lambda_{k}=\tilde{\lambda}$ for all $k \geq 1$ sufficiently large, then $\Lambda_{1}=\left(\lambda_{0}<\lambda \leq \tilde{\lambda}\right)$ and we are done. On the other hand, if $\left(\lambda_{k}\right)_{k \geq 1}$ has infinitely many distinct terms, then we have $d\left(p_{\lambda_{k}}, n_{\lambda_{k}}\right) \rightarrow 0$. Thus,

$$
\begin{equation*}
\lim _{\lambda \in \Lambda_{1} \uparrow} p_{\lambda}=\lim _{k \rightarrow \infty} n_{\lambda_{k}}=\lim _{k \rightarrow \infty} p_{\lambda_{k}}=p_{\lambda_{0}} . \tag{72}
\end{equation*}
$$

(The first equality follows from gap $[\Lambda]=0$ ). As we explained before, this gives a contradiction.

Using the same ideas we can prove a slightly more general result:
Lemma 42 Let $[\Lambda]$ be a simple cycle. Given $\mu_{1}<\nu_{2} \leq \nu_{2}<\mu_{2}$ in $\Lambda$, let $q$ be a point in the closure of the set

$$
\bigcup_{\nu_{1} \leq \lambda \leq \nu_{2}}\left\{p_{\lambda}\right\} \cup\left\{n_{\lambda}\right\} .
$$

If $\left(\lambda_{k}\right)_{k \geq 1}$ is a sequence in $\Lambda$ such that $p_{\lambda_{k}} \rightarrow q$, then there exists $k_{0} \geq 1$ such that

$$
\begin{equation*}
\mu_{1}<\lambda_{k}<\mu_{2} \quad \forall k \geq k_{0} \tag{73}
\end{equation*}
$$

This lemma will be used to prove our next result:
Proposition 43 Assume $[\Lambda]$ is a simple cycle. Then

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}=\mathcal{H}^{1}\left(S_{[\Lambda]}\right), \tag{74}
\end{equation*}
$$

where

$$
S_{[\Lambda]}:=\overline{\bigcup_{\lambda \in \Lambda}\left\{p_{\lambda}\right\} \cup\left\{n_{\lambda}\right\}}
$$

PROOF. We split the proof into three steps:
Step 1. Given $\mu_{1}<\mu_{2}$ in $\Lambda$, we consider the segment $\tilde{\Lambda}:=\left(\mu_{1}<\lambda<\mu_{2}\right)$. Then we have

$$
\begin{equation*}
\operatorname{diam} S_{[\tilde{\Lambda}]} \leq \ell_{[\tilde{\Lambda}]}+\ell_{[\tilde{\Lambda}]}^{*}-d\left(p_{\mu_{1}}, n_{\mu_{2}}\right) \tag{75}
\end{equation*}
$$

Since $S_{[\tilde{\Lambda}]}$ is compact, for any $\eta>0$ we can find an $\varepsilon$-chain

$$
\tilde{\Lambda}_{\varepsilon}=\left(\tilde{\lambda}_{1} \cdots \tilde{\lambda}_{k}\right) \subset \tilde{\Lambda}
$$

such that

$$
\begin{equation*}
S_{[\tilde{\Lambda}]} \subset \bigcup_{i=1}^{k}\left[B_{\eta}\left(p_{\tilde{\lambda}_{i}}\right) \cup B_{\eta}\left(n_{\tilde{\lambda}_{i}}\right)\right] \tag{76}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\operatorname{diam} S_{[\tilde{\Lambda}]} & \leq d\left(n_{\tilde{\lambda}_{1}}, p_{\tilde{\lambda}_{1}}\right)+d\left(p_{\tilde{\lambda}_{1}}, n_{\tilde{\lambda}_{2}}\right)+\cdots+d\left(n_{\tilde{\lambda}_{k}}, p_{\tilde{\lambda}_{k}}\right)+2 \eta \\
& =\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}+\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}^{*}-d\left(n_{\tilde{\lambda}_{1}}, p_{\tilde{\lambda}_{k}}\right)+2 \eta  \tag{77}\\
& \leq \ell_{[\tilde{\Lambda}]}+\ell_{[\tilde{\Lambda}]}^{\tilde{\Lambda}}-d\left(n_{\tilde{\lambda}_{1}}, p_{\tilde{\lambda}_{k}}\right)+2 \eta .
\end{align*}
$$

We first let $\varepsilon \downarrow 0$. Then gap $[\Lambda]=0$ implies that $n_{\tilde{\lambda}_{1}} \rightarrow p_{\mu_{1}}$ and $p_{\tilde{\lambda}_{k}} \rightarrow n_{\mu_{2}}$. Since $\eta>0$ is arbitrary, (75) follows.

## Step 2.

$$
\begin{equation*}
\mathcal{H}^{1}\left(S_{[\Lambda]}\right) \leq \ell_{[\Lambda]}^{*} . \tag{78}
\end{equation*}
$$

(This inequality holds even if $[\Lambda]$ is just a closed cycle).
Let $\delta>0$ fixed. Given an $\varepsilon$-chain $\Lambda_{\varepsilon}=\left(\lambda_{1} \cdots \lambda_{k}\right) \subset \Lambda$, we define the segments

$$
\begin{equation*}
\Lambda_{i}:=\left(\lambda_{i}<\lambda<\lambda_{i+1}\right) \quad i=1, \ldots, k . \tag{79}
\end{equation*}
$$

(we use the convention that $\lambda_{k+1}:=\lambda_{1}$ ).
By taking $\varepsilon>0$ sufficiently small, we can assume that

$$
\begin{equation*}
\operatorname{diam} S_{\left[\Lambda_{i}\right]} \leq \delta \quad \forall i=1, \ldots, k \tag{80}
\end{equation*}
$$

Since gap $[\Lambda]=0$, we have

$$
\begin{equation*}
S_{[\Lambda]}=\bigcup_{i=1}^{k} S_{\left[\Lambda_{i}\right]} . \tag{81}
\end{equation*}
$$

It follows from the previous step and (61) that

$$
\begin{align*}
\mathcal{H}_{\delta}^{1}\left(S_{[\Lambda]}\right) & \leq \sum_{i=1}^{k} \operatorname{diam} S_{\left[\Lambda_{i}\right]} \\
& \leq \sum_{i=1}^{k}\left\{\ell_{\left[\Lambda_{i}\right]}+\left(\ell_{\left[\Lambda_{i}\right]}^{*}-d\left(p_{\lambda_{i}}, n_{\lambda_{i+1}}\right)\right)\right\}  \tag{82}\\
& \leq \sum_{i \notin \Lambda_{\varepsilon}} d\left(p_{i}, n_{i}\right)+\ell_{[\Lambda]}^{*} .
\end{align*}
$$

Taking $\varepsilon \downarrow 0$, we conclude that

$$
\begin{equation*}
\mathcal{H}^{1}\left(S_{[\Lambda]}\right)=\lim _{\delta \downarrow 0} \mathcal{H}_{\delta}^{1}\left(S_{[\Lambda]}\right) \leq \ell_{[\Lambda]}^{*} . \tag{83}
\end{equation*}
$$

## Step 3.

$$
\begin{equation*}
\ell_{[\Lambda]}^{*} \leq \mathcal{H}^{1}\left(S_{[\Lambda]}\right) . \tag{84}
\end{equation*}
$$

Given an $\varepsilon$-chain $\Lambda_{\varepsilon}=\left(\lambda_{1} \cdots \lambda_{k}\right) \subset \Lambda$, we consider the segments $\Lambda_{i}$ given by (79). Since $[\Lambda]$ is simple, the sets $S_{\left[\Lambda_{i}\right]}$ are disjoint (see Lemma 42). Let $\delta>0$ be such that

$$
\begin{equation*}
d\left(S_{\left[\Lambda_{1}\right]}, S_{\left[\Lambda_{i_{2}}\right]}\right)>2 \delta \quad \forall i_{1} \neq i_{2} . \tag{85}
\end{equation*}
$$

Take $\left(B_{j}\right)_{j \in J}$ to be a finite open covering of $S_{[\Lambda]}$ so that diam $B_{j}<\delta$ for every $j \in J$.

We claim we can select
(i) a new $\varepsilon$-chain $\tilde{\Lambda}_{\varepsilon}=\left(\tilde{\lambda}_{1} \cdots \tilde{\lambda}_{l}\right)$ containing $\Lambda_{\varepsilon}$;
(ii) $l$ distinct elements from the family $\left(B_{j}\right)_{j \in J}$, say $\tilde{B}_{1}, \ldots, \tilde{B}_{l}$,
such that

$$
\begin{equation*}
p_{\tilde{\lambda}_{i}}, n_{\tilde{\lambda}_{i+1}} \in \tilde{B}_{i} \quad \forall i=1, \ldots, l \tag{86}
\end{equation*}
$$

We proceed as follows:
We first define the segments

$$
\begin{equation*}
\Gamma_{j}:=\left\{\lambda: \mu_{1} \leq \lambda \leq \mu_{2} \text { for some } \mu_{1}<\mu_{2} \text { such that } p_{\mu_{1}}, n_{\mu_{2}} \in B_{j}\right\} . \tag{87}
\end{equation*}
$$

Note that if $B_{j} \cap S_{[\Lambda]} \neq \phi$, then $\Gamma_{j} \neq \phi$. In fact, assume for instance that $p_{\mu} \in B_{j}$. Since gap $[\Lambda]=0$, then either there exists $\mu_{2}>\mu$ such that $n_{\mu_{2}}=p_{\mu}$ or we can find a decreasing sequence $\mu_{j} \downarrow \mu$ such that $n_{\mu_{j}} \rightarrow p_{\mu}$. The set $B_{j}$ being open, we conclude that $n_{\mu_{j_{0}}} \in B_{j}$ for some $\mu_{j_{0}}>\mu$. Thus, in both case we have $\Gamma_{j} \neq \phi$. Moreover, (85) implies that $\Gamma_{j}$ is contained in some segment $\left(\lambda_{i} \leq \lambda \leq \lambda_{i+1}\right)$.

We also define

$$
r_{j}:=\lim _{\lambda \in \Gamma_{j} \uparrow} p_{\lambda}
$$

to be the upper endpoint of $\Gamma_{j}$.
Let $\tilde{\lambda}_{1}:=\lambda_{1}$ and $\tilde{B}_{1}$ be an element of the family $\left(B_{j}\right)_{j \in J}$ containing $p_{\lambda_{1}}$. By abuse of notation, we denote by $\Gamma_{1}$ the segment $\Gamma_{j}$ corresponding to $\tilde{B}_{1}$. We have two possibilities for $\Gamma_{1}$ :
(a) $\Gamma_{1}$ has a largest element $\tilde{\lambda}_{2}$ : in this case, we have

$$
n_{\tilde{\lambda}_{2}} \in \tilde{B}_{1} \quad \text { and } \quad p_{\tilde{\lambda}_{2}} \notin \tilde{B}_{1}
$$

(since $\tilde{B}_{1}$ is open and gap $[\Lambda]=0$ ), we then choose $\tilde{B}_{2} \in\left(B_{j}\right)_{j \in J}$ such that $p_{\tilde{\lambda}_{2}} \in \tilde{B}_{2}$;
$\left(\mathrm{b}_{1}\right) \Gamma_{1}$ does not have a largest element: this implies the existence of an increasing sequence $\left(\mu_{j}\right)$ in $\Gamma_{1}$ such that

$$
n_{\mu_{j}} \in \tilde{B}_{1} \quad \forall j \geq 1 \quad \text { and } \quad p_{\mu_{j}} \rightarrow r_{1}
$$

moreover, $d\left(p_{\mu_{j}}, n_{\mu_{j}}\right) \rightarrow 0$.
Let $\tilde{B}_{2} \in\left(B_{j}\right)_{j \in J}$ be such that $r_{1} \in \tilde{B}_{2}$. Since $\tilde{B}_{1}$ and $\tilde{B}_{2}$ are both open, we can find $j_{0} \geq 1$ sufficiently large so that

$$
n_{\mu_{j_{0}}} \in \tilde{B}_{1} \cap \tilde{B}_{2} \quad \text { and } \quad p_{\mu_{j_{0}}} \in \tilde{B}_{2}
$$

We then take $\tilde{\lambda}_{2}:=\mu_{j_{0}}$.
Note that in both cases we have

$$
\tilde{B}_{1} \neq \tilde{B}_{2}, \quad n_{\tilde{\lambda}_{2}} \in \tilde{B}_{1} \quad \text { and } \quad p_{\tilde{\lambda}_{2}} \in \tilde{B}_{2} .
$$

We can now repeat the construction above with $\tilde{\lambda}_{2}$ and $\tilde{B}_{2}$, and so on until we get $n_{\lambda_{1}} \in \tilde{B}_{l}$. In order to see this will be indeed the case, it suffices to prove the following:

We claim that $\tilde{\Lambda}_{\varepsilon}:=\left(\tilde{\lambda}_{1} \cdots \tilde{\lambda}_{l}\right) \supset \Lambda_{\varepsilon}$.
Let us check for instance that $\lambda_{2} \in \tilde{\Lambda}_{\varepsilon}$ (the general case follows by induction): let $1<l_{1}<l$ be such that $\tilde{\lambda}_{l_{1}}<\lambda_{2} \leq \tilde{\lambda}_{l_{1}+1}$. Since

$$
p_{\tilde{\lambda}_{1}}, n_{\tilde{\lambda}_{l_{1}+1}} \in \tilde{B}_{l_{1}} \quad \text { and } \quad p_{\tilde{\lambda}_{l_{1}}} \in S_{\left[\Lambda_{1}\right]},
$$

we have $n_{\tilde{\lambda}_{l_{1}+1}} \in S_{\left[\Lambda_{1}\right]}$. On the other hand, $\lambda_{2} \leq \tilde{\lambda}_{l_{1}+1}$ implies that $p_{\tilde{\lambda}_{l_{1}+1}} \notin$ $S_{\left[\Lambda_{1}\right]}$. In particular,

$$
d\left(p_{\tilde{\lambda}_{l_{1}+1}}, n_{\tilde{\lambda}_{l_{1}+1}}\right)>2 \delta,
$$

from which we conclude that $\tilde{\lambda}_{l_{1}+1}=\lambda_{2}$.
By construction, the sets $\tilde{B}_{1}, \ldots, \tilde{B}_{l}$ are all distinct and (86) holds. It follows from (86) that

$$
\begin{equation*}
d\left(p_{\tilde{\lambda}_{i}}, n_{\tilde{\lambda}_{i+1}}\right) \leq \operatorname{diam} \tilde{B}_{i} . \tag{88}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\ell_{\left[\tilde{\Lambda}_{\varepsilon}\right]}^{*}=\sum_{i=1}^{l} d\left(p_{\tilde{\lambda}_{i}}, n_{\tilde{\lambda}_{i+1}}\right) \leq \sum_{i=1}^{l} \operatorname{diam} \tilde{B}_{i} \leq \sum_{j \in J} \operatorname{diam} B_{j} . \tag{89}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\inf _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*} \leq \sum_{j \in J} \operatorname{diam} B_{j} . \tag{90}
\end{equation*}
$$

where the infimum is taken over all $\varepsilon$-chains of $[\Lambda]$. We now take the infimum with respect to all (finite) open coverings $\left(B_{j}\right)_{j \in J}$ of $S_{[\Lambda]}$ with diam $B_{j}<\delta$ for all $j \in J$. We get

$$
\begin{equation*}
\inf _{\Lambda_{\varepsilon}} \ell_{\left[\Lambda_{\varepsilon}\right]}^{*} \leq \mathcal{H}_{\delta}^{1}\left(S_{[\Lambda]}\right) \tag{91}
\end{equation*}
$$

Note that this estimate holds for $\varepsilon>0$ fixed and every $\delta>0$ sufficiently small. Taking $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$, we obtain (84).

We conclude this section with the following result which will be used in the proof of Theorem 6:

Proposition 44 Let $[\Lambda]$ be a co-minimal cycle. If $T_{[\Lambda]}$ is singular, then we can write $\Lambda=\bigcup_{j} \Lambda_{j}$ as a disjoint union of subchains, where each $\left[\Lambda_{j}\right]$ is a simple cycle and

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}=\sum_{j} \ell_{\left[\Lambda_{j}\right]}^{*} . \tag{92}
\end{equation*}
$$

In particular, $\left[\Lambda_{j}\right]$ is a co-minimal cycle and $T_{\left[\Lambda_{j}\right]}$ is singular for every $j$.

PROOF. Consider the family

$$
\mathcal{F}:=\left\{\begin{array}{l|l}
\left(\Lambda_{k}\right)_{k \in \Lambda} & \begin{array}{l}
\Lambda_{k} \text { is a subchain of }[\Lambda] \text { and } k \in \Lambda_{k} \\
\text { if } \Lambda_{k_{1}} \cap \Lambda_{k_{2}} \neq \phi, \text { then } \Lambda_{k_{1}}=\Lambda_{k_{2}}, \\
\sum_{\Lambda_{k}} \ell_{\left[\Lambda_{k}\right]}^{*} \leq \ell_{[\Lambda]}^{*}
\end{array} \tag{93}
\end{array}\right\}
$$

(The sum $\sum_{\Lambda_{k}}$ is taken over all disjoint components of $\left.\left(\Lambda_{k}\right)_{k \in \Lambda}\right)$.
Since $(\Lambda)_{k \in \Lambda} \in \mathcal{F}$ (i.e. we take $\Lambda_{k}=\Lambda$ for each $k$ ), $\mathcal{F}$ is nonempty. We consider the order relation $\leq$ in $\mathcal{F}$ given by $\left(\tilde{\Lambda}_{k}\right)_{k} \leq\left(\hat{\Lambda}_{k}\right)_{k}$ iff $\tilde{\Lambda}_{k} \supset \hat{\Lambda}_{k}$ for every $k \in \Lambda$.

Step 1. If $\left(\Lambda_{k}\right)_{k \in \Lambda} \in \mathcal{F}$, then $\left[\Lambda_{k}\right]$ is a co-minimal cycle and $T_{\left[\Lambda_{k}\right]}$ is singular for every $k \in \Lambda$. Moreover,

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}=\sum_{\Lambda_{k}} \ell_{\left[\Lambda_{k}\right]}^{*} . \tag{94}
\end{equation*}
$$

Since $[\Lambda]$ is co-minimal cycle, it follows from the triangle inequality applied to $T_{[\Lambda]}=\sum_{\Lambda_{k}} T_{\left[\Lambda_{k}\right]}$ that

$$
\begin{equation*}
L_{[\Lambda]} \leq \sum_{\Lambda_{k}} L_{\left[\Lambda_{k}\right]} \leq \sum_{\Lambda_{k}} \ell_{\left[\Lambda_{k}\right]}^{*} \leq \ell_{[\Lambda]}^{*}=L_{[\Lambda]} . \tag{95}
\end{equation*}
$$

Therefore, equality holds everywhere in (95) and we have

$$
\begin{equation*}
\ell_{[\Lambda]}^{*}=\sum_{\Lambda_{k}} \ell_{\left[\Lambda_{k}\right]}^{*} \quad \text { and } \quad L_{\left[\Lambda_{k}\right]}=\ell_{\left[\Lambda_{k}\right]}^{*} \quad \forall k \in \Lambda . \tag{96}
\end{equation*}
$$

In particular, $\left[\Lambda_{k}\right]$ is a co-minimal cycle. Since $L_{[\Lambda]}=\sum_{\Lambda_{k}} L_{\left[\Lambda_{k}\right]}$ and $T_{\Lambda}$ is singular, we conclude that every $T_{\left[\Lambda_{k}\right]}$ is singular (see Remark 61 (b)).

Step 2. $\mathcal{F}$ has a maximal element.
By Zorn's Lemma, it suffices to show that if $\left(\left(\Lambda_{k, \alpha}\right)_{k \in \Lambda}\right)_{\alpha}$ is a linearly ordered family in $\mathcal{F}$, then it has an upper bound.

For each $k \in \Lambda$ we set $\Lambda_{k}:=\bigcap_{\alpha} \Lambda_{k, \alpha}$.
Clearly, the first two properties in (93) are satisfied by $\left(\Lambda_{k}\right)_{k \in \Lambda}$. We now check the last one.

Let $\Lambda_{k_{1}}, \ldots, \Lambda_{k_{l}}$ be the first $l$ disjoint subchains in $\left(\Lambda_{k}\right)_{k \in \Lambda}$. Take $\alpha_{0}$ sufficiently large so that $\Lambda_{k_{1}, \alpha_{0}}, \ldots, \Lambda_{k_{l}, \alpha_{0}}$ are disjoint. Applying Corollary 26 we get

$$
\begin{align*}
\sum_{i=1}^{l} \ell_{\left[\Lambda_{k_{i}}\right]}^{*} & \leq \sum_{i=1}^{l} \ell_{\left[\Lambda \lambda_{i}, \alpha_{0}\right]}^{*}+\sum_{i=1}^{l}\left(\ell_{\left[\Lambda_{k_{i}, \alpha_{0}}\right]}-\ell_{\left[\Lambda k_{i}\right]}\right)  \tag{97}\\
& \leq \ell_{[\Lambda]}^{*}+\ell_{[\Lambda]}-\ell_{\left[\Lambda_{k_{1}} \cup \cdots \cup \Lambda_{k_{l}}\right]} .
\end{align*}
$$

Since $l$ was arbitrary, we conclude that $\sum_{\Lambda_{k}} \ell_{\left[\Lambda_{k}\right]}^{*} \leq \ell_{[\Lambda]}^{*}$. Thus, $\left(\Lambda_{k}\right)_{k \in \Lambda} \in \mathcal{F}$.
We can now invoke Zorn's Lemma to conclude that $\mathcal{F}$ has a maximal element.
Step 3. Proof of the proposition completed.
Let $\left(\Lambda_{k}\right)_{k \in \Lambda}$ be a maximal element of $\mathcal{F}$. We claim that $\left[\Lambda_{k}\right]$ is simple for every $k \in \Lambda$.

Suppose by contradiction that $\Lambda_{k}$ is not simple. By definition, we can split $\Lambda_{k}=\Lambda_{k, 1} \cup \Lambda_{k, 2}$ so that both $\Lambda_{k, 1}$ and $\Lambda_{k, 2}$ are segments of $\left[\Lambda_{k}\right]$ and gap $\left[\Lambda_{k, 1}\right]=$ 0 . Since gap $[\Lambda]=0$, we also have gap $\left[\Lambda_{k, 2}\right]=0$. It follows from Remark 35 that

$$
\begin{equation*}
\ell_{\left[\Lambda_{k}\right]}^{*}=\ell_{\left[\Lambda_{k, 1}\right]}^{*}+\ell_{\left[\Lambda_{k, 2}\right]}^{*}, \tag{98}
\end{equation*}
$$

but this contradicts the maximality of $\left(\Lambda_{k}\right)_{k \in \Lambda}$ in $\mathcal{F}$.
The proposition follows from Step 1 after relabeling and removing the repeated components of $\left(\Lambda_{k}\right)_{k \in \Lambda}$.

## 5 Paths and loops

Let $\Gamma$ be a chain such that $\ell_{[\Gamma]}^{*}<\infty$.
It follows from Proposition 33 that both limits

$$
\begin{equation*}
r:=\lim _{\lambda \in \Gamma \uparrow} p_{\lambda} \quad \text { and } \quad q:=\lim _{\lambda \in \Gamma \downarrow} n_{\lambda} \tag{99}
\end{equation*}
$$

exist. Clearly, we have $\ell_{[\Gamma]}^{*} \geq d(r, q)$.
Definition $45 \Gamma$ is a path from $q$ to $r$ if

$$
\begin{equation*}
\ell_{[\Gamma]}^{*}=d(r, q) . \tag{100}
\end{equation*}
$$



Fig. 3. A finite path $\Gamma$ from $q$ to $r$ and the cycle $[\Lambda]$ associated to $\Gamma$
We can give an equivalent definition of a path in terms of loops (see Definition 28 (c)). In fact, let $p_{0}:=q$ and $n_{0}:=r$. We consider the chain $\Lambda:=\Gamma \cup\{0\}$, where 0 is the largest element in $\Lambda$. Then $\Gamma$ is a path from $q$ to $r$ iff $[\Lambda]$ is a loop. In particular, all results for loops can be translated in terms of paths, and conversely.

Example 46 Assume $\Gamma=(1 \cdots k)$ is a finite path from $q$ to $r$. Then $q=n_{1}$ and $r=p_{k}$; moreover, we have $p_{i}=n_{i+1}$ for every $i=1, \ldots, k-1$. Figure 3 shows a finite path $\Gamma$ (with $k=4$ ) and the cycle $[\Lambda]$ associated to it.

A less trivial example is given by Example 5 with $\alpha=0$. In this case, we may take $\Gamma=\mathbb{N}$ oriented from left to right in Figure 1. It is easy to see that $\Gamma$ is a path from 0 to 1 .

Remark 47 If $\Gamma$ is a path from $q$ to $r$, then it follows from (46) that

$$
\begin{equation*}
T_{[\Gamma]}=\delta_{r}-\delta_{q} \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{101}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
L_{[\Gamma]}=\ell_{[\Gamma]}^{*}=d(r, q) \leq \ell_{[\Lambda]} . \tag{102}
\end{equation*}
$$

Remark 48 Assume $\Gamma_{1}$ is a segment of $\Gamma$, and let $r_{1}$ and $q_{1}$ be the corresponding endpoints. We claim that $\Gamma_{1}$ is a path from $q_{1}$ to $r_{1}$.

Suppose for simplicity that $\Gamma_{2}:=\Gamma \backslash \Gamma_{1}$ is also a segment and $\Gamma_{1} \leq \Gamma_{2}$. Note that $q_{1}=q$ and $r_{2}=r$. Applying (60) with $\Lambda$ replaced by $\Gamma$ we get

$$
\begin{equation*}
\left(\ell_{\left[\Gamma_{1}\right]}^{*}-d\left(r_{1}, q_{1}\right)\right)+d\left(r_{1}, q_{2}\right)+\left(\ell_{\left[\Gamma_{2}\right]}^{*}-d\left(r_{2}, q_{2}\right)\right)=0 . \tag{103}
\end{equation*}
$$

Since each one of these terms is nonnegative, we must have $\ell_{\left[\Gamma_{1}\right]}^{*}=d\left(r_{1}, q_{1}\right)$ (note also that $d\left(r_{1}, q_{2}\right)=0$ ).

The general case follows from the above since $\Gamma \backslash \Gamma_{1}$ is a union of at most two segments.

A simple consequence of Proposition 31 is the following:
Corollary 49 Assume

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=0 \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{104}
\end{equation*}
$$

Then we can write $\mathbb{N}=\bigcup_{j} \Lambda_{j}$ as a disjoint union, where each $\left[\Lambda_{j}\right]$ is a loop.
The corollary below is just a restatement of the previous one in terms of paths:
Corollary 50 If

$$
\begin{equation*}
\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=\delta_{r}-\delta_{q} \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{105}
\end{equation*}
$$

for some $r, q \in X, r \neq q$, then there exists a path $\Gamma$ from $q$ to $r$.
As a consequence, we can now prove Lemma 9:

Proof of Lemma 9 Assume that $q_{2}$ cannot be connected to $r_{1}$ by any path. We write

$$
\begin{equation*}
\left(\delta_{p_{0}}-\delta_{n_{0}}\right)+\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)=\delta_{r_{1}}-\delta_{q_{1}} \quad \text { in }[\operatorname{Lip}(X)]^{*}, \tag{106}
\end{equation*}
$$

where $p_{0}:=q_{2}$ and $n_{0}:=r_{2}$. It follows from Corollary 50 that there exists a path $\Gamma$ from $q_{1}$ to $r_{1}$. We claim that $0 \notin \Gamma$. In fact, otherwise the segment $[\lambda>0] \subset \Gamma$ would be a path from $p_{0}=q_{2}$ to $r_{1}$, which cannot be the case by assumption. The result now follows directly from (101).

Combining Corollaries 10 and 50 we get the proposition below, which is especially suitable to study irreducible representations (see e.g. Lemma 57).

Proposition 51 Assume $\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ is reducible and $p_{i} \neq n_{j}$ for every $i, j$. Then there exist $r, q \in X$ and an infinite path $\Gamma$ from $q$ to $r$.

The following lemma will be used in the proof of Theorem 8:
Lemma 52 Let $\tilde{\mathbb{N}} \subset \mathbb{N}$. For each $j \in \tilde{\mathbb{N}}$ there exists a path $\Gamma_{\tilde{\mathbb{N}}, j}$ which is maximal among all paths in $\tilde{\mathbb{N}}$ containing $j$.

PROOF. This is a simple application of Zorn's Lemma. In fact, note that $(j)$ is a path containing $j$. Moreover, if $\left(\Gamma_{\alpha}\right)$ is a linearly ordered set of paths
containing $j$, then we define $\Gamma:=\bigcup_{\alpha} \Gamma_{\alpha}$, equipped with the order relation induced from each $\Gamma_{\alpha}$. We claim that $\Gamma$ is a path.

In fact, let $\left(\alpha_{j}\right)$ be an increasing sequence such that $\Gamma=\bigcup_{j} \Gamma_{\alpha_{j}}$. On the one hand, Corollary 27 says that

$$
\begin{equation*}
\ell_{[\Gamma]}^{*}=\lim _{j \rightarrow \infty} \ell_{\left[\Gamma_{\alpha_{j}}\right]}^{*} . \tag{107}
\end{equation*}
$$

In particular, $\ell_{[\Gamma]}^{*} \leq \ell_{[\Gamma]}<\infty$. We conclude from Proposition 33 that both limits

$$
\begin{equation*}
r:=\lim _{\lambda \in \Gamma \uparrow} p_{\lambda} \quad \text { and } \quad q:=\lim _{\lambda \in \Gamma \downarrow} n_{\lambda} \tag{108}
\end{equation*}
$$

exist.
On the other hand, each $\Gamma_{\alpha_{j}}$ is a path from some $q_{j}$ to some $r_{j}$. In addition, it follows from Remark 48 that $\Gamma \backslash \Gamma_{\alpha_{j}}=\Gamma_{1} \cup \Gamma_{2}$ is the union of two segments: $\Gamma_{1}$ goes from $q$ to $q_{j}$, while $\Gamma_{2}$ goes from $r_{j}$ to $r$. Applying (102), we conclude that $q_{j} \rightarrow q$ and $r_{j} \rightarrow r$. In view of (107), we have $\ell_{[\Gamma]}^{*}=d(r, q)$, which shows that $\Gamma$ is a path.

The statement now follows from Zorn's Lemma.

## 6 Examples

Throughout this section, we shall use the same notation as in the Examples 5 and 30 .

The example below shows that the converse to Theorem 3 does not hold in general. Namely, $T \in \mathcal{Z}$ may be regular and yet we can have $\mathcal{H}^{1}(\operatorname{supp} T)>0$.

Example 53 Assume $X=[0,1]$. We consider the chain $\Lambda:=\mathbb{N}$ oriented so that $k_{1} \leq k_{2}$ iff $p_{k_{1}} \leq p_{k_{2}}$ in $[0,1]$ (see Figure 4). We claim that

$$
T_{[\Lambda]}=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { is irreducible. }
$$

In view of Proposition 51, it suffices to show that if $\Gamma$ is a path containing $i_{0} \geq 1$, then $\Gamma=\left(i_{0}\right)$. Let $r, q \in C_{\alpha}, q \leq r$, be the endpoints of $\Gamma$. It is easy to see that the inclusion $\Gamma \subset \Lambda$ is order preserving and that $\Gamma$ is a segment of $\Lambda$. Thus,

$$
\ell_{[\Gamma]}^{*}=d(r, q)+\left|[q, r] \backslash \bigcup_{i \in \Gamma} J_{i}\right|=d(r, q)+\left|C_{\alpha} \cap[q, r]\right| .
$$



Fig. 4. Dipoles $\delta_{p_{i}}-\delta_{n_{i}}$ in Example 53
Since $\Gamma$ is a path, the second term in the right-hand side has to vanish. In other words, we must have $(q, r) \subset J_{i_{0}}$, which implies that $\Gamma=\left(i_{0}\right)$. This proves the claim.

Note that Proposition 18 implies that

$$
L_{[\Lambda]}=\sum_{i=1}^{\infty} d\left(p_{i}, n_{i}\right)=1-\alpha .
$$

In particular, $[\Lambda]$ is a minimal cycle and $T_{[\Lambda]}$ is regular.
In the next example we show that $T_{[\Lambda]}+\left(\delta_{0}-\delta_{1}\right)$ is singular.
Example 54 As in Example 30, we consider the chain $\Lambda_{0}:=\Lambda \cup\{0\}$ so that $\left[\Lambda_{0}\right]$ is oriented clockwise with respect to Figure 1.

According to the previous example,

$$
T_{\left[\Lambda_{0}\right]}=\sum_{i=0}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { is irreducible } .
$$

Moreover, it follows from Proposition 18 that

$$
L_{\left[\Lambda_{0}\right]}=\ell_{\left[\Lambda_{0}\right]}^{*}=\alpha .
$$

In particular, $\left[\Lambda_{0}\right]$ is a co-minimal cycle.
We claim that $T_{\left[\Lambda_{0}\right]}$ is singular.
Let $\zeta_{k}$ be the Lipschitz function such that $\zeta_{k}(t)=0$ if $t \leq n_{k}, \zeta_{k}(t)=d\left(p_{k}, n_{k}\right)$ if $t \geq p_{k}$, and $\zeta_{k}$ is affine linear on $J_{k}$. We define

$$
\zeta(t):=t-\sum_{k=1}^{\infty} \zeta_{k}(t)
$$

(by construction, $\zeta$ is constant on each $J_{k}$ ). Note that $|\zeta|_{\text {Lip }} \leq 1$ and

$$
L_{\left[\Lambda_{0}\right]}=\alpha=\sum_{i=0}^{\infty}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right]
$$

In other words, $\zeta$ is a function which achieves the supremum in (8).


Fig. 5. Dipoles $\delta_{r_{i}}-\delta_{q_{i}}$ in Example 55

Given $r, q \in[0,1], r \neq q$, we have $|\zeta(r)-\zeta(q)|<d(r, q)$. Thus,

$$
\left\|T_{\left[\Lambda_{0}\right]}-\left(\delta_{r}-\delta_{q}\right)\right\| \geq \sum_{i=0}^{\infty}\left[\zeta\left(p_{i}\right)-\zeta\left(n_{i}\right)\right]-[\zeta(r)-\zeta(q)]>\left\|T_{\left[\Lambda_{0}\right]}\right\|-d(r, q)
$$

This proves our claim (see Lemma 60).
We now combine somewhat Examples 53 and 54:
Example 55 Let $X=S^{1}$ equipped with its geodesic distance. We shall identify $\mathbb{R}^{2}$ with the complex plane $\mathbb{C}$. Using the same notation as above, we define

$$
r_{k}:=\mathrm{e}^{2 \pi p_{k} i} \quad \text { and } \quad q_{k}:=\mathrm{e}^{2 \pi n_{k} i} \quad \forall k \geq 1 .
$$

We consider the chain $\Lambda=\mathbb{N}$ oriented anticlockwise with respect to Figure 5.
Note that

$$
\ell_{[\Lambda]}=2 \pi(1-\alpha) \quad \text { and } \quad \ell_{[\Lambda]}^{*}=2 \pi \alpha .
$$

On the other hand, applying Proposition 18, we get

$$
L_{[\Lambda]}=2 \pi \min \{\alpha, 1-\alpha\} .
$$

Thus,
(a) if $0<\alpha<\frac{1}{2}$, then $[\Lambda]$ is a co-minimal cycle and $T_{[\Lambda]}$ is singular (we proceed as in the previous example);
(b) if $\frac{1}{2} \leq \alpha<1$, then $[\Lambda]$ is a minimal cycle and $T_{[\Lambda]}$ is regular.

## 7 Proof of Theorem 8

It suffices to consider the case when

$$
T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}
$$

is an infinite sum of dipoles. The strategy will be to construct a sequence of disjoint paths $\Gamma_{1}, \Gamma_{2}, \ldots$ and sets $\mathbb{N}=: \mathbb{N}_{1} \supset \mathbb{N}_{2} \supset \cdots$ inductively as follows.

Let $\Gamma_{1}$ be a maximal path containing 1 (such a path exists by Lemma 52 ). Set $\mathbb{N}_{2}:=\left\{j \in \mathbb{N}: j \notin \Gamma_{1}\right\}$.

Given $k \geq 2$ such that $\mathbb{N}_{k} \neq \phi$, let $j_{k}$ be the smallest integer in $\mathbb{N}_{k}$ and let $\Gamma_{k}$ be a maximal path among those in $\mathbb{N}_{k}$ containing $j_{k}$. Set $\mathbb{N}_{k+1}:=\left\{j \in \mathbb{N}_{k}\right.$ : $\left.j \notin \Gamma_{k}\right\}$.

By construction, each $\Gamma_{k}$ is a path from some $\hat{n}_{k}$ to some $\hat{p}_{k}$, and these paths are disjoint from each other. It follows from (101) that

$$
\begin{equation*}
T=\sum_{k \in \mathbb{N}^{\prime}} T_{\left[\Gamma_{k}\right]}=\sum_{k \in \mathbb{N}^{\prime}}\left(\delta_{\hat{p}_{k}}-\delta_{\hat{n}_{k}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}, \tag{109}
\end{equation*}
$$

where $\mathbb{N}^{\prime}:=\left\{k: \hat{p}_{k} \neq \hat{n}_{k}\right\}$.
We claim this representation is irreducible.
Suppose by contradiction it is reducible. By maximality, we must have $\hat{p}_{i} \neq \hat{n}_{j}$ for all $i, j \in \mathbb{N}^{\prime}$. Then, according to Proposition 51, we can find an infinite path $\Gamma^{\prime}$ from $q$ to $r\left(\Gamma^{\prime}\right.$ is a path with respect to the dipoles $\left.\delta_{\hat{p}_{k}}-\delta_{\hat{n}_{k}}\right)$. In particular,

$$
\begin{equation*}
\sum_{k \in \Gamma^{\prime}}\left(\delta_{\hat{p}_{k}}-\delta_{\hat{n}_{k}}\right)=\delta_{r}-\delta_{q} \quad \text { in }[\operatorname{Lip}(X)]^{*} . \tag{110}
\end{equation*}
$$

Consider $\Gamma:=\bigcup_{k \in \Gamma^{\prime}} \Gamma_{k}$ with the order induced by $\Gamma^{\prime}$, i.e. $\lambda_{1} \leq \lambda_{2}$ in $\Gamma$ iff one of the following conditions hold:
(i) $\lambda_{1}, \lambda_{2} \in \Gamma_{k}$ for some $k \in \Gamma^{\prime}$ and $\lambda_{1} \leq \lambda_{2}$ in $\Gamma_{k}$;
(ii) $\lambda_{1} \in \Gamma_{k_{1}}, \lambda_{2} \in \Gamma_{k_{2}}$ and $k_{1}<k_{2}$ in $\Gamma^{\prime}$.

Then one can easily check that $\Gamma$ is a path from $q$ to $r$ (associated to the dipoles $\delta_{p_{k}}-\delta_{n_{k}}$ ). But this contradicts the maximality of $\Gamma_{k_{0}}$, where $k_{0}$ is the smallest integer in $\Gamma^{\prime}$. This concludes the proof of the theorem.

Remark 56 Since $d\left(\hat{p}_{k}, \hat{n}_{k}\right) \leq \sum_{j \in \mathbb{N}_{k}} d\left(p_{j}, n_{j}\right)$ for every $k$, we conclude that
the points $\hat{p}_{k}, \hat{n}_{k} \in X$ constructed above also satisfy the estimate

$$
\begin{equation*}
\sum_{k} d\left(\hat{p}_{k}, \hat{n}_{k}\right) \leq \sum_{i} d\left(p_{i}, n_{i}\right) . \tag{111}
\end{equation*}
$$

## 8 A lemma on irreducible representations

Lemma 57 Assume $T \in \mathcal{Z}$ and $T \neq 0$ in $[\operatorname{Lip}(X)]^{*}$. Let

$$
\begin{equation*}
T=\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \tag{112}
\end{equation*}
$$

be an irreducible representation of $T$.
Then given any $\delta>0$ and $i_{0} \geq 1$, there exists $\zeta \in \operatorname{BLip}(X)$ such that

$$
\begin{equation*}
\|\zeta\|_{\infty} \leq 1, \quad \operatorname{supp} \zeta \subset \overline{B_{\delta}\left(p_{i_{0}}\right)} \quad \text { and } \quad\langle T, \zeta\rangle \geq \frac{1}{4} \tag{113}
\end{equation*}
$$

PROOF. If the representation in (112) is a finite sum of Dirac masses, then we are done. Therefore, we can assume that

$$
T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}
$$

and $i_{0}=1$.
Let $A:=X \backslash B_{\delta}\left(p_{1}\right)$. We consider the quotient space $X / A$ endowed with the metric

$$
\begin{equation*}
d(\bar{x}, \bar{y}):=\min \{d(x, y), d(x, A)+d(y, A)\} \quad \forall x, y \in X \tag{114}
\end{equation*}
$$

The quotient map $\pi: X \rightarrow X / A$ induces the linear functional

$$
\begin{equation*}
\bar{T}=\sum_{i=1}^{\infty}\left(\delta_{\bar{p}_{i}}-\delta_{\bar{n}_{i}}\right) \quad \text { in }[\operatorname{Lip}(X / A)]^{*} \tag{115}
\end{equation*}
$$

Since the representation in (112) is irreducible, we have $\bar{T} \neq 0$ in $[\operatorname{Lip}(X / A)]^{*}$. In fact, suppose by contradiction that $\bar{T}=0$. Applying Corollary 50 to the identity

$$
\sum_{i=2}^{\infty}\left(\delta_{\bar{p}_{i}}-\delta_{\bar{n}_{i}}\right)=\delta_{\bar{n}_{1}}-\delta_{\bar{p}_{1}} \quad \text { in }[\operatorname{Lip}(X / A)]^{*},
$$

we can find a path $\Gamma$ starting at $\bar{p}_{1}$. Since $\bar{p}_{1} \neq \bar{n}_{j}$ for every $j \geq 1$, $\Gamma$ has no smallest index. In particular, the path $\left(\lambda \leq \lambda_{0}\right) \subset \Gamma$ contains infinitely many
indices $\forall \lambda_{0} \in \Gamma$. Choosing $\lambda_{0}$ appropriately, we can assume that

$$
\begin{equation*}
\ell_{\left[\lambda \leq \lambda_{0}\right]}=\sum_{i \in\left(\lambda \leq \lambda_{0}\right)} d\left(\bar{p}_{i}, \bar{n}_{i}\right)<\frac{r}{2} . \tag{116}
\end{equation*}
$$

Therefore, after replacing $\Gamma$ by $\left(\lambda \leq \lambda_{0}\right)$ if necessary, we may assume that $\ell_{[\Gamma]}<\frac{r}{2}$. In particular, $\bar{p}_{i}, \bar{n}_{i} \in B_{r / 2}\left(\bar{p}_{1}\right)$ for every $i \in \Gamma$. Since the restriction of the quotient map $\pi: B_{r / 2}\left(p_{1}\right) \rightarrow B_{r / 2}\left(\bar{p}_{1}\right)$ is an isometry, $\Gamma$ induces a path in $X$ starting at $p_{1}$ in the family of dipoles $\delta_{p_{i}}-\delta_{n_{i}}$. But this contradicts the fact that the representation of $T$ is irreducible. We conclude that $\bar{T} \neq 0$ in $[\operatorname{Lip}(X / A)]^{*}$.

Let $\bar{L}>0$ be the length of the minimal connection of $\bar{T}$. By Proposition 18 there exist $k \geq 1, \sigma \in S_{k}$ and $\tilde{\zeta} \in \operatorname{Lip}(X / A),|\tilde{\zeta}|_{\text {Lip }} \leq 1$, such that

$$
\begin{equation*}
\frac{2 \bar{L}}{3} \leq \sum_{i=1}^{k} d\left(\bar{p}_{i}, \bar{n}_{\sigma(i)}\right)=\sum_{i=1}^{k}\left[\tilde{\zeta}\left(\bar{p}_{i}\right)-\tilde{\zeta}\left(\bar{n}_{i}\right)\right] \leq \frac{4 \bar{L}}{3} . \tag{117}
\end{equation*}
$$

By taking $k$ large enough, we may also assume that

$$
\begin{equation*}
\sum_{i \geq k+1} d\left(\bar{p}_{i}, \bar{n}_{i}\right) \leq \frac{\bar{L}}{3} \tag{118}
\end{equation*}
$$

For the sake of normalization we set $\tilde{\zeta}(A)=0$. We claim that $\tilde{\zeta}$ can be chosen so that

$$
\begin{equation*}
\|\tilde{\zeta}\|_{\infty} \leq \frac{4 \bar{L}}{3} \tag{119}
\end{equation*}
$$

In fact, for each $i=1, \ldots, k$, we define the intervals

$$
J_{i}:=\left[\tilde{\zeta}\left(\bar{n}_{\sigma(i)}\right), \tilde{\zeta}\left(\bar{p}_{i}\right)\right] \subset \mathbb{R} .
$$

(note that $\tilde{\zeta}\left(\bar{p}_{i}\right) \geq \tilde{\zeta}\left(\bar{n}_{\sigma(i)}\right)$ by (117)).
Let $h: \mathbb{R} \rightarrow \mathbb{R}$ continuous such that $h(0)=0, h$ is constant outside $\bigcup_{i} J_{i}$ and $h$ is affine linear with slope 1 on each $J_{i}$. It is easy to see that $h \circ \tilde{\zeta}$ satisfies $|h \circ \tilde{\zeta}|_{\text {Lip }} \leq 1$ and (117). Moreover, since $\sum_{i}\left|J_{i}\right| \leq \frac{4}{3} \bar{L}$, we have $\|h \circ \tilde{\zeta}\|_{\infty} \leq \frac{4}{3} \bar{L}$. This proves our claim.

We now let $\bar{\zeta}:=\frac{3}{4 \bar{L}} \tilde{\zeta}$. Then $\|\bar{\zeta}\|_{\infty} \leq 1$ and

$$
\begin{equation*}
\langle\bar{T}, \bar{\zeta}\rangle=\sum_{i=1}^{k}\left[\bar{\zeta}\left(\bar{p}_{i}\right)-\bar{\zeta}\left(\bar{n}_{i}\right)\right]+\sum_{i \geq k+1}\left[\bar{\zeta}\left(\bar{p}_{i}\right)-\bar{\zeta}\left(\bar{n}_{i}\right)\right] \geq \frac{1}{2}-\frac{1}{4}=\frac{1}{4} . \tag{120}
\end{equation*}
$$

The lemma now follows by taking the pull-back of $\bar{\zeta}$, namely $\zeta:=\bar{\zeta} \circ \pi$.

Remark 58 An inspection of the proof shows that one can construct $\zeta$ so that (113) holds with $1 / 4$ replaced by any number $\theta \in(0,1)$.

## $9 \quad$ Proof of Theorems 11 and 13

Theorems 11 and 13 can now be immediately derived from Lemma 57:

Proof of Theorem 11. As we have already pointed out, we always have $\operatorname{supp} T \subset \overline{\bigcup_{i}\left\{\hat{p}_{i}\right\} \cup \bigcup_{i}\left\{\hat{n}_{i}\right\}}$, even if the representation is not irreducible. To prove the reverse inclusion, let $B \subset X$ be an open set containing some $\hat{p}_{i_{0}}$ or some $\hat{n}_{i_{0}}$. Using the previous lemma we can find $\zeta \in \operatorname{Lip}(X)$ such that $\langle T, \zeta\rangle>0$ and $\operatorname{supp} \zeta \subset B$. In other words, $B \cap \operatorname{supp} T \neq \phi$.

Proof of Theorem 13. Assume the irreducible representation of $T$ has infinitely many terms. We shall show that there is no $C>0$ for which (21) is true.

Without loss of generality, we may assume there are infinitely many distinct $p_{i}$ 's, say $\tilde{p}_{1}, \tilde{p}_{2}, \ldots$. Given $M>0$, let $\delta>0$ be such that the balls $B_{\delta}\left(\tilde{p}_{i}\right)$ are disjoint for $i=1, \ldots, M$. Applying the lemma above to these balls, then for each $i_{0}$ we get a bounded Lipschitz function $\zeta_{i_{0}}$ satisfying (113). The function $\zeta:=\sum_{i=1}^{M} \zeta_{i}$ satisfies

$$
\|\zeta\|_{\infty} \leq 1 \quad \text { and } \quad\langle T, \zeta\rangle \geq \frac{M}{4}
$$

Since $M$ can be chosen arbitrarily large, the theorem follows.

## 10 Some comments about Definition 4

In this section we present some properties related to regular and singular functionals in $\mathcal{Z}$ (in the sense of Definition 4). At the end, we shall prove that every $T \in \mathcal{Z}$ can be decomposed in terms of a regular and singular part.

We first show that Definition 4 is intrinsic in the sense that it does not depend on the ambient space $X$. More precisely, we have

Proposition 59 Let $T \in \mathcal{Z}$. Then
(a) $T$ is regular in $X$ iff $T$ is regular in $\operatorname{supp} T$;
(b) $T$ is singular in $X$ iff $T$ is singular in $\operatorname{supp} T$.

In particular, the minimization problem (10) has a solution in $X$ if, and only if, it has a solution in $\operatorname{supp} T$.

## PROOF.

Step 1. Proof of (a).
Assume $T$ is regular in $\operatorname{supp} T$. By definition, there exist $\left(p_{i}\right),\left(n_{i}\right) \subset \operatorname{supp} T$ such that

$$
\begin{equation*}
\|T\|=\sum_{i} d\left(p_{i}, n_{i}\right) \quad \text { and } \quad T=\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \text { in }[\operatorname{Lip}(\operatorname{supp} X)]^{*} . \tag{121}
\end{equation*}
$$

Since the number $\|T\|$ is the same, whether we compute it using $X$ or $\operatorname{supp} T$ as the ambient space, we conclude that $T$ is regular in $X$.

Suppose now that $T$ is regular in $X$. Then we can find sequences $\left(p_{i}\right),\left(n_{i}\right) \subset X$ such that

$$
\begin{equation*}
\|T\|=\sum_{i} d\left(p_{i}, n_{i}\right) \quad \text { and } \quad T=\sum_{i}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \text { in }[\operatorname{Lip}(X)]^{*} . \tag{122}
\end{equation*}
$$

It follows from Remark 56 that we can construct an irreducible representation of $T$ :

$$
\begin{equation*}
T=\sum_{j}\left(\delta_{\hat{p}_{j}}-\delta_{\hat{n}_{j}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}, \tag{123}
\end{equation*}
$$

so that

$$
\begin{equation*}
\|T\| \leq \sum_{i} d\left(\hat{p}_{i}, \hat{n}_{i}\right) \leq \sum_{i} d\left(p_{i}, n_{i}\right)=\|T\| \tag{124}
\end{equation*}
$$

Since $\hat{p}_{j}, \hat{n}_{j} \in \operatorname{supp} T$ for every $j$ (see Theorem 11), we conclude that $T$ is regular in $\operatorname{supp} T$.

Step 2. Proof of (b).
Assume $T$ is not singular in supp $T$. Then we can find $T_{1}, T_{2} \in \mathcal{Z}(\operatorname{supp} T)$ such that $T=T_{1}+T_{2},\|T\|=\left\|T_{1}\right\|+\left\|T_{2}\right\|$ and $T_{1} \neq 0$ is regular in $\operatorname{supp} T$. By (a), $T_{1}$ is also regular in $X$. We conclude that $T$ is not singular in $X$.

The converse statement is a trivial consequence of the following lemma:
Lemma 60 If $T \in \mathcal{Z}$ is not singular in $X$, then there exist $r, q \in \operatorname{supp} T$, $r \neq q$, such that

$$
\begin{equation*}
\|T\|=d(r, q)+\left\|T-\left(\delta_{r}-\delta_{q}\right)\right\| . \tag{125}
\end{equation*}
$$

## PROOF.

Step 1. (125) holds for some $r, q \in X, r \neq q$.
Let $T_{1}=\sum_{i}\left(\delta_{r_{i}}-\delta_{q_{i}}\right) \in \mathcal{Z}$ be regular and nonzero such that

$$
\begin{equation*}
\|T\|=\left\|T_{1}\right\|+\left\|T-T_{1}\right\|=\sum_{i} d\left(r_{i}, q_{i}\right)+\left\|T-T_{1}\right\| . \tag{126}
\end{equation*}
$$

Without loss of generality, we may assume that $r_{1} \neq q_{1}$. Then applying the triangle inequality we have

$$
\begin{align*}
\|T\| & \leq\left\|\left(\delta_{r_{1}}-\delta_{q_{1}}\right)\right\|+\left\|T-\left(\delta_{r_{1}}-\delta_{q_{1}}\right)\right\| \\
& \leq\left\|\left(\delta_{r_{1}}-\delta_{q_{1}}\right)\right\|+\left\|\sum_{i \neq 1}\left(\delta_{r_{i}}-\delta_{q_{i}}\right)\right\|+\left\|T-\sum_{i}\left(\delta_{r_{i}}-\delta_{q_{i}}\right)\right\|  \tag{127}\\
& \leq \sum_{i} d\left(r_{i}, q_{i}\right)+\left\|T-T_{1}\right\|=\|T\| .
\end{align*}
$$

Therefore, equality must hold everywhere. Since $d\left(r_{1}, q_{1}\right)=\left\|\left(\delta_{r_{1}}-\delta_{q_{1}}\right)\right\|$, we conclude that (125) holds with $r:=r_{1}$ and $q:=q_{1}$.

Step 2. Proof of the lemma completed.
Let $r=: n_{0}$ and $q=: p_{0}$ be two distinct points in $X$ for which (125) holds, and let $T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right)$ in $[\operatorname{Lip}(X)]^{*}$ be an irreducible representation of $T$.

Applying Proposition 31 to

$$
\tilde{T}:=T-\left(\delta_{n_{0}}-\delta_{p_{0}}\right)=\sum_{i=0}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*},
$$

we can decompose $\tilde{T}$ in terms of disjoint cycles:

$$
\tilde{T}=\sum_{j} \tilde{T}_{\left[\Lambda_{j}\right]} \quad \text { in }[\operatorname{Lip}(X)]^{*}
$$

such that

$$
\begin{equation*}
\|\tilde{T}\|=\sum_{j} \ell_{\left[\Lambda_{j}\right]}^{*} . \tag{128}
\end{equation*}
$$

Without loss of generality, we may assume that $0 \in \Lambda_{1}$.
We claim that $\Lambda_{1} \backslash\{0\}$ is nonempty. In fact, assume by contradiction that $\Lambda_{1}=(0)$. Then we would have

$$
\begin{equation*}
\|T\|=d\left(p_{0}, n_{0}\right)+\|\tilde{T}\|=2 d\left(p_{0}, n_{0}\right)+\sum_{j \neq 1} \ell_{\left[\Lambda_{j}\right]}^{*} \geq 2 d\left(p_{0}, n_{0}\right)+\|T\| . \tag{129}
\end{equation*}
$$

(In the last step we use that $T=\sum_{j \neq 1} T_{\left[\Lambda_{j}\right]}$, and so

$$
\left.\|T\| \leq \sum_{j \neq 1}\left\|T_{\left[\Lambda_{j}\right]}\right\| \leq \sum_{j \neq 1} \ell_{\left[\Lambda_{j}\right]}^{*}\right) .
$$

Therefore we must have $p_{0}=n_{0}$, which is a contradiction.
By taking another representative of $\left[\Lambda_{1}\right]$ if necessary, we may assume that 0 is the largest element in $\Lambda_{1}$. Let

$$
\begin{equation*}
\tilde{r}:=\lim _{\lambda \in \Lambda_{1} \backslash\{0\} \uparrow} p_{\lambda} \quad \text { and } \quad \tilde{q}:=\lim _{\lambda \in \Lambda_{1} \backslash\{0\} \downarrow} n_{\lambda} . \tag{130}
\end{equation*}
$$

Since $p_{\lambda}, n_{\lambda} \in \operatorname{supp} T$ for every $\lambda \neq 0$, we have $\tilde{r}, \tilde{q} \in \operatorname{supp} T$.
We claim that (125) holds with $\tilde{r}$ and $\tilde{q}$.
By a slight abuse of notation, let us denote by $\left[\Lambda_{1}\right]_{\text {new }}$ the cycle $\left[\Lambda_{1}\right]$ where $\delta_{p_{0}}-\delta_{n_{0}}$ is replaced by the dipole $\delta_{\tilde{q}}-\delta_{\tilde{r}}$. It is easy to see that (see e.g. Remark 35)

$$
\begin{equation*}
\ell_{\left[\Lambda_{1}\right]}^{*}=\ell_{\left[\Lambda_{1}\right]_{\text {new }}}^{*}+d\left(\tilde{r}, n_{0}\right)+d\left(p_{0}, \tilde{q}\right) . \tag{131}
\end{equation*}
$$

We then have

$$
\begin{align*}
\|T\| & =d\left(p_{0}, n_{0}\right)+\|\tilde{T}\| \\
& =d\left(p_{0}, n_{0}\right)+d\left(\tilde{r}, n_{0}\right)+d\left(p_{0}, \tilde{q}\right)+\ell_{\left[\Lambda_{1}\right]_{\mathrm{new}}}^{*}+\sum_{j \neq 1} \ell_{\left[\Lambda_{j}\right]}^{*}  \tag{132}\\
& \geq d(\tilde{r}, \tilde{q})+\left\|T+\left(\delta_{\tilde{q}}-\delta_{\tilde{r}}\right)\right\| .
\end{align*}
$$

Thus,

$$
\begin{equation*}
\|T\| \geq d(\tilde{r}, \tilde{q})+\left\|T-\left(\delta_{\tilde{r}}-\delta_{\tilde{q}}\right)\right\| \geq\|T\| . \tag{133}
\end{equation*}
$$

This establishes the lemma.

A simple consequence of Lemma 60 is the following: assume we are given some $T \in \mathcal{Z}$ and we want to show that $T$ is singular; then it suffices to show that if $r, q \in \operatorname{supp} T$ satisfies (125), then $r=q$. We have already used this fact in Section 6.

We now state some properties related to Definition 4:
Remark 61 Assume $T=T_{1}+T_{2}$, where $T_{1}, T_{2} \in \mathcal{Z}$ and $\|T\|=\left\|T_{1}\right\|+\left\|T_{2}\right\|$. Then we have:
(a) if $T_{1}$ and $T_{2}$ are both regular, then so is $T$;
(b) if $T$ is singular, then $T_{1}$ and $T_{2}$ are singular as well.
(a) and (b) follow immediately from Definition 4 (in fact, they still hold in the case of infinite sums). Note however that


Fig. 6. Dipoles in Remark 61 (d)
(c) if we know that $T$ and $T_{1}$ are regular, then we cannot conclude that $T_{2}$ is regular; take for instance

$$
T_{1}:=\sum_{i \geq 1}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { and } \quad T_{2}:=-\sum_{i \geq 0}\left(\delta_{p_{i}}-\delta_{n_{i}}\right),
$$

where $p_{i}$ and $n_{i}$ are given by Example 5; then $T_{1}$ and $T=\delta_{1}-\delta_{0}$ are regular, but $T_{2}$ is singular; note also that

$$
\operatorname{supp} T=\{0,1\} \varsubsetneqq C_{\alpha}=\operatorname{supp} T_{1} \cup \operatorname{supp} T_{2}
$$

(d) it is possible to construct $T_{1}, T_{2} \in \mathcal{Z}$, both singular, such that $T=T_{1}+T_{2}$ is regular. Let $X=S^{1}$ equipped with its geodesic distance. We consider two sequences in $S^{1}$ (see Figure 6):

$$
\begin{array}{llll}
r_{k}:=\mathrm{e}^{\pi p_{k} i} & \text { and } & q_{k}:=\mathrm{e}^{\pi n_{k} i} & \forall k>0 \\
r_{k}:=\mathrm{e}^{\pi\left(p_{k}+1\right) i} & \text { and } \quad q_{k}:=\mathrm{e}^{\pi\left(n_{k}+1\right) i} & \forall k<0,
\end{array}
$$

where $p_{k}$ and $n_{k}$ belong to the Cantor set $C_{\alpha}$ as before. Then

$$
\begin{aligned}
& T_{1}:=\sum_{k \geq 1}\left(\delta_{r_{k}}-\delta_{q_{k}}\right)+\left(\delta_{(1,0)}-\delta_{(-1,0)}\right), \\
& T_{2}:=\sum_{k \leq-1}\left(\delta_{r_{k}}-\delta_{q_{k}}\right)+\left(\delta_{(-1,0)}-\delta_{(1,0)}\right)
\end{aligned}
$$

are both singular for every $\alpha \in(0,1)$ and $\left\|T_{1}\right\|=\left\|T_{2}\right\|=\alpha \pi$.
We now take $\alpha=\frac{1}{2}$. Then (see e.g. Example 55)

$$
T=T_{1}+T_{2} \quad \text { is regular } \quad \text { and } \quad\|T\|=\pi=\left\|T_{1}\right\|+\left\|T_{2}\right\| .
$$

The proposition below gives the first part of Theorem 6:
Proposition 62 Given $T \in \mathcal{Z}$, there exist $T_{\text {reg }}, T_{\text {sing }} \in \mathcal{Z}$ such that $T_{\text {reg }}$ is regular, $T_{\text {sing }}$ is singular,

$$
\begin{equation*}
T=T_{\text {reg }}+T_{\text {sing }} \quad \text { and } \quad\|T\|=\left\|T_{\text {reg }}\right\|+\left\|T_{\text {sing }}\right\| . \tag{134}
\end{equation*}
$$

Moreover, $T_{\text {reg }}$ and $T_{\text {sing }}$ can be chosen so that

$$
\begin{equation*}
\operatorname{supp} T=\operatorname{supp} T_{\text {reg }} \cup \operatorname{supp} T_{\text {sing }} \tag{135}
\end{equation*}
$$

PROOF. In view of Proposition 59, it suffices to prove the result for $X=$ $\operatorname{supp} T$, in which case (135) is automatically satisfied.

We proceed by transfinite induction.
Let $T_{0}:=T$ and $T_{0,1}:=0$.
Let $\alpha \geq 1$ be a nonlimit countable ordinal. If $T_{\alpha-1}$ is not singular, then we can find $T_{\alpha}, T_{\alpha, 1} \in \mathcal{Z}$ such that $T_{\alpha, 1}$ is regular, $T_{\alpha, 1} \neq 0$ in $[\operatorname{Lip}(X)]^{*}$,

$$
\begin{equation*}
T_{\alpha-1}=T_{\alpha, 1}+T_{\alpha} \quad \text { and } \quad\left\|T_{\alpha-1}\right\|=\left\|T_{\alpha, 1}\right\|+\left\|T_{\alpha}\right\| \tag{136}
\end{equation*}
$$

Assume for instance that $\alpha=k \in \mathbb{N}$. Summing (136) over $\alpha$ replaced by $j$ we get

$$
\begin{equation*}
T=\sum_{j=1}^{k} T_{j, 1}+T_{k} \quad \text { and } \quad\|T\|=\sum_{j=1}^{k}\left\|T_{j, 1}\right\|+\left\|T_{k}\right\| . \tag{137}
\end{equation*}
$$

If $\alpha$ is a limit countable ordinal, then we take

$$
\begin{equation*}
T_{\alpha}:=T-\sum_{\beta<\alpha} T_{\beta, 1} \quad \text { and } \quad T_{\alpha, 1}=0 . \tag{138}
\end{equation*}
$$

By construction, for every $\alpha$ we have

$$
\begin{equation*}
T=\sum_{\beta \leq \alpha} T_{\beta, 1}+T_{\alpha} \quad \text { and } \quad\|T\|=\sum_{\beta \leq \alpha}\left\|T_{\beta, 1}\right\|+\left\|T_{\alpha}\right\| . \tag{139}
\end{equation*}
$$

In particular, (see Remark 61 (a))

$$
\begin{equation*}
\sum_{\beta \leq \alpha} T_{\beta, 1} \quad \text { is regular. } \tag{140}
\end{equation*}
$$

On the other hand, note that if $T_{\alpha}$ is not singular, then we have the strict inequality $\left\|T_{\alpha}\right\|>\left\|T_{\alpha+1}\right\|$. In other words, the family $\left(\left\|T_{\alpha}\right\|\right)_{\alpha}$ is strictly decreasing, so it can only have countably many terms. Therefore, our construction has to stop at some countable ordinal $\alpha_{0}$, which means that $T_{\alpha_{0}}$ is singular. Thus,

$$
\begin{equation*}
T=T_{\mathrm{reg}}+T_{\text {sing }}, \quad \text { where } \quad T_{\mathrm{reg}}:=\sum_{\beta \leq \alpha_{0}} T_{\beta, 1} \quad \text { and } \quad T_{\text {sing }}:=T_{\alpha_{0}} . \tag{141}
\end{equation*}
$$



Fig. 7. The cycle [ $\Lambda$ ] in Example 63
We now show that the decomposition of $T$ in terms of a regular and a singular part need not be unique.

Example 63 Let $X=S^{1}$ equipped with its geodesic distance as before. We consider two sequences $\left(r_{k}\right)_{k \in \mathbb{Z}},\left(q_{k}\right)_{k \in \mathbb{Z}} \subset S^{1}$ given by (see Figure 7)

$$
\begin{array}{llll}
r_{k}:=\mathrm{e}^{\pi p_{k} i} & \text { and } & q_{k}:=\mathrm{e}^{\pi n_{k} i} & \forall k \geq 0, \\
r_{k}:=\mathrm{e}^{-\pi p_{k} i} & \text { and } & q_{k}:=\mathrm{e}^{-\pi n_{k} i} & \forall k<0 .
\end{array}
$$

Then

$$
T:=\sum_{k=-\infty}^{\infty}\left(\delta_{r_{k}}-\delta_{q_{k}}\right) \text { is irreducible and }\|T\|=\pi .
$$

Moreover,

$$
T=\sum_{k<0}\left(\delta_{r_{k}}-\delta_{q_{k}}\right)+\sum_{k \geq 0}\left(\delta_{r_{k}}-\delta_{q_{k}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}
$$

is a decomposition of $T$ in terms of a regular and a singular part. By symmetry, we also have a second decomposition, namely $T=\sum_{k>0}+\sum_{k \leq 0}$.

## 11 Proof of Theorem 6 completed

In view of Proposition 62, we are left to show that (13) holds, where each $\operatorname{supp} T_{i}$ is homeomorphic to the Cantor set in $\mathbb{R}$.

Without loss of generality, we may assume that $T$ is singular. Let

$$
\begin{equation*}
T=\sum_{i=1}^{\infty}\left(\delta_{p_{i}}-\delta_{n_{i}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{142}
\end{equation*}
$$

be an irreducible representation of $T$.

Applying Proposition 31, we can find a sequence of disjoint co-minimal cycles ([ $\left.\Lambda_{j}\right]$ ) such that

$$
\begin{equation*}
T=\sum_{j} T_{\left[\Lambda_{j}\right]} \quad \text { and } \quad\|T\|=\sum_{j}\left\|T_{\left[\Lambda_{j}\right]}\right\| . \tag{143}
\end{equation*}
$$

Since $T$ is singular, so is $T_{\left[\Lambda_{j}\right]}$ for each $j$. Moreover, Proposition 44 implies that we can further split each $\left[\Lambda_{j}\right]$ in terms of simple cycles so that (92) holds. Therefore, we can assume that each $\left[\Lambda_{j}\right]$ is a simple cycle.

Since the representation

$$
\begin{equation*}
T_{j}:=T_{\left[\Lambda_{j}\right]}=\sum_{\lambda \in \Lambda_{j}}\left(\delta_{p_{\lambda}}-\delta_{n_{\lambda}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{144}
\end{equation*}
$$

is also irreducible, we have $S_{\left[\Lambda_{j}\right]}=\operatorname{supp} T_{j}$. In particular, we conclude from Proposition 43 that

$$
\begin{equation*}
\left\|T_{j}\right\|=\mathcal{H}^{1}\left(\operatorname{supp} T_{j}\right) \tag{145}
\end{equation*}
$$

Assertion (13) is an immediate consequence of (143)-(145). Note also that

$$
\begin{equation*}
\operatorname{supp} T=\overline{\bigcup_{j} \operatorname{supp} T_{j}} \tag{146}
\end{equation*}
$$

The the last part of the theorem follows from the proposition below:
Proposition 64 Assume that $[\Lambda]$ is a simple cycle and

$$
\begin{equation*}
T_{[\Lambda]}=\sum_{\lambda \in \Lambda}\left(\delta_{p_{\lambda}}-\delta_{n_{\lambda}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*} \tag{147}
\end{equation*}
$$

is an irreducible representation of $T_{[\Lambda]}$. Then $\operatorname{supp} T_{[\Lambda]}$ is homeomorphic to the Cantor set in $\mathbb{R}$.

PROOF. Since $\Lambda$ is infinite, we can assume that $\Lambda=\mathbb{N} \cup\{0\}$ and 0 is its largest element. The fact that the representation of $T_{[\Lambda]}$ is irreducible and gap $[\Lambda]=0$ implies that $\Lambda$ cannot have a smallest element. We now take $\Lambda_{1}:=\Lambda \backslash\{0\}$.

Let $C \subset[0,1]$ be the standard Cantor set in $\mathbb{R}$. We denote by $\left(J_{k}\right)_{k \geq 1}, J_{k}=$ $\left(a_{k}, b_{k}\right)$, the sequence of disjoint open intervals which are removed from $[0,1]$ in the construction of $C$. We define $\Omega:=\mathbb{N}$ to be an ordered set so that $k_{1} \leq k_{2}$ iff $a_{k_{1}} \leq a_{k_{2}}$.

We claim there exists a bijection $\sigma: \Omega \rightarrow \Lambda_{1}$ which preserves the order of $\Omega$, i.e. if $k_{1} \leq k_{2}$ in $\Omega$, then $\sigma\left(k_{1}\right) \leq \sigma\left(k_{2}\right)$ in $\Lambda_{1}$.

In fact, let $\sigma(1):=1$. We next define

$$
\begin{aligned}
& \sigma(2):=\text { smallest integer in }\left\{\lambda \in \Lambda_{1}: \lambda<1\right\}, \\
& \sigma(3):=\text { smallest integer in }\left\{\lambda \in \Lambda_{1}: \lambda>1\right\} .
\end{aligned}
$$

Note that $\sigma(2)<\sigma(1)<\sigma(3)$. Moreover, we can keep this construction indefinitely since each of the sets of the form

$$
\left\{\lambda \in \Lambda_{1}: \lambda<\lambda_{0}\right\}, \quad\left\{\lambda \in \Lambda_{1}: \lambda>\lambda_{0}\right\} \quad \text { and } \quad\left\{\lambda \in \Lambda_{1}: \lambda_{0}<\lambda<\lambda_{1}\right\}
$$

has no smallest nor largest element. This proves our claim.
We define the map

$$
\begin{align*}
h: \bigcup_{k}\left\{a_{k}\right\} \cup\left\{b_{k}\right\} & \longrightarrow \operatorname{supp} T_{[\Lambda]} \\
a_{k} & \longmapsto n_{\sigma(k)}  \tag{148}\\
b_{k} & \longmapsto p_{\sigma(k)}
\end{align*} .
$$

Note that $h$ is uniformly continuous (since $\ell_{[\Lambda]}+\ell_{[\Lambda]}^{*}<\infty$ ), and so it can be extended by continuity as a map $h: C \rightarrow \operatorname{supp} T_{[\Lambda]}$. It is easy to see that $h$ is surjective.

We claim that $h$ is injective. Suppose by contradiction $h$ is not injective. We can find $c<d$ in $C$ such that $h(c)=h(d)$. Let $\Omega_{1}:=\left\{k: J_{k} \subset(c, d)\right\}$. Then $\sigma\left(\Omega_{1}\right)$ is a segment of $\Lambda_{1} \varsubsetneqq \Lambda$ such that gap $\sigma\left(\Omega_{1}\right)=0$. In other words, $\left[\sigma\left(\Omega_{1}\right)\right]$ is a closed cycle, which is a contradiction.

We conclude that $h$ is a continuous bijection between $C$ and $\operatorname{supp} T_{[\Lambda]}$. Since $C$ is compact, $h$ is a homeomorphism.

## Appendix. Compact subsets of $\mathcal{Z}$

We start with the following (see [2])
Proposition $65 \mathcal{Z}$ is a complete metric space.

PROOF. It suffices to show that if the series $T:=\sum_{k} T_{k}$ converges absolutely and $T_{k} \in \mathcal{Z}$ for each $k \geq 1$, then $T \in \mathcal{Z}$.

For $k \geq 1$ fixed, it follows from Proposition 18 that we can find sequences $\left(p_{i}^{k}\right)_{i}$ and $\left(n_{i}^{k}\right)_{i}$ in $X$ such that

$$
\begin{gathered}
T_{k}=\sum_{i}\left(\delta_{p_{i}^{k}}-\delta_{n_{i}^{k}}\right) \quad \text { in }[\operatorname{Lip}(X)]^{*}, \\
\sum_{i} d\left(p_{i}^{k}, n_{i}^{k}\right) \leq\left\|T_{k}\right\|+\frac{1}{2^{k}} .
\end{gathered}
$$

Thus,

$$
\sum_{k} \sum_{i} d\left(p_{i}^{k}, n_{i}^{k}\right) \leq \sum_{k}\left\|T_{k}\right\|+1<\infty,
$$

from which we conclude that

$$
T=\sum_{k} \sum_{i}\left(\delta_{p_{i}^{k}}-\delta_{n_{i}^{k}}\right) \in \mathcal{Z}
$$

In order to describe the compact subsets of $\mathcal{Z}$, we first introduce a definition:
Definition $66 \mathcal{A} \subset \mathcal{Z}$ is equisummable if, and only if, $\mathcal{A}$ is bounded and, for each $\varepsilon>0$, there exist $k_{\varepsilon} \geq 1$ and $K_{\varepsilon} \subset X$ compact such that the following holds: for every $T \in \mathcal{A}$ we can find $T_{1}, T_{2} \in \mathcal{Z}, T=T_{1}+T_{2}$ in $\mathcal{Z}$, where
(i) $T_{1}$ can be written as a sum of at most $k_{\varepsilon}$ dipoles supported in $K_{\varepsilon}$;
(ii) $\left\|T_{2}\right\|<\varepsilon$.

Note that this definition is satisfied if $\mathcal{A}$ is finite. More generally, we have the following

Theorem $67 \mathcal{A} \subset \mathcal{Z}$ is relatively compact if, and only if, $\mathcal{A}$ is equisummable.

PROOF. Assume $\mathcal{A}$ is relatively compact in $\mathcal{Z}$, and let $\left(T_{k}\right)$ in $\mathcal{A}$ be such that $T_{k} \rightarrow T \in \mathcal{Z}$. It suffices to show that $\left(T_{k}\right)$ is equisummable. Without loss of generality, we may assume that $T=0$; in other words, $\left\|T_{k}\right\| \rightarrow 0$. Given $\varepsilon>0$, let $k_{0} \geq 1$ be such that $\left\|T_{k}\right\|<\varepsilon$ for every $k \geq k_{0}$. On the other hand, Definition 66 clearly holds for the finite set $\left\{T_{1}, \ldots, T_{k_{0}-1}\right\}$. We conclude that $\left(T_{k}\right)$ is equisummable.

We now prove the converse statement. By assumption, given $\varepsilon>0$ there exist $k_{\varepsilon} \geq 1$ and a compact set $K_{\varepsilon} \subset X$ such that for each $T \in \mathcal{A}$ we can write $T=T_{1}+T_{2}$ in $\mathcal{Z}$, where

$$
\begin{equation*}
T_{1}=\sum_{i=1}^{k_{\varepsilon}}\left(\delta_{p_{i}}-\delta_{n_{i}}\right), \quad p_{i}, n_{i} \in K_{\varepsilon} \quad \text { and } \quad\left\|T_{2}\right\| \leq \varepsilon \tag{149}
\end{equation*}
$$

Since $\mathcal{A}$ is bounded and $K_{\varepsilon}$ is compact, $\left\{T_{1}\right\}_{T \in \mathcal{A}}$ is relatively compact in $\mathcal{Z}$. In particular, there exists a finite number of balls $B_{\varepsilon}\left(S_{1}\right), \ldots, B_{\varepsilon}\left(S_{n}\right)$ in $\mathcal{Z}$ which
cover $\left\{T_{1}\right\}_{T \in \mathcal{A}}$. By (149), the balls $B_{2 \varepsilon}\left(S_{1}\right), \ldots, B_{2 \varepsilon}\left(S_{n}\right)$ cover $\mathcal{A}$, which means that $\mathcal{A}$ is totally bounded. Since $X$ is complete, $\mathcal{A}$ is relatively compact.

In contrast with the previous result, the example below shows that $\mathcal{Z}$ is not closed in $[\operatorname{Lip}(X)]^{*}$ with respect to the weak* topology:

Example 68 Let $X=[0,1]$. For each $k \geq 1$ we define

$$
T_{k}:=\sum_{\substack{j=0 \\ j \text { even }}}^{2^{k}-2}\left(\delta_{\frac{j+1}{2^{k}}}-\delta_{\frac{j}{2^{k}}}\right) .
$$

It is easy to see that

$$
T_{k} \stackrel{*}{\rightharpoonup} \frac{1}{2}\left(\delta_{1}-\delta_{0}\right) \notin \mathcal{Z} .
$$

Recall that, in general, $\operatorname{Lip}(X)$ is not separable, which implies that the unit ball in $[\operatorname{Lip}(X)]^{*}$ is not metrizable with respect to the weak* topology. The example below shows that bounded sequences in $\mathcal{Z}$ do not necessarily converge in the weak* topology of $[\operatorname{Lip}(X)]^{*}$.

Example 69 We take $X=[0,1] \subset \mathbb{R}$. Let

$$
T_{k}=k\left(\delta_{1 / k}-\delta_{0}\right)
$$

If $\zeta \in \operatorname{Lip}(X)$ has a derivative at 0 , then we have

$$
\begin{equation*}
\left\langle T_{k}, \zeta\right\rangle=\frac{\zeta(1 / k)-\zeta(0)}{1 / k} \rightarrow \zeta^{\prime}(0) \tag{150}
\end{equation*}
$$

In particular, $\left(T_{k}\right)$ has no subsequence converging in $[\operatorname{Lip}(X)]^{*}$.
However, because $\left\|T_{k}\right\|=1$, we can find a subnet $\left(T_{k_{\alpha}}\right)_{\alpha \in A}$ such that

$$
T_{k_{\alpha}} \stackrel{*}{\rightharpoonup} T
$$

for some $T \in \overline{\operatorname{conv}}(\mathcal{Z})$. Since supp $T=\{0\}$, it follows from Corollary 12 that $T \notin \mathcal{Z}$ (otherwise $T$ would be expressed in terms of finitely many dipoles, which clearly cannot be the case).

An alternative approach to show that $T \notin \mathcal{Z}$ (without making use of irreducible representations) is the following. Assume by contradiction that $T \in \mathcal{Z}$. Then, given $\varepsilon>0$, there exists $M_{\varepsilon}>0$ such that

$$
|\langle T, \zeta\rangle| \leq M_{\varepsilon}\|\zeta\|_{\infty}+\varepsilon|\zeta|_{\text {Lip }} \quad \forall \zeta \in \operatorname{Lip}(X) .
$$

This estimate implies that given any sequence $\left(\zeta_{j}\right)_{j \geq 1}$ in $\operatorname{Lip}(X),\left|\zeta_{j}\right|_{\text {Lip }} \leq 1$ $\forall j \geq 1$, such that $\zeta_{j} \rightarrow 0$ uniformly in $X$, then

$$
\lim _{j \rightarrow \infty}\left\langle T, \zeta_{j}\right\rangle=0,
$$

which contradicts the fact that $\langle T, \zeta\rangle=\zeta^{\prime}(0)$ for every $\zeta \in C^{1}[0,1]$.

## Acknowledgements

The author is deeply grateful to H . Brezis for his encouragement and also for very interesting discussions.

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[^0]:    ${ }^{1}$ Supported by CAPES, Brazil, under grant no. BEX1187/99-6.

