$W^{1,1}$-MAPS WITH VALUES INTO S^1

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Dedicated to François Treves with esteem and friendship

1. Introduction.

Let $G \subset \mathbb{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. In [BBM2] we studied properties of

$$H^{1/2}(\Omega; S^1) = \big\{ g \in H^{1/2}(\Omega; \mathbb{R}^2) \ ; \ |g| = 1 \text{ a.e. on } \Omega \big\}.$$

(In what follows, we identify $\mathbb{R}^2$ with $\mathbb{C}$.)
The space $W^{1,1} \cap L^\infty$ shares some properties with $H^{1/2}$ and it is natural to investigate

$$W^{1,1}(\Omega; S^1) = \big\{ g \in W^{1,1}(\Omega; \mathbb{R}^2) \ ; \ |g| = 1 \text{ a.e. on } \Omega \big\}.$$

One of the issues that we shall discuss is the question of existence of a lifting and, more precisely, "optimal" liftings. If $g \in W^{1,1}(\Omega; S^1) \cap C^0(\Omega; S^1)$, then g admits a "canonical" lifting $\varphi \in W^{1,1}(\Omega; \mathbb{R}) \cap C^0(\Omega; \mathbb{R})$ satisfying

$$\int_\Omega |\nabla \varphi| = \int_\Omega |\nabla g|. \tag{1.1}$$

(Since $g \in C^0$ and Ω is simply connected, there exists a $\varphi \in C^0$ such that $g = \mathrm{e}^{i\varphi}$ and (1.1) holds for this φ.) However, if one removes the continuity assumption, then a general $g \in W^{1,1}(\Omega; S^1)$ need not have a lifting φ in $W^{1,1}(\Omega; \mathbb{R})$. This obstruction phenomenon — which also holds for other Sobolev spaces — is due to topological singularities of g and has been extensively studied in [BBM1] ; see also earlier results of Schoen-Uhlenbeck [SU] and Bethuel [B2].

It has been established by Giaquinta-Modica-Souček [GMS2] that every map $g \in W^{1,1}(\Omega; S^1)$ admits a lifting in $BV(\Omega; \mathbb{R})$. However, as we shall see below, for some maps g in $W^{1,1}$ we may have

$$\mathrm{Min} \left\{ \int_\Omega |D\varphi| \ ; \ \varphi \in BV(\Omega; \mathbb{R}) \text{ and } g = \mathrm{e}^{i\varphi} \text{ a.e.} \right\} > \int_\Omega |\nabla g|,$$

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where the measure $D\varphi$ is the distributional derivative of φ.

As we shall prove (see Corollary 6 below), there is always a $\varphi \in BV(\Omega;\mathbb{R})$ such that $g = \mathrm{e}^{i\varphi}$ and

$$\int_\Omega |D\varphi| \leq 2 \int_\Omega |\nabla g|. \tag{1.2}$$

The constant 2 in (1.2) is optimal (see Remark 2 below). Inequality (1.2) has been extended by Dávila-Ignat [DI] to maps $g \in BV(\Omega; S^1)$ (here, Ω can be an arbitrary domain in $\mathbb{R}^N$) ; the striking fact is that (1.2), with constant 2, holds in any dimension.

It is natural to study, for a given $g \in W^{1,1}(\Omega; S^1)$, the quantity

$$E(g) = \operatorname{Min}\left\{ \int_\Omega |D\varphi| \ ; \ \varphi \in BV(\Omega;\mathbb{R}) \text{ and } g = \mathrm{e}^{i\varphi} \text{ a.e.} \right\}. \tag{1.3}$$

Another quantity which is commonly studied in the framework of Sobolev maps with values into manifolds (see [BBC], and also [GMS2]) is the relaxed energy

$$E_{\mathrm{rel}}(g) = \operatorname{Inf}\left\{ \liminf_{n\to\infty} \int |\nabla g_n| \ ; \ g_n \in C^\infty(\Omega; S^1) \text{ and } g_n \to g \text{ a.e.} \right\}. \tag{1.4}$$

It is not difficult to prove (see Proposition 2) that

$$E_{\mathrm{rel}}(g) = E(g), \quad \forall\, g \in W^{1,1}(\Omega; S^1).$$

As we shall establish in Section 3, the gap

$$E(g) - \int_\Omega |\nabla g| \tag{1.5}$$

can be easily computed in terms of the minimal connection $L(g)$ of the topological singularities of g. For example, if $g \in C^\infty(\Omega \setminus \{P, N\}; S^1) \cap W^{1,1}$, $\deg(g, P) = +1$ and $\deg(g, N) = -1$, then $L(g)$ is the geodesic distance in Ω between N and P, and the gap (1.5) equals $2\pi L(g)$. For the definition of $L(g)$ when g is an arbitrary element of $W^{1,1}(\Omega; S^1)$, see (1.9) below. The concept of a minimal connection connecting the topological singularities has its source in [BCL].

One of our main results is

Theorem 1. *Let $g \in W^{1,1}(\Omega; S^1)$. We have*

$$E(g) - \int_\Omega |\nabla g| = 2\pi L(g). \tag{1.6}$$

The first result of this kind (see [BBC]) concerned the Dirichlet integral $\int |\nabla g|^2$ and maps g from a 3-d domain into S^2. Inequality $\leq$ in (1.6) has been known for some time (see [DH] and [GMS2]) ; it relies on the dipole construction introduced in [BCL]. More generally, the [BCL] dipole construction has been adapted to a large variety of problems involving singularities (points and beyond) ; see e.g. [ABO]. The exact lower bound for the relaxed energy is always a more delicate issue. For $W^{1,2}(S^3; S^2)$ the corresponding lower bound obtained in [BBC] asserts that

$$E_{\mathrm{rel}}(g) \geq \int_{S^3} |\nabla g|^2 + 8\pi L(g).$$

The same argument applies to $W^{1,N}(S^{N+1}; S^N)$, $N \geq 3$, and yields

$$E_{\mathrm{rel}}(g) \geq \int_{S^{N+1}} |\nabla g|^N + c_N L(g), \quad c_N > 0.$$

The properties of L^p, $1 < p < \infty$, are heavily used in these arguments. However, the space L^1 is different and it is not possible to adapt the proof of [BBC] to obtain a lower bound of the form

$$E_{\mathrm{rel}}(g) \geq \int_\Omega |\nabla g| + \alpha L(g),$$

for some $\alpha > 0$. Such a lower bound can presumably be proved using the theory of Cartesian currents of [GMS2] ; however, the precise relationship between the formalism of [GMS2] and (1.6) is yet to be clarified.

We call the attention of the reader to the fact that, in the $H^{1/2}$-setting studied in [BBM2], the analog of Theorem 1 is open ; we only have

$$E_{\mathrm{rel}}(g) - |g|^2_{H^{1/2}} \sim L(g).$$

A useful quantity which plays a central role in our analysis is $g \wedge \nabla g$. More precisely, given $g \in W^{1,1}(\Omega; \mathbb{R}^2)$, consider the vector field $g \wedge \nabla g$ defined in a local frame by

$$g \wedge \nabla g = (g \wedge g_x, g \wedge g_y).$$

[This is the 2-d analog of the vector field D associated to $H^1(B^3; S^2)$ maps, originally introduced in [BCL] ; there is a natural analog of D in the $W^{1,N}(S^{N+1}; S^N)$ context, for each N.]
When g is smooth with values into S^1, $g \wedge \nabla g$ is a gradient map since we may always write $g = \mathrm{e}^{i\varphi}$, so that $g \wedge \nabla g = \nabla\varphi$. However, if $g \in W^{1,1}(\Omega; S^1)$, then $g \wedge \nabla g$ is an L^1-vector field which need not be a gradient map, e.g., when $g(x) \sim (x-a)/|x-a|$ near a point $a \in \Omega$, then $g \wedge \nabla g$ is not a gradient map since

$$(g \wedge g_x)_y \neq (g \wedge g_y)_x \quad \text{in } \mathcal{D}'(\Omega).$$

The following result gives an interpretation of $L(g)$ as the "L^1-distance" of $g \wedge \nabla g$ to the class of gradient maps :

Theorem 2. *For every $g \in W^{1,1}(\Omega; S^1)$, we have*

$$L(g) = \frac{1}{2\pi} \operatorname*{Inf}_{\psi \in C^\infty(\Omega;\mathbb{R})} \int_\Omega |g \wedge \nabla g - \nabla \psi| = \frac{1}{2\pi} \operatorname*{Min}_{\psi \in BV(\Omega;\mathbb{R})} \int_\Omega |g \wedge \nabla g - D\psi|. \tag{1.7}$$

There are many minimizers ψ in (1.7) ; however, at least one of them satisfies $g = \mathrm{e}^{i\psi}$ a.e. in Ω.

Let $g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$. Following the ideas of [BCL] (or, more specifically, [DH] for this particular setting), we introduce the distribution $T(g) \in \mathcal{D}'(\Omega; \mathbb{R})$, defined by its action on $\operatorname{Lip}(\Omega; \mathbb{R})$ through the formula

$$\langle T(g), \zeta \rangle = \int (g \wedge \nabla g) \cdot \nabla^\perp \zeta, \tag{1.8}$$

where $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$. In other words,

$$T(g) = -(g \wedge g_x)_y + (g \wedge g_y)_x = 2 \operatorname{Det}(\nabla g),$$

where $\operatorname{Det}(\nabla g)$ denotes the distributional Jacobian of g. We then set

$$L(g) = \frac{1}{2\pi} \operatorname*{Max}_{\|\nabla \zeta\|_{L^\infty} \leq 1} \langle T(g), \zeta \rangle. \tag{1.9}$$

We first state some analogs of the results in [BBM2] :

Theorem 3. *Assume $g \in W^{1,1}(\Omega; S^1)$. There exist two sequences $(P_i), (N_i)$ in Ω such that $\sum_i |P_i - N_i| < \infty$ and*

$$T(g) = 2\pi \sum (\delta_{P_i} - \delta_{N_i}). \tag{1.10}$$

Moreover,

$$L(g) = \operatorname{Inf} \sum_j d(\tilde{P}_j, \tilde{N}_j) \quad \Big(\leq \frac{1}{2\pi} \int_\Omega |\nabla g| \Big), \tag{1.11}$$

where d denotes the geodesic distance in Ω, and the infimum is taken over all possible sequences $(\tilde{P}_j), (\tilde{N}_j)$ satisfying

$$\sum (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) = \sum (\delta_{P_i} - \delta_{N_i}) \quad \text{in } (W^{1,\infty})^*.$$

Conversely, given two sequences $(P_i),(N_i)$ in Ω such that $\sum_i |P_i - N_i| < \infty$, there is always a map $g \in W^{1,1}(\Omega; S^1)$ such that (1.10) holds ; this is the "generalized dipole" construction (see [BBM, Lemma 15] and Lemma 4 below). Furthermore (see Theorem 10) the length of the minimal connection (as given by the right-hand side of (1.11)) equals Inf $\left\{\frac{1}{2\pi}\int|\nabla g|\right\}$, where the infimum is taken over all maps g such that (1.10) holds.

As was already pointed out in [BBM2, Lemma 20], we have

$$\langle T(g), \zeta\rangle = 2\pi \int_{\mathbb{R}} \deg(g, \Gamma_\lambda)\, d\lambda,$$

where $\Gamma_\lambda = \{x \in \Omega\ ;\ \zeta(x) = \lambda\}$ is equipped with the appropriate orientation (Lemma 20 in [BBM2] is stated for $g \in H^{1/2}$, but the proof also covers the case where $g \in W^{1,1}$). Here is a new property

Theorem 4. *Assume $g \in W^{1,1}(\Omega; S^1)$, and let $\zeta \in \mathrm{Lip}\,(\Omega;\mathbb{R})$ with $\|\nabla\zeta\|_{L^\infty} \le 1$. Then*

$$\int_{\mathbb{R}} |\deg(g, \Gamma_\lambda)|\, d\lambda \le L(g). \tag{1.12}$$

In particular, if ζ is a maximizer in (1.9), then

$$\deg(g, \Gamma_\lambda) \ge 0 \quad \textit{for a.e. } \lambda. \tag{1.13}$$

Finally, we study a notion of relaxed Jacobian determinants in the spirit of Fonseca-Fusco-Marcellini [FFM], and also Giaquinta-Modica-Souček [GMS1]. Given $g \in W^{1,1}(\Omega; S^1)$, we set (using the same notation as in [FFM])

$$TV(g) = \mathrm{Inf}\left\{\liminf_{n\to\infty} \int_\Omega |g_{nx} \wedge g_{ny}|\ ;\ g_n \in C^\infty(\Omega;\mathbb{R}^2) \text{ and } g_n \to g \text{ in } W^{1,1}\right\}. \tag{1.14}$$

Of course this number is possibly infinite. The following is a far-reaching extension of some results in [FFM]

Theorem 5. *Let $g \in W^{1,1}(\Omega; S^1)$. Then*

$$TV(g) < \infty \quad \Longleftrightarrow \quad \mathrm{Det}\,(\nabla g) \quad \textit{is a measure.}$$

In this case, we have

$$\mathrm{Det}\,(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i})$$

and

$$TV(g) = |\operatorname{Det}(\nabla g)|_{\mathcal{M}}.$$

In particular, $\frac{1}{\pi}TV(g)$ *is an integer which equals the number of topological singularities of* g *(counting their multiplicities).*

Here, for any Radon measure μ,

$$|\mu|_{\mathcal{M}} = \operatorname{Sup}\big\{\langle \mu, \varphi\rangle \ ; \ \varphi \in C(\Omega;\mathbb{R}),\ \|\varphi\|_{L^\infty} \le 1\big\}.$$

Remark 1. The conclusion of Theorem 5 still holds if one replaces the strong $W^{1,1}$-convergence in (1.14) by the weak $W^{1,1}$-convergence. There are numerous variants and extensions of Theorem 5 in Sections 4 and 5.

The paper is organized as follows :

1. Introduction
2. Properties of $W^{1,1}(S^1;S^1)$
3. Properties of $W^{1,1}(\Omega;S^1)$. Proofs of Theorems 1–4
4. $W^{1,1}(\Omega;S^1)$ and relaxed Jacobians
5. Further directions and open problems
   - 5.1. Some examples of BV-functions with jumps
   - 5.2. Some analogs of Theorems 1, 3, and 5 for bounded domains in $\mathbb{R}^2$
   - 5.3. Extensions of Theorems 1, 2, and 3 to higher dimensions
   - 5.4. Extension of TV to higher dimensions and to fractional Sobolev spaces
   - 5.5. Extension of Theorem 3 to maps with values into a curve

2. Properties of $W^{1,1}(S^1;S^1)$.

Even though the core of the paper deals with maps from a two dimensional manifold Ω with values into S^1, it is illuminating to start with the study of $W^{1,1}$-maps from S^1 into itself.

Let $g \in W^{1,1}(S^1;S^1)$. There are two natural quantities associated with g ; namely,

$$E(g) = \operatorname{Min}\big\{|\varphi|_{BV} \ ; \ \varphi \in BV(S^1;\mathbb{R}),\ g = \mathrm{e}^{i\varphi} \text{ a.e.}\big\} \tag{2.1}$$

and

$$E_{\mathrm{rel}}(g) = \operatorname{Inf}\left\{\liminf_{n\to\infty} \int_{S^1} |\dot g_n| \ ; \ g_n \in C^\infty(S^1;S^1),\ \deg g_n = 0,\ g_n \to g \text{ a.e.}\right\}. \tag{2.2}$$

It turns out that the two quantities are equal and that they can be easily computed in terms of g :

Theorem 6. *Let $g \in W^{1,1}(S^1; S^1)$. Then*

$$E_{\text{rel}}(g) = E(g) = \int_{S^1} |\dot g| + 2\pi|\deg g|. \tag{2.3}$$

Proof. First equality in (2.3) : "$\geq$" Let $(g_n) \subset C^\infty(S^1; S^1)$ be such that $\deg g_n = 0$ and $g_n \to g$ a.e. Then we may write $g_n = \mathrm{e}^{i\psi_n}$, with $\psi_n \in C^\infty(S^1; \mathbb{R})$ and $\int_{S^1} |\dot\psi_n| = \int_{S^1} |\dot g_n|$. Subtracting a suitable integer multiple of 2π, we may assume (ψ_n) bounded in $W^{1,1}(S^1; \mathbb{R})$. After passing to a subsequence, we may further assume that $\psi_n \to \psi$ a.e. for some $\psi \in BV(S^1; \mathbb{R})$. Therefore,

$$\liminf_{n\to\infty} \int_{S^1} |\dot g_n| = \liminf_{n\to\infty} \int_{S^1} |\dot\psi_n| \geq \int_{S^1} |\dot\psi|$$

and, clearly, $\mathrm{e}^{i\psi} = g$ a.e.

"$\leq$" Let $\psi \in BV(S^1; \mathbb{R})$ be such that

$$|\psi|_{BV} = \text{Min}\left\{|\varphi|_{BV}\ ;\ g = \mathrm{e}^{i\varphi} \text{ a.e.}\right\}.$$

Consider a sequence $(\psi_n) \subset C^\infty(S^1; \mathbb{R})$ such that $\psi_n \to \psi$ a.e. and $\int_{S^1} |\dot\psi_n| \to |\psi|_{BV}$. If we set $g_n = \mathrm{e}^{i\psi_n}$, then clearly $g_n \in C^\infty(S^1; S^1)$, $\deg g_n = 0$ and $g_n \to g$ a.e. Moreover,

$$\lim_{n\to\infty} \int_{S^1} |\dot g_n| = \lim_{n\to\infty} \int_{S^1} |\dot\psi_n| = |\psi|_{BV}.$$

Second equality in (2.3) : "$\geq$" This assertion has been established under slightly more general assumptions in [BBM2, Section 4.3]. Here is an alternative approach. Let $g \in W^{1,1}(S^1; S^1)$. We prove that, if $\varphi \in BV(S^1; \mathbb{R})$ satisfies $g = \mathrm{e}^{i\varphi}$ a.e., then

$$|\varphi|_{BV} \geq \int_{S^1} |\dot g| + 2\pi|\deg g|. \tag{2.4}$$

The main ingredient is the chain rule formula for BV-maps, due to Vol'pert ; see [V], and also [AFP].

Chain rule. Let $\varphi \in BV(S^1; \mathbb{R})$. Recall that there is a representative φ_0 of φ which is continuous except at (at most) countably many points $a_n \in S^1$; in the sequel, we take φ to be φ_0 itself. Moreover, at the points a_n, φ admits limits from the "right" and from the "left", say $\varphi(a_n+)$ and $\varphi(a_n-)$.

Let $\dot\varphi$ be the distributional derivative of φ, which is a Borel measure. The diffuse part of $\dot\varphi$ is

$$\dot\varphi_d = \dot\varphi - \sum_n (\varphi(a_n+) - \varphi(a_n-))\delta_{a_n}.$$

Vol'pert's chain rule for BV-maps on a bounded interval (or a closed curve) asserts that, if $F \in C^1(\mathbb{R};\mathbb{R})$, then

$$\dot{\overline{F \circ \varphi}} = F'(\varphi)\dot\varphi_d + \sum_n \big(F(\varphi(a_n+)) - F(\varphi(a_n-))\big)\delta_{a_n}.$$

A more general version of the chain rule, which is valid in $\mathbb{R}^N$, is stated and explained in the proof of Lemma 5 in Section 3 below.

We now return to the proof of (2.4). By the chain rule formula, we have

$$\dot g = i\mathrm{e}^{i\varphi}\dot\varphi_d + \sum_n \left(\mathrm{e}^{i\varphi(a_n+)} - \mathrm{e}^{i\varphi(a_n-)}\right)\delta_{a_n}.$$

Using the continuity of g, we have $g(a_n) = \mathrm{e}^{i\varphi(a_n+)} = \mathrm{e}^{i\varphi(a_n-)}$ for each n. Hence,

$$\dot g = i\mathrm{e}^{i\varphi}\dot\varphi_d.$$

Since $\dot g \in L^1$ and $\mathrm{e}^{i\varphi} = g$ a.e., we thus find that

$$g \wedge \dot g = \frac{1}{ig}\dot g = \dot\varphi_d.$$

Consequently,

$$|\dot\varphi|_{\mathcal{M}} = |\dot\varphi_d|_{\mathcal{M}} + |\dot\varphi - \dot\varphi_d|_{\mathcal{M}} = |g\wedge\dot g|_{\mathcal{M}} + |g\wedge\dot g - \dot\varphi|_{\mathcal{M}} = \int_{S^1}|\dot g| + |g\wedge\dot g - \dot\varphi|_{\mathcal{M}}. \tag{2.5}$$

On the other hand,

$$|g\wedge\dot g - \dot\varphi|_{\mathcal{M}} \geq |\langle g\wedge\dot g - \dot\varphi, 1\rangle| = |\langle g\wedge\dot g, 1\rangle| = 2\pi|\deg g|. \tag{2.6}$$

(The last equality is clear when g is smooth ; the case of a general $W^{1,1}$-map follows by approximation.) Finally, by combining (2.5) and (2.6) we find that

$$|\varphi|_{BV} \geq \int_{S^1}|\dot g| + 2\pi|\deg g|,$$

as claimed.

Second equality in (2.3) : "$\leq$" Since $S^1\setminus\{1\}$ is simply connected, we may write $g = \mathrm{e}^{i\varphi}$ on $S^1\setminus\{1\}$, for some $\varphi \in W^{1,1}(S^1\setminus\{1\};\mathbb{R})$ such that $|\dot\varphi| = |\dot g|$ in $S^1\setminus\{1\}$. Since φ is continuous, we have

$$\varphi(1-) - \varphi(1+) = 2\pi\deg g.$$

Passing to the full S^1, we have

$$|\varphi|_{BV} = \int_{S^1\setminus\{1\}}|\dot\varphi| + |\varphi(1-) - \varphi(1+)| = \int_{S^1}|\dot g| + 2\pi|\deg g|.$$

As a consequence of Theorem 6, we have

Corollary 1. *For every $g \in W^{1,1}(S^1; S^1)$,*

$$E(g) \leq 2|g|_{W^{1,1}}. \tag{2.7}$$

Remark 2. The constant 2 in (2.7) is optimal. Indeed, for $g = \mathrm{Id}$, we have $|g|_{W^{1,1}} = 2\pi$, while $E(g) = 4\pi$ by Theorem 6.

It is easy to see from the definition of the relaxed energy that E_{rel} is lower semicontinuous with respect to the pointwise a.e. convergence in S^1. In view of Theorem 6, we have the following :

Corollary 2. *Let $(g_n) \subset W^{1,1}(S^1; S^1)$ be such that $g_n \to g$ a.e. for some $g \in W^{1,1}(S^1; S^1)$. Then*

$$\int_{S^1} |\dot{g}| + 2\pi|\deg g| \leq \liminf_{n\to\infty} \left(\int_{S^1} |\dot{g}_n| + 2\pi|\deg g_n| \right). \tag{2.8}$$

Remark 3. The constant 2π in (2.8) cannot be improved. In fact, assume that (2.8) holds with 2π replaced by some C. In particular, for any sequence $(g_n) \subset C^\infty(S^1; S^1)$ such that $\deg g_n = 0$ and $g_n \to \mathrm{Id}$ a.e., we have

$$2\pi + C = \int_{S^1} |\dot{g}| + C|\deg g| \leq \liminf_{n\to\infty} \left(\int_{S^1} |\dot{g}_n| + C|\deg g_n| \right) = \liminf_{n\to\infty} \int_{S^1} |\dot{g}_n|. \tag{2.9}$$

On the other hand, according to Theorem 6, the sequence (g_n) can be chosen so that

$$\lim_{n\to\infty} \int_{S^1} |\dot{g}_n| = \int_{S^1} |\dot{g}| + 2\pi|\deg g| = 4\pi. \tag{2.10}$$

A comparison between (2.9) and (2.10) implies $C \leq 2\pi$.

Inequality (2.8) still holds if one replaces $|\deg g|$ and $|\deg g_n|$ by $\deg g$ and $\deg g_n$, under the additional assumption that the sequence (g_n) is **bounded** in $W^{1,1}$. This assumption is essential ; see Remark 4 below. More precisely, we have

Proposition 1 ([BBM2]). *Let $g_n, g \in W^{1,1}(S^1; S^1)$ be such that $g_n \to g$ a.e and*

$$\sup_n |g_n|_{BV} < \infty.$$

Then

$$\int_{S^1} |\dot{g}| + 2\pi \deg g \leq \liminf_{n\to\infty} \left(\int_{S^1} |\dot{g}_n| + 2\pi \deg g_n \right). \tag{2.11}$$

We present here an alternative proof based on Corollary 2.

Proof. Assume $|g_n|_{BV} \le C$, $\forall\, n$. In particular,

$$|\deg g_n| \le \frac{1}{2\pi}\int_{S^1} |\dot g_n| \le \frac{C}{2\pi}.$$

Since $\deg g_n$ takes only integer values, after passing to a subsequence, we can assume that $d = \deg g_n$, $\forall\, n$. Given $\varepsilon > 0$, let $h \in C^\infty(S^1; S^1)$ be such that $\deg h = -d$ and $h(x) = 1$, $\forall\, x \in S^1\backslash B_\varepsilon(1)$. Clearly,

$$hg_n \to hg \quad \text{a.e. in } S^1 \quad \text{and} \quad \deg hg_n = 0, \quad \forall\, n.$$

It follows from Corollary 2 that

(2.12)
$$\int_{S^1} |\dot g h + g\dot h| + 2\pi(\deg g - d) \le \liminf_{n\to\infty} \int_{S^1} |\dot g_n h + g_n \dot h| \le \liminf_{n\to\infty} \int_{S^1} |\dot g_n| + \int_{S^1} |\dot h|.$$

On the other hand, since $h(x) = 1$ for $x \in S^1\backslash B_\varepsilon(1)$, we have

$$\begin{aligned}
\int_{S^1} |\dot g h + g\dot h| &= \int_{S^1\backslash B_\varepsilon(1)} |\dot g| + \int_{S^1\cap B_\varepsilon(1)} |\dot g h + g\dot h| \\
&\ge \int_{S^1\backslash B_\varepsilon(1)} |\dot g| - \int_{S^1\cap B_\varepsilon(1)} |\dot g| + \int_{S^1\cap B_\varepsilon(1)} |\dot h| \qquad (2.13) \\
&= \int_{S^1} |\dot g| - 2\int_{S^1\cap B_\varepsilon(1)} |\dot g| + \int_{S^1} |\dot h|.
\end{aligned}$$

Comparison between (2.12) and (2.13) yields

$$\int_{S^1} |\dot g| - 2\int_{S^1\cap B_\varepsilon(1)} |\dot g| + 2\pi(\deg g - d) \le \liminf_{n\to\infty} \int_{S^1} |\dot g_n|.$$

Taking $\varepsilon \to 0$, we obtain (2.11).

An immediate consequence of Proposition 1 is

Corollary 3. *Under the assumptions of Proposition 1, we have*

$$\int_{S^1} |\dot g| \le \liminf_{n\to\infty} \left(\int_{S^1} |\dot g_n| - 2\pi|\deg g_n - \deg g| \right).$$

Remark 4. Proposition 1 (or, equivalently, Corollary 3) is **false** without the assumption $\sup_n |g_n|_{BV} < \infty$. Here is an example. Let $n \ge 1$ be a fixed integer.

Given $0 \le j \le n-1$, let $a_{j,n} = \frac{2\pi j}{n}$ and $I_{j,n} = [a_{j,n}, a_{j+1,n} - \frac{1}{2^n}] \subset \mathbb{R}$. On each interval $I_{j,n}$, we define $f_n(t) = 2\pi j - a_{j,n}$. We then extend f_n continuously to $[0, 2\pi]$, so that f_n is affine linear outside the set $\bigcup_j I_{j,n}$, and $f_n(2\pi) = 2\pi(n-1)$. By construction, f_n is Lipschitz, nondecreasing, and $f_n(2\pi) - f_n(0) \in 2\pi\mathbb{Z}$. Note that

$$d\big(f_n(t), -t + 2\pi\mathbb{Z}\big) \le |a_{j+1,n} - a_{j,n}| = \frac{2\pi}{n} \quad \forall t \in \bigcup_j I_{j,n} \ ;$$

$$\big|[0,2\pi] \backslash \bigcup_j I_{j,n}\big| = \frac{n}{2^n}.$$

Set $g_n(\theta) = \mathrm{e}^{-if_n(\theta)}$. Then, we have $g_n \to g$ a.e., where $g = \mathrm{Id}$; however,

$$\int_{S^1} |\dot{g}| + 2\pi \deg g = 4\pi,$$

while

$$\int_{S^1} |\dot{g}_n| + 2\pi \deg g_n = 0, \quad \forall\ n.$$

3. Properties of $W^{1,1}(\Omega; S^1)$.

We start with the rigorous definitions of $T(g)$ and of the class Lip mentioned in the Introduction. If $g \in W^{1,1}(\Omega; \mathbb{R}^2)$, we set

$$|\nabla g| = \left[\left(\frac{\partial g_1}{\partial x}\right)^2 + \left(\frac{\partial g_1}{\partial y}\right)^2 + \left(\frac{\partial g_2}{\partial x}\right)^2 + \left(\frac{\partial g_2}{\partial y}\right)^2 \right]^{1/2},$$

where (x, y) is any orthonormal frame at some point on Ω, and we let

$$|g|_{W^{1,1}} = \int_\Omega |\nabla g|.$$

Recall that we defined $T(g)$ by

$$\langle T(g), \zeta \rangle = \int_\Omega \left((g \wedge g_x)\zeta_y - (g \wedge g_y)\zeta_x\right), \quad \forall\ \zeta \in \mathrm{Lip}\,(\Omega; \mathbb{R}).$$

Here, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1$, and the integrand is computed in any orthonormal frame (x, y) such that (x, y, n) is direct, where n is the outward normal to G. (This integrand is frame invariant.) The class of testing functions, $\mathrm{Lip}\,(\Omega; \mathbb{R})$, is the set of functions which are Lipschitz with respect to the geodesic distance d in Ω. For such a map, we set

$$|\zeta|_{\mathrm{Lip}} = \operatorname*{Sup}_{x \ne y} \frac{|\zeta(x) - \zeta(y)|}{d(x, y)} = \|\nabla\zeta\|_{L^\infty}.$$

We next collect some straightforward properties of $T(g)$ and $L(g)$:

Lemma 1. *We have*

a) $T(\bar{g}) = -T(g)$, $\forall\ g \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L^\infty$;

b) $T(gh) = T(g) + T(h)$, $\forall\ g, h \in W^{1,1}(\Omega; S^1)$;

c) $L(g) \leq \dfrac{1}{2\pi} |g|_{W^{1,1}} \|g\|_\infty$, $\forall\ g \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L^\infty$;

d) If g_n, $g \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L^\infty$ *are such that* $g_n \to g$ *in* $W^{1,1}$ *and* $\|g_n\|_{L^\infty} \leq C$, *then* $L(g_n) \to L(g)$.

Proof. The only property that requires a proof is d). Since

$$|\langle T(g_n), \zeta\rangle - \langle T(g), \zeta\rangle| \leq \int_\Omega |g_n|\ |\nabla(g_n - g)|\ |\nabla\zeta| + \int_\Omega |g_n - g|\ |\nabla g|\ |\nabla \zeta|,$$

we have

$$|L(g_n) - L(g)| \leq C|g_n - g|_{W^{1,1}} + \|(g_n - g)\nabla g\|_{L^1}$$

and d) follows by dominated convergence.

Recall the following density result of Bethuel-Zheng [BZ] :

Lemma 2. *The class*

$$\mathcal{R} = \big\{g \in W^{1,1}(\Omega; S^1)\ ;\ g \in C^\infty(\Omega \setminus A; S^1),\ \text{where } A \text{ is some finite set}\big\}$$

is dense in $W^{1,1}(\Omega; S^1)$.

When $g \in \mathcal{R}$, a straightforward adaptation of the proof of Lemma 2 in [BBM2] yields the following :

Lemma 3. *If* $g \in W^{1,1}(\Omega; S^1)$, $g \in C^\infty(\Omega \setminus \{a_1, \ldots, a_k\}; S^1)$, *then*

$$T(g) = 2\pi \sum_{j=1}^k d_j \delta_{a_j}.$$

Here, $d_j = \deg(g, a_j)$ is the topological degree of g restricted to any small circle around a_j, positively oriented with respect to the outward normal. Moreover, $L(g)$ is the length of the minimal connection associated to the configuration (a_j, d_j) and to the geodesic distance on Ω (see Remark 5 below).

Remark 5. By the definition of $T(g)$, we have $\langle T(g), 1\rangle = 0$. Thus, $\sum_{j=1}^k d_j = 0$, by Lemma 3. Therefore, we may write the collection of points (a_j) (repeated with multiplicity $|d_j|$) as $(P_1, \ldots, P_\ell, N_1, \ldots, N_\ell)$, where $\ell = \frac{1}{2}\sum_{j=1}^k |d_j|$; the points of degree 0 do not appear in this list, a_j is counted among the points P_i if $d_j > 0$, and among the points N_i otherwise. Then

$$L(g) = \operatorname*{Min}_{\sigma \in S_\ell} \sum_{j=1}^{\ell} d(P_j, N_{\sigma(j)}).$$

This formula first appeared in the context of S^2-valued maps ; see [BCL].

Using the density of $\mathcal{R}$ in $W^{1,1}(\Omega; S^1)$, one can easily obtain Theorem 3 from Lemma 3. The analog of Theorem 3 for $H^{1/2}(\Omega; S^1)$ was proved in [BBM2], and the arguments there also apply to our case.

A converse to Theorem 3 is also true. Namely, for any sequence of points (P_i), (N_i) satisfying $\sum_i |P_i - N_i| < \infty$, one can find $g \in W^{1,1}(\Omega; S^1)$ such that (1.10) holds ; see [BBM2]. Motivated by this, we state the following :

Open Problem 1. Let $1 < p < 2$. Given $g \in W^{1,p}(\Omega; S^1)$, can one find (P_i), (N_i) such that $\sum_i |P_i - N_i|^{2/p-1} < \infty$ and (1.10) holds ?

Open Problem 2. Given two sequences (P_i), (N_i) such that $\sum_i |P_i - N_i|^{2/p-1} < \infty$ for some $1 < p < 2$, does there exist some $g \in W^{1,p}(\Omega; S^1)$ such that (1.10) holds ? If the answer is negative (as we suspect), what is the right condition on the points P_i, N_i (in terms of capacity ?) which guarantees the existence of g ?

We now consider the following class

$$Y = \overline{C^\infty(\Omega; S^1)}^{W^{1,1}};$$

this class is properly contained in $W^{1,1}(\Omega; S^1)$ (see Remark 7 below).

It turns out that maps in Y can be characterized in terms of their distribution $T(g)$:

Theorem 7. *Let $g \in W^{1,1}(\Omega; S^1)$. Then the following properties are equivalent :*

a) $g \in Y$;

b) $T(g) = 0$;

c) there exists $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ such that $g = e^{i\varphi}$.

Remark 6. When Ω is a smooth bounded open set in $\mathbb{R}^2$, the equivalence a) $\Leftrightarrow$ b) was established by Demengel [D]. We could adapt the argument in [D] to our case, but we present below a different approach, based on an idea of Carbou [C].

Remark 7. Using Theorem 7, it is easy to construct maps in $W^{1,1}(\Omega; S^1) \setminus Y$. Assume, e.g., that $\Omega = S^2$, and let $g(x, y, z) = \dfrac{(x, y)}{|(x, y)|}$. By Lemma 3, we have

$T(g) = 2\pi(\delta_N - \delta_S)$, where N, S are the North and South pole of S^2. By Theorem 7, this implies that $g \notin Y$.

Proof of Theorem 7.
a) $\Rightarrow$ *b*) By Lemma 3, we have $T(g) = 0$ if $g \in C^\infty(\Omega; S^1)$. By Lemma 1, $g \mapsto T(g)$ is continuous with respect to $W^{1,1}$-convergence, and thus $T(g) = 0$, $\forall\, g \in Y$.

b) $\Rightarrow$ c) We argue as in [C] ; see also [BBM1]. Let $x_0 \in \Omega$ and assume that $\Omega \subset \mathbb{R}^2$ near x_0. Since $T(g) = 0$, the L^1-vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} g \wedge g_x \\ g \wedge g_y \end{pmatrix}$$

satisfies, near x_0, $\dfrac{\partial F1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ in the sense of distributions. By a variant of the Poincaré Lemma (see [BBM1]), we may find a neighborhood ω of x_0 and a function $\psi \in W^{1,1}(\omega; \mathbb{R})$ such that $g = \mathrm{e}^{i(\psi + C)}$ in ω, for some constant C (see [BBM1]).

Consider a finite covering of Ω with open sets ω_j such that

(i) in each ω_j we may write $g = \mathrm{e}^{i\varphi_j}$ for some $\varphi_j \in W^{1,1}(\omega_j; \mathbb{R})$;

(ii) $\omega_j \cap \omega_k$ is connected, $\forall\, j, \forall\, k$.

In $\omega_j \cap \omega_k$, the map $\varphi_j - \varphi_k$ belongs to $W^{1,1}$ and is $2\pi\mathbb{Z}$-valued ; thus, it has to be constant a.e. Since Ω is simply connected, we may therefore find a map φ in $W^{1,1}(\Omega; \mathbb{R})$ such that $\varphi - \varphi_j$ is, a.e. in ω_j, a constant integer multiple of 2π. In particular, $g = \mathrm{e}^{i\varphi}$ in Ω.

c) $\Rightarrow$ a) Let $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$ be such that $\varphi_n \to \varphi$ in $W^{1,1}$. Set $g_n = \mathrm{e}^{i\varphi_n}$. Then, clearly, $g_n \in C^\infty(\Omega; S^1)$ and $g_n \to g$ in $W^{1,1}$.

Remark 8. It follows from Theorem 7 that, given a map $g \in W^{1,1}(\Omega; S^1)$, in general we may **not** write $g = \mathrm{e}^{i\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$; consider, for example, the map g in Remark 7. However, it follows from Theorem 2 that we may write $g = \mathrm{e}^{i\varphi}$ for some $\varphi \in BV(\Omega; \mathbb{R})$. This conclusion still holds for maps $g \in BV(\Omega; S^1)$; see [GMS2] and [DI].

Before starting the proof of Theorem 2, we recall the "generalized dipole" construction presented in [BBM2] :

Lemma 4. *Let $g \in W^{1,1}(\Omega; S^1)$. Then, for each $\varepsilon > 0$, there is some $h = h_\varepsilon \in W^{1,1}(\Omega; S^1)$ such that*

(i) $|h|_{W^{1,1}} \leq 2\pi L(g) + \varepsilon$;

(ii) $T(h) = T(g)$;

(iii) there is a function $\psi = \psi_\varepsilon \in BV(\Omega;\mathbb{R})$ *such that* $h = e^{i\psi}$ *a.e. and* $|\psi|_{BV} \le 4\pi L(g) + \varepsilon$ *;*

(iv) $\text{meas}\,(\text{Supp}\,\psi) = \text{meas}\,(\text{Supp}\,(h-1)) < \varepsilon$.

Proof of Theorem 2. Let $\psi \in BV(\Omega;\mathbb{R})$ and $\zeta \in C^\infty(\Omega;\mathbb{R})$ be such that $|\nabla\zeta| \le 1$. Then

$$|g\wedge\nabla g - D\psi|_{\mathcal{M}(\Omega)} \ge \int_\Omega (g\wedge\nabla g)\cdot\nabla^\perp\zeta - \int_\Omega D\psi\cdot\nabla^\perp\zeta = \langle T(g),\zeta\rangle,$$

so that

$$\frac{1}{2\pi}|g\wedge\nabla g - D\psi|_{\mathcal{M}(\Omega)} \ge L(g),$$

by taking the supremum over ζ.

It thus remains to construct, for each $\varepsilon > 0$, a map $\psi \in C^\infty(\Omega;\mathbb{R})$ such that

$$\int_\Omega |g\wedge\nabla g - \nabla\psi| \le 2\pi L(g) + \varepsilon.$$

Recall that, by Lemma 4, we may find some $h \in W^{1,1}(\Omega;S^1)$ such that $T(h) = T(g)$ and

$$\int_\Omega |\nabla h| \le 2\pi L(g) + \varepsilon/2.$$

Set $k = g\bar{h}$, so that $k \in Y$, by Lemma 1 and Theorem 7. Write $k = e^{i\varphi}$ for some $\varphi \in W^{1,1}$ and let $\psi \in C^\infty(\Omega;\mathbb{R})$ be such that $\displaystyle\int_\Omega |\nabla\varphi - \nabla\psi| < \frac{\varepsilon}{2}$.

Then

$$\begin{aligned}\int_\Omega |g\wedge\nabla g - \nabla\psi| &= \int_\Omega |(hk)\wedge\nabla(hk) - \nabla\psi| = \int_\Omega |h\wedge\nabla h + k\wedge\nabla k - \nabla\psi| \\ &= \int_\Omega |h\wedge\nabla h + \nabla\varphi - \nabla\psi| \le \int_\Omega |h\wedge\nabla h| + \int_\Omega |\nabla\varphi - \nabla\psi| \\ &\le \int_\Omega |\nabla h| + \frac{\varepsilon}{2} \le 2\pi L(g) + \varepsilon.\end{aligned}$$

In order to complete the proof of Theorem 2, it suffices to prove the following

Claim. Given $g \in W^{1,1}(\Omega;S^1)$, there exists some $\varphi \in BV(\Omega;\mathbb{R})$ such that

$$g = e^{i\varphi} \quad \text{a.e. in } \Omega \tag{3.1}$$

and

$$|g\wedge\nabla g - D\varphi|_{\mathcal{M}(\Omega)} = 2\pi L(g). \tag{3.2}$$

In other words, in (1.7), one may restrict the minimization to the class of functions $\psi \in BV(\Omega;\mathbb{R})$ such that $g = \mathrm{e}^{i\psi}$.

Using the same argument as above, we can write g as

$$g = h_n \mathrm{e}^{i\varphi_n} \quad \text{in } \Omega, \tag{3.3}$$

where $\varphi_n \in W^{1,1}(\Omega;\mathbb{R})$, $h_n \in W^{1,1}(\Omega;S^1)$ and

$$|h_n|_{W^{1,1}} \le 2\pi L(g) + \frac{1}{n}.$$

Moreover, in view of (iv) in Lemma 4, we can also assume that $h_n \to 1$ a.e.

Note that

$$\int_\Omega |g \wedge \nabla g - \nabla\varphi_n| = \int_\Omega |h_n \wedge \nabla h_n| = \int_\Omega |\nabla h_n| \le 2\pi L(g) + \frac{1}{n}. \tag{3.4}$$

Subtracting a suitable integer multiple of 2π from φ_n, we may assume that (φ_n) is bounded in $W^{1,1}(\Omega;\mathbb{R})$. After passing to a subsequence if necessary, we can find $\varphi \in BV(\Omega;\mathbb{R})$ such that

$$\varphi_n \to \varphi \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla\varphi_n \overset{*}{\rightharpoonup} D\varphi \quad \text{in } \mathcal{M}(\Omega).$$

Since $h_n \to 1$ a.e. in Ω, it follows from (3.3) that $g = \mathrm{e}^{i\varphi}$ a.e. in Ω. Letting $n \to \infty$ in (3.4), we obtain

$$\int_\Omega |g \wedge \nabla g - D\varphi| \le \liminf_{n\to\infty} \int_\Omega |g \wedge \nabla g - \nabla\varphi_n| \le 2\pi L(g).$$

This establishes "$\le$" in (3.2). The reverse inequality follows trivially from (1.7).

Remark 9. Here is an example which shows that a minimizing function ψ in (1.7) is not necessarily a lifting of g (modulo constants). Assume for simplicity Ω is flat and consider a map g having four singular points in Ω, say $P_1 = (0,0)$, $P_2 = (1,1)$, $N_1 = (1,0)$ and $N_2 = (0,1)$. Then $S = P_1N_1P_2N_2$ is a square. We may write $g = \mathrm{e}^{i\psi_1} = \mathrm{e}^{i\psi_2}$, where

$$\psi_1 \in C^\infty(\Omega \setminus ([P_1,N_1] \cup [P_2,N_2])) \quad \text{and} \quad \psi_2 \in C^\infty(\Omega \setminus ([P_1,N_2] \cup [P_2,N_1])).$$

Then $|g \wedge \nabla g - D\psi_1| = 2\pi\nu_1$ (resp. $|g \wedge \nabla g - D\psi_2| = 2\pi\nu_2$), where ν_1 (resp. ν_2) denotes the 1-dimensional Hausdorff measure on $[P_1,N_1] \cup [P_2,N_2]$ (resp. $[P_1,N_2] \cup [P_2,N_1]$).

It follows from Theorem 2 that ψ_1, ψ_2 are minimizers in (1.7). Moreover, we may assume that $\psi_1 = \psi_2$ in the square S. By convexity, the function $\psi = (\psi_1 + \psi_2)/2$ is also a minimizer. Outside $\bar{S}$, ψ is smooth and, clearly, $g = \alpha e^{i\psi}$ in $\Omega \setminus \bar{S}$ for some $\alpha \in S^1$. One may check that $\alpha = -1$, and thus

$$e^{i\psi} = \begin{cases} g, & \text{in } S \\ -g, & \text{in } \Omega \setminus \bar{S} \end{cases},$$

so that ψ is not a lifting of g.

Going back to the general situation, let K be the set of minimizers of the problem

$$\operatorname*{Min}_{\psi \in BV} \int |g \wedge \nabla g - D\psi|$$

satisfying $\int \psi = 0$. Clearly, K is convex and compact in $L^1(\Omega; \mathbb{R})$.

Open Problem 3. Is it true that

$$\psi \text{ is an extreme point of } K \quad \Longleftrightarrow \quad g = e^{i(\psi + C)} \text{ for some constant } C \text{ ?}$$

Another result, closely related to Theorem 1, is the following :

Theorem 8. *Let $g \in W^{1,1}(\Omega; S^1)$. Then,*

$$\text{(3.5)} \quad \operatorname{Inf} \Big\{ |\varphi_2|_{BV} \ ; \ g = e^{i(\varphi_1 + \varphi_2)}, \ \varphi_1 \in W^{1,1}(\Omega; \mathbb{R}), \ \varphi_2 \in BV(\Omega; \mathbb{R}) \Big\} = 4\pi L(g).$$

The analog of Theorem 8 for the space $H^{1/2}(\Omega; S^1)$ was established in [BBM2], and the arguments there can be adapted to our case. The proof we present below for "$\geq$" in (3.5) is however different.

Proof of Theorem 8.

Proof of "$\leq$" in (3.5). With $\varepsilon > 0$ fixed and h given by Lemma 4, we write $g = hk$, where $k = g\bar{h}$. By Lemma 1 a), b), we have $T(k) = 0$. Therefore, by Theorem 7 we may write $k = e^{i\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$. It follows that $g = e^{i(\varphi + \psi)}$, with ψ given by Lemma 4. Inequality "$\leq$" in (3.5) follows from (iii) in Lemma 4.

Proof of "$\geq$" in (3.5). We rely on the following

Lemma 5. *Let $\varphi \in BV(\Omega;\mathbb{R})$ be such that $g = \mathrm{e}^{i\varphi} \in W^{1,1}(\Omega;S^1)$. Then*

$$|D\varphi|_{\mathcal{M}(\Omega)} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}.$$

Proof. We split the measure $D\varphi$ as

$$D\varphi = (D\varphi)_{ac} + (D\varphi)_C + (D\varphi)_J, \tag{3.6}$$

where ac, C, J stand respectively for the absolutely continuous, Cantor and jump part. Applying Volpert's chain rule to the composition $f(\varphi)$, where $f(t) = \mathrm{e}^{it}$, we obtain

$$Dg = D(f \circ \varphi) = f'(\varphi)(D\varphi)_{ac} + f'(\varphi)(D\varphi)_C + \frac{f(\varphi^+) - f(\varphi^-)}{\varphi^+ - \varphi^-}(D\varphi)_J. \tag{3.7}$$

The meaning of this identity is the following : recall that, for every function $\varphi \in BV(\Omega)$, the Lebesgue set of φ is the complement of a set of σ-finite $\mathcal{H}^1$-measure. We may assume that φ coincides with its precise representative on the Lebesgue set of φ. Since $|(D\varphi)_{ac}|(A) = |(D\varphi)_C|(A) = 0$ whenever $\mathcal{H}^1(A) < \infty$, the first two terms in the right-hand side of (3.7) are well-defined (i.e., independently of the choice of the representative of φ). The last term in (3.7) is to be understood as follows : the jump set J of φ is a countable union of Lipschitz curves $\mathcal{C}_i$ and, at $\mathcal{H}^1$-a.e. point x of $\mathcal{C}_i$, $\mathcal{C}_i$ has a normal vector and φ has one-sided limits at x along the normal direction ; the quantities φ^+ and φ^- stand for the two one-sided limits. See [AFP] for a proof of (3.7).

Since $g \in W^{1,1}$, it follows that $(Dg)_C = (Dg)_J = 0$, so that $(D\varphi)_C = 0$ and

$$\nabla g = f'(\varphi)(D\varphi)_{ac} = ig(D\varphi)_{ac}. \tag{3.8}$$

From (3.8), we obtain that

$$g \wedge \nabla g = -i\bar{g}\,\nabla g = (D\varphi)_{ac}. \tag{3.9}$$

Thus

$$(D\varphi)_J = D\varphi - g \wedge \nabla g.$$

Since the decomposition (3.6) consists of mutually orthogonal measures, we have

$$\begin{aligned} |D\varphi| = |(D\varphi)_{ac}| + |(D\varphi)_J| &= |i\bar{g}\,\nabla g|_{\mathcal{M}(\Omega)} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \\ &= |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}. \end{aligned}$$

Proof of Theorem 8 completed. Write $g = \mathrm{e}^{i(\varphi_1+\varphi_2)}$, with $\varphi_1 \in W^{1,1}$, $\varphi_2 \in BV$. Then, with $h = g\mathrm{e}^{-i\varphi_1}$, we have $h = \mathrm{e}^{i\varphi_2}$, $h \in W^{1,1}$ and $T(h) = T(g)$. Theorem 2 and Lemma 5 yield

$$\begin{aligned}|D\varphi_2|_{\mathcal{M}(\Omega)} &= |h|_{W^{1,1}} + |h \wedge \nabla h - D\varphi_2|_{\mathcal{M}(\Omega)}\\ &\geq |h|_{W^{1,1}} + 2\pi L(h) \geq 4\pi L(h) = 4\pi L(g),\end{aligned}$$

since $2\pi L(h) \leq |h|_{W^{1,1}}$, by Lemma 1.

Maps in $W^{1,1}(\Omega; S^1)$ need not belong to $H^{1/2}(\Omega; S^1)$. However, we have the following link between $W^{1,1}$ and $H^{1/2}$:

Theorem 9. *Let $g \in W^{1,1}(\Omega; S^1)$. Then there exist $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$ and $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ such that $g = \mathrm{e}^{i\varphi} h$.*

The analog of Theorem 9 for $H^{1/2}(\Omega; S^1)$ was established in [BBM2].

Proof. We rely on the following additional property of the maps $h = h_\varepsilon$ constructed in Lemma 4 (see [BBM2]) :

(v) $h \in H^{1/2}(\Omega; S^1)$.

Pick any of the maps h as in Lemma 4. Then $T(g\bar{h}) = 0$, so that, by Theorem 7, we may write $g\bar{h} = \mathrm{e}^{i\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$. The decomposition $g = \mathrm{e}^{i\varphi} h$ has all the required properties.

From Theorem 2, we have

Corollary 4. *Each $g \in W^{1,1}(\Omega; S^1)$ may be written as $g = \mathrm{e}^{i\varphi}$ for some $\varphi \in BV(\Omega; \mathbb{R})$.*

Corollary 5 ([GMS2]). *For each $g \in W^{1,1}(\Omega; S^1)$, one can find a sequence $(g_n) \subset C^\infty(\Omega; S^1)$, bounded in $W^{1,1}$, such that $g_n \to g$ a.e.*

We now establish

Proposition 2. *For each $g \in W^{1,1}(\Omega; S^1)$, we have*

$$E_{\mathrm{rel}}(g) = E(g).$$

Proof. "$\leq$" Let $\varphi \in BV(\Omega; \mathbb{R})$ be such that $g = \mathrm{e}^{i\varphi}$. Let $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$ be such that $\varphi_n \to \varphi$ a.e. and $\int_\Omega |\nabla \varphi_n| \to |\varphi|_{BV}$. We define $g_n = \mathrm{e}^{i\varphi_n} \in C^\infty(\Omega; S^1)$. Then $g_n \to g$ a.e. and $\int_\Omega |\nabla g_n| = \int_\Omega |\nabla \varphi_n| \to |\varphi|_{BV}$, so that "$\leq$" follows.

"$\geq$" Let $(g_n) \subset C^\infty(\Omega; S^1)$ be such that $g_n \to g$ a.e. and $\int_\Omega |\nabla g_n| \to E_{\mathrm{rel}}(g)$. Since Ω is simply connected, we may write $g_n = \mathrm{e}^{i\varphi_n}$, with $\varphi_n \in C^\infty(\Omega; \mathbb{R})$. Since

$\int_\Omega |\nabla g_n| = \int_\Omega |\nabla \varphi_n|$, we may find some $\varphi \in BV(\Omega;\mathbb{R})$ such that, after subtracting an integer multiple of 2π from φ_n and up to some subsequence, $\varphi_n \to \varphi$ a.e. ; we then conclude that $|\varphi|_{BV} \leq \liminf\limits_{n\to\infty} \int_\Omega |\nabla \varphi_n| = E_{\mathrm{rel}}(g)$.

The relaxed energy is also related to the minimal connection $L(g)$. This is the content of Theorem 1 :

$$E_{\mathrm{rel}}(g) = \int_\Omega |\nabla g| + 2\pi L(g), \quad \forall\, g \in W^{1,1}(\Omega; S^1). \tag{3.10}$$

Proof of Theorem 1. Inequality "$\leq$" in (3.10) was proved in [DH] when Ω is a smooth bounded open set in $\mathbb{R}^2$, and their argument could be easily adapted to our situation. Here is another way. By Theorem 2, we may find some $\varphi_1 \in BV$ such that $g = \mathrm{e}^{i\varphi_1}$ and

$$|g \wedge \nabla g - D\varphi_1|_{\mathcal{M}} = 2\pi L(g).$$

Combining with Lemma 5 yields

$$|D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + 2\pi L(g).$$

By Proposition 2, we finally get

$$E_{\mathrm{rel}}(g) \leq |D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + 2\pi L(g).$$

For the reverse inequality "$\geq$" in (3.10), we argue as follows. By Proposition 2, we know that

$$E_{\mathrm{rel}}(g) = |D\varphi_0|_{\mathcal{M}}$$

for some $\varphi_0 \in BV(\Omega;\mathbb{R})$ such that $g = \mathrm{e}^{i\varphi_0}$. By Lemma 5 and Theorem 2, we have

$$|D\varphi_0|_{\mathcal{M}} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi_0|_{\mathcal{M}} \geq |g|_{W^{1,1}} + 2\pi L(g).$$

Corollary 6. *For each $g \in W^{1,1}(\Omega; S^1)$, there is some $\varphi \in BV(\Omega;\mathbb{R})$ such that $g = \mathrm{e}^{i\varphi}$ a.e. and $|\varphi|_{BV} \leq 2|g|_{W^{1,1}}$.*

Corollary 6 is a special case of a much more general result of Dávila and Ignat [DI] which asserts that the same conclusion holds for maps $g \in BV(\Omega; S^1)$.

Proof. The corollary follows from Proposition 2, Theorem 1 and the inequality $L(g) \leq \dfrac{1}{2\pi}|g|_{W^{1,1}}$, $\forall\, g \in W^{1,1}(\Omega; S^1)$ (this last estimate is an immediate consequence of the definition (1.9) of $L(g)$).

We now present a coarea type formula proved in [BBM2], which relates the quantity $\langle T(g), \zeta\rangle$ and the degree of $g \in H^{1/2}(\Omega; S^1)$ with respect to the level sets of ζ (in [BBM2] the result is stated for $H^{1/2}$-maps, but it is actually proved for $W^{1,1}$). More precisely, let $\zeta \in C^\infty(\Omega; \mathbb{R})$. If $\lambda \in \mathbb{R}$ is a regular value of ζ, let

$$\Gamma_\lambda = \big\{ x \in \Omega \; ; \; \zeta(x) = \lambda \big\}.$$

We orient Γ_λ such that, for each $x \in \Gamma_\lambda$, the basis $\big(\tau(x), \nabla\zeta(x), n(x)\big)$ is direct, where $n(x)$ denotes the outward normal to Ω at x.

Given $g \in H^{1/2}(\Omega; S^1)$, the restriction of g to the level set Γ_λ belongs to $W^{1,1} \subset C^0$ for a.e. λ ; this follows from the coarea formula. Therefore, $\deg(g; \Gamma_\lambda)$ makes sense for a.e. λ, and Γ_λ is a union of simple curves, say $\Gamma_\lambda = \bigcup \gamma_j$; then we set

$$\deg(g; \Gamma_\lambda) = \sum \deg(g; \gamma_j).$$

In [BBM2], the authors proved that for every $g \in W^{1,1}(\Omega; S^1)$ we have

$$\langle T(g), \zeta \rangle = 2\pi \int_{\mathbb{R}} \deg(g; \Gamma_\lambda)\, d\lambda. \tag{3.11}$$

We point out that this formula still holds if $\zeta \in \mathrm{Lip}(\Omega; \mathbb{R})$. If we assume in addition that $|\zeta|_{\mathrm{Lip}} \le 1$, then a simple corollary of (3.11) is the inequality :

$$\left| \int_{\mathbb{R}} \deg(g; \Gamma_\lambda)\, d\lambda \right| \le L(g). \tag{3.12}$$

The novelty in Theorem 4 is that this estimate remains true if one replaces $\deg(g; \Gamma_\lambda)$ by its absolute value inside the integral in (3.12).

Proof of Theorem 4. We shall first establish (1.12) for functions g in the class $\mathcal{R}$, and then we argue by density.

Let $g \in \mathcal{R}$ and $\zeta \in \mathrm{Lip}(\Omega; \mathbb{R})$, with $|\zeta|_{\mathrm{Lip}} \le 1$. By Lemma 3, we can find finitely many points P_i, N_i such that

$$T(g) = 2\pi \sum_{i=1}^{k} (\delta_{P_i} - \delta_{N_i}).$$

Let $\lambda \in \mathbb{R}$ be a regular value of ζ such that $\lambda \ne \zeta(P_i), \zeta(N_i)$ for any $i \in \{1, \dots, k\}$. Then, we have

$$\deg(g; \Gamma_\lambda) = \mathrm{card}\big\{ i \; ; \; \zeta(P_i) > \lambda \big\} - \mathrm{card}\big\{ i \; ; \; \zeta(N_i) > \lambda \big\},$$

so that

$$\deg(g; \Gamma_\lambda) = \frac{1}{2} \sum_{i=1}^{k} \Big\{ \mathrm{sgn}\big[\zeta(P_i) - \zeta\big] - \mathrm{sgn}\big[\zeta(N_i) - \zeta\big] \Big\}.$$

After relabeling the negative points N_i if necessary, we can assume that $L(g) = \sum_{i=1}^{k} d(P_i, N_i)$. Let γ_i be a geodesic arc in Ω connecting P_i to N_i. Clearly,

$$\frac{1}{2}\Big| \operatorname{sgn}\big[\zeta(P_i) - \zeta\big] - \operatorname{sgn}\big[\zeta(N_i) - \zeta\big] \Big| \leq \operatorname{card}\big\{x \in \gamma_i \ ; \ \zeta(x) = \lambda\big\}.$$

Using the area formula, we obtain

$$\int_{\mathbb{R}} |\deg{(g; \Gamma_\lambda)}|\, d\lambda \leq \sum_{i=1}^{k} \int_{\mathbb{R}} \operatorname{card}\big\{x \in \gamma_i \ ; \ \zeta(x) = \lambda\big\}\, d\lambda = \sum_{i=1}^{k} \int_{\gamma_i} \left|\frac{\partial \zeta}{\partial \tau}\right| \leq L(g).$$

This establishes (1.12) for maps $g \in \mathcal{R}$.

For a general $g \in W^{1,1}(\Omega; S^1)$, it follows from Lemma 2 that we can find a sequence $(g_n) \subset \mathcal{R}$ such that $g_n \to g$ strongly in $W^{1,1}$. In particular, by Lemma 1 d) we have

$$L(g_n) \to L(g).$$

Passing to a subsequence, we may assume that $u_{n|\Gamma_\lambda}$ converges to $u_{|\Gamma_\lambda}$ in $W^{1,1}$, and hence uniformly, for a.e. λ. Thus,

$$\deg{(g_n; \Gamma_\lambda)} \to \deg{(g; \Gamma_\lambda)} \quad \text{for a.e. } \lambda.$$

Applying Fatou's lemma, we find

$$\int_{\mathbb{R}} |\deg{(g; \Gamma_\lambda)}|\, d\lambda \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |\deg{(g_n; \Gamma_\lambda)}|\, d\lambda \leq \lim_{n\to\infty} L(g_n) = L(g).$$

This proves (1.12). Note that (1.13) follows immediately from (1.12). In fact, if ζ maximizes (1.9), then

$$L(g) = \int_{\mathbb{R}} \deg{(g; \Gamma_\lambda)}\, d\lambda \leq \int_{\mathbb{R}} |\deg{(g; \Gamma_\lambda)}|\, d\lambda \leq L(g).$$

Therefore, $\deg{(g; \Gamma_\lambda)} = |\deg{(g; \Gamma_\lambda)}| \geq 0$ for a.e. λ.

Given two (infinite) sequences of points (P_i) and (N_i) in Ω such that

$$\sum_{i=1}^{\infty} d(P_i, N_i) < \infty, \tag{3.13}$$

we may introduce the distribution

$$T = 2\pi \sum_{i=1}^{\infty} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } (W^{1,\infty})^*, \tag{3.14}$$

and the number

$$L = \frac{1}{2\pi} \operatorname*{Max}_{|\zeta|_{\mathrm{Lip}} \le 1} \langle T, \zeta \rangle, \tag{3.15}$$

where the best Lipschitz constant $|\zeta|_{\mathrm{Lip}}$ refers to the geodesic distance d in Ω. The distribution T admits many representations, and it has been proved in [BBM2, Lemma 12′] (see also [P]) that

$$L = \operatorname{Inf} \Big\{ \sum_j d(\tilde{P}_j, \tilde{N}_j) \; ; \; \sum_j (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \text{ in } (W^{1,\infty})^* \Big\}.$$

We also recall that if the sequences $(P_i), (N_i)$ consist of a **finite** number of points $P_1, P_2, \dots, P_k$, $N_1, N_2, \dots, N_k$, then

$$L = \operatorname*{Min}_{\sigma} \sum_{i=1}^{k} d(P_i, N_{\sigma(i)}), \tag{3.16}$$

where the minimum in (3.16) is taken over all permutations of the integers $\{1, 2, \dots, k\}$.

In our next result, we are **given** points $(P_i), (N_i)$ satisfying (3.13), and we ask what is the least "$W^{1,1}$-energy" needed to produce singularities of degree $+1$ at the points P_i, and degree -1 at the points N_i ; more precisely, we consider the class of all maps g in $W^{1,1}(\Omega; S^1)$ such that

$$T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}). \tag{3.17}$$

[We know (see Lemma 16 in [BBM2]) that such class of maps g is not empty.]

The answer is given by

Theorem 10. *Let $P_i, N_i \in \Omega$ be such that $\sum_i d(P_i, N_i) < \infty$. Then*

$$\operatorname{Inf} \Big\{ \int_\Omega |\nabla g| \; ; \; g \in W^{1,1}(\Omega; S^1) \textit{ satisfying } (3.17) \Big\} = 2\pi L. \tag{3.18}$$

In particular,

(3.19)
$$\begin{aligned} d(P,N) &= \frac{1}{2\pi}\,\mathrm{Inf}\left\{\int_\Omega |\nabla g|\ ;\ g\in W^{1,1}(\Omega;S^1),\ T(g)=2\pi(\delta_P-\delta_N)\right\} \\ &= \frac{1}{2\pi}\,\mathrm{Inf}\left\{\int_\Omega |\nabla g|\ \middle|\ \begin{array}{l} g\in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{P,N\};S^1), \\ \deg(g,P)=+1 \ \textit{and}\ \deg(g,N)=-1 \end{array}\right\}. \end{aligned}$$

Proof. Given P_i, N_i as above, we fix some $g_0 \in W^{1,1}(\Omega;S^1)$ such that

$$T(g_0) = T = 2\pi\sum_i(\delta_{P_i}-\delta_{N_i}).$$

By Lemma 4, for each $\varepsilon > 0$ we may find a map $h \in W^{1,1}(\Omega;S^1)$ such that $T(h) = T(g_0) = T$ and

$$\int_\Omega |\nabla h| \le 2\pi L(g_0) + \varepsilon = 2\pi L + \varepsilon,$$

which implies "$\le$" in (3.18). Inequality "$\ge$" in (3.18) follows from Lemma 1 c).

To prove the second equality in (3.19), it suffices to apply Lemma 15 in [BBM2].

In view of Theorem 10, it is natural to define, for every $P, N \in \Omega$,

$$\rho(P,N) = \frac{1}{2\pi}\,\mathrm{Inf}\left\{[g]_{W^{1,1}}\ ;\ g\in W^{1,1}(\Omega;S^1),\ T(g)=2\pi(\delta_P-\delta_N)\right\}.$$

Here, $[\ ]_{W^{1,1}}$ is a general given semi-norm on $W^{1,1}(\Omega;\mathbb{C})$ equivalent to $|\ |_{W^{1,1}}$. Of course, ρ depends on the choice of $[\ ]_{W^{1,1}}$. We require from $[\ ]_{W^{1,1}}$ some structural properties :

(P1) $[\alpha g]_{W^{1,1}} = [g]_{W^{1,1}},\ \forall\ g\in W^{1,1}(\Omega;\mathbb{C}),\ \forall\ \alpha\in S^1$;

(P2) $[\bar g]_{W^{1,1}} = [g]_{W^{1,1}},\ \forall\ g\in W^{1,1}(\Omega;\mathbb{C})$;

(P3) $[gh]_{W^{1,1}} \le \|g\|_{L^\infty}[h]_{W^{1,1}} + \|h\|_{L^\infty}[g]_{W^{1,1}},\ \forall\ g,h\in W^{1,1}(\Omega;\mathbb{C})\cap L^\infty$.

It follows easily from (P3) that ρ is a distance.

Example 1. The semi-norm

$$[g]_{W^{1,1}} = \int_\Omega |\nabla g|\, w,$$

where w is a positive smooth function defined on Ω, satisfies (P1), (P2) and (P3). Exercise : compute ρ in this case.

One may define a new relaxed energy associated to $[\ ]_{W^{1,1}}$ by setting, for every $g \in W^{1,1}(\Omega; S^1)$,

$$\widetilde{E}_{\rm rel}(g) = \operatorname{Inf}\Big\{ \liminf_{n\to\infty} [g_n]_{W^{1,1}}\ ;\ g_n \in C^\infty(\Omega; S^1),\ g_n \to g \text{ a.e.}\Big\},$$

and also

$$\widetilde{L}(g) = \frac{1}{2\pi} \operatorname{Sup}\Big\{ \langle T(g), \zeta\rangle\ ;\ \big|\zeta(x) - \zeta(y)\big| \le \rho(x,y),\ \forall\ x, y \in \Omega\Big\}.$$

We end this section with the following

Open Problem 4. Is it true that, for every $g \in W^{1,1}(\Omega; S^1)$,

$$\widetilde{E}_{\rm rel}(g) = [g]_{W^{1,1}} + 2\pi \widetilde{L}(g)\ ?$$

4. $W^{1,1}(\Omega; S^1)$ and relaxed Jacobians.

Given any function $g \in W^{1,p}(\Omega; \mathbb{R}^2)$, with $p \ge 1$, a natural concept associated to g is the following

$$TV_\tau(g) = \operatorname{Inf}\left\{ \liminf_{n\to\infty} \int_\Omega |g_{nx} \wedge g_{ny}|\ ;\ g_n \in C^\infty(\Omega; \mathbb{R}^2),\ g_n \to g \text{ with respect to } \tau\right\},$$

for some topology τ.

There are several topologies τ of interest. For example, given $1 \le p < 2$ and $g \in W^{1,p}(\Omega; \mathbb{R}^2)$, we consider

$$TV_{p,s}(g) = TV \text{ computed with respect to the strong } W^{1,p}\text{-topology},$$
$$TV_{p,w}(g) = TV \text{ computed with respect to the weak } W^{1,p}\text{-topology}.$$

In the case $p = 1$, for every $g \in W^{1,1}(\Omega; \mathbb{R}^2)$, we also define

$$TV_{1,w^*}(g) = TV \text{ computed with respect to the weak}^*\ W^{1,1}\text{-topology}.$$

In what follows, we are going to work with the weak $W^{1,1}$-topology and simply write TV for the total variation $TV_{1,w}$. But we will also state results for $TV_{p,w}$ and $TV_{p,s}$ for every $1 \le p < 2$, and for TV_{1,w^*} ; see Remarks 10 and 12 below.

Let us start with a simple

Proposition 3. *Assume* $g \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L^\infty$ *and* $TV(g) < \infty$. *Then* $\operatorname{Det}(\nabla g) \in \mathcal{M}(\Omega)$ *and*

$$|\operatorname{Det}(\nabla g)|_{\mathcal{M}} \le TV(g). \tag{4.1}$$

Recall that $\operatorname{Det}(\nabla g)$ is the distributional Jacobian of g and that $T(g) = 2\operatorname{Det}(\nabla g)$ (see (1.8)).

Proof. Given $\varepsilon > 0$, there exists a sequence $(g_n) \subset C^\infty(\Omega;\mathbb{R}^2)$ such that

$$g_n \rightharpoonup g \quad \text{weakly in } W^{1,1}, \tag{4.2}$$

$$\int_\Omega |g_{nx} \wedge g_{ny}| \le TV(g) + \varepsilon, \quad \forall n. \tag{4.3}$$

Let $M = \|g\|_{L^\infty}$ and let $P : \mathbb{R}^2 \to B_M$ be the orthogonal projection onto B_M. Set $\tilde{g}_n = Pg_n$. It is easy to see (using Dunford-Pettis' theorem) that $\tilde{g}_n$ satisfies (4.2) and (4.3). Moreover, by a standard regularization argument, we may assume that the functions $\tilde{g}_n$ are smooth. In what follows, we will denote $\tilde{g}_n$ by g_n, and so we also have

$$\|g_n\|_{L^\infty} \le \|g\|_{L^\infty}. \tag{4.4}$$

We claim that

$$g_n \wedge \nabla g_n \rightharpoonup g \wedge \nabla g \quad \text{weakly in } L^1.$$

In fact, it suffices to notice that

$$\int_\Omega |g_n - g||\nabla g_n| \to 0,$$

which follows from Egorov's and Dunford-Pettis' theorems. Hence

$$g_{nx} \wedge g_{ny} = \frac{1}{2}\Big[(g_n \wedge g_{ny})_x + (g_{nx} \wedge g_n)_y\Big]$$

converges to $\operatorname{Det}(\nabla g)$ in the sense of distributions. We deduce from (4.3) that $\operatorname{Det}(\nabla g) \in \mathcal{M}(\Omega)$ and that (4.1) holds.

Remark 10. The conclusion of Proposition 3 is no longer true if we compute the total variation of g with respect to the weak*-topology of $W^{1,1}$, $TV_{1,w^*}(g)$. In fact, assume $g \in W^{1,1}(\Omega;S^1)$. It follows from Corollary 5 that there exists $(g_n) \subset C^\infty(\Omega;S^1)$ such that $g_n \overset{*}{\rightharpoonup} g$ in $W^{1,1}$. Since $g_{nx} \wedge g_{ny} = 0$ for each n, we conclude that $TV_{1,w^*}(g) = 0$. On the other hand, for some maps g in $W^{1,1}(\Omega;S^1)$ we have

$\operatorname{Det}(\nabla g) = \dfrac{1}{2}T(g) \neq 0$; see Theorem 11 below. A fortiori, the conclusion of Proposition 3 fails if τ is the strong L^1-topology (or the convergence pointwise a.e.).

In general, the inequality in (4.1) is strict. This fact was pointed out by an example in [M] ; see also [GMS1]. There, the map $g \in W^{1,1}(\Omega;\mathbb{R}^2)$ takes its values in an eight-shaped curve and satisfies $\operatorname{Deg}(\nabla g) = 0$ in the sense of distributions, while $TV(g) > 0$. It is therefore remarkable that equality in (4.1) holds whenever the map g takes its values in S^1. This is the content of our next result, which is stronger than Theorem 5 :

Theorem 11. *Assume $g \in W^{1,p}(\Omega;S^1)$, $1 \leq p < 2$, is such that $\operatorname{Det}(\nabla g) \in \mathcal{M}$. Then there exists a sequence $(g_n) \subset C^\infty(\Omega;\mathbb{R}^2)$ such that*

$$g_n \to g \quad \textit{strongly in } W^{1,p}$$

and

$$TV(g) = \lim_{n\to\infty} \int_\Omega |g_{nx} \wedge g_{ny}| = |\operatorname{Det}(\nabla g)|_{\mathcal{M}}.$$

Moreover, in this case,

$$\operatorname{Det}(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}).$$

In particular, $\dfrac{1}{\pi}|\operatorname{Det}(\nabla g)|_{\mathcal{M}}$ equals the number of topological singularities of g, taking into account their multiplicities.

Remark 11. Theorem 11 extends and clarifies some of the results of [FFM]. Although in their case Ω is a smooth bounded domain in $\mathbb{R}^2$, the above results, stated for $\Omega = \partial G$, adapt easily to bounded domains ; see Section 5.2 below.

Proof of Theorem 11. The fact that

$$\operatorname{Det}(\nabla g) \quad \text{measure} \quad \Longrightarrow \quad \operatorname{Det}(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i})$$

is a consequence of Theorem 3 and a result of Smets [S] ; see also [P]. Let us assume, for simplicity, that $\operatorname{Det}(\nabla g) = \pi(\delta_P - \delta_N)$; the argument below still applies to the general case. Suppose, in addition, that Ω is flat and horizontal near P and N. We start by defining, near P and N, a map h by setting

$$h(x) = \left(\frac{x-P}{|x-P|}\right)^{\pm 1} \quad \text{near } P, \qquad h(x) = \left(\frac{x-N}{|x-N|}\right)^{\mp 1} \quad \text{near } N.$$

For appropriate choices of $\pm$, we have $\deg(h,P) = +1$ and $\deg(h,N) = -1$. Then h extends to a map in $C^\infty(\Omega\backslash\{P,N\}; S^1) \cap W^{1,p}(\Omega; S^1)$, $1 \le p < 2$. Set

$$h_n(x) = \begin{cases} h(x), & \text{if } d(x,P) \ge 1/n \text{ and } d(x,N) \ge 1/n \\ n\,d(x,P)h(x), & \text{if } d(x,P) < 1/n \\ n\,d(x,N)h(x), & \text{if } d(x,N) < 1/n \end{cases}.$$

Clearly, $h_n \to h$ in $W^{1,p}$ and

$$\int_\Omega |h_{nx} \wedge h_{ny}| = 2\pi.$$

Let $k = g\bar{h}$. Since $T(k) = 0$, we may write $k = \mathrm{e}^{i\varphi}$ for some $\varphi \in W^{1,1}$ (see Theorem 7). Moreover, $g, h \in W^{1,p} \cap L^\infty$ implies $k \in W^{1,p}$. From this, we easily conclude that $\varphi \in W^{1,p}$.

Let $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$ be such that $\varphi_n \to \varphi$ in $W^{1,p}$. Since a point has zero $W^{1,2}$-capacity, we may also assume that $\varphi_n(x) = 0$ if $d(x,P) \le 1/n$ or $d(x,N) \le 1/n$. Clearly, $g_n = h_n \mathrm{e}^{i\varphi_n}$ belongs to $C^\infty(\Omega; \mathbb{R}^2)$ and $g_n \to g$ in $W^{1,p}$. Since $g_{nx} \wedge g_{ny} = h_{nx} \wedge h_{ny}$, we obtain

$$\int_\Omega |g_{nx} \wedge g_{ny}| = 2\pi = |\operatorname{Det}(\nabla g)|_{\mathcal{M}},$$

which shows that

$$TV(g) \le |\operatorname{Det}(\nabla g)|_{\mathcal{M}}.$$

The reverse inequality follows from Proposition 3.

Remark 12. Theorem 11 and Proposition 3 imply that, for every $p \in [1,2)$,

$$TV_{p,w}(g) = TV_{p,s}(g) = TV(g), \quad \forall\, g \in W^{1,p}(\Omega; S^1).$$

We do not know whether the same holds without assuming that g is S^1-valued :

Open Problem 5. Let $g \in W^{1,1}(\Omega; \mathbb{R}^2)$. Is it true that

$$TV_{1,w}(g) = TV_{1,s}(g)\,?$$

Assume in addition that $g \in W^{1,p}(\Omega; \mathbb{R}^2)$ for some $1 < p < 2$. Does one have

$$TV_{1,w}(g) = TV_{1,s}(g) = TV_{p,w}(g) = TV_{p,s}(g)\,?$$

Remark 13. The analog of Remark 12 for $p \ge 2$ is true, but uninteresting. Indeed, every $g \in W^{1,p}(\Omega; S^1)$, with $p \ge 2$, is a strong limit in $W^{1,p}$ of a sequence (g_n) in $C^\infty(\Omega; S^1)$ (see, e.g., [BZ]). Thus, $TV(g) = 0$ and $TV_{p,w}(g) = TV_{p,s}(g) = 0$ for every $g \in W^{1,p}(\Omega; S^1)$.

5. Further directions and open problems.

5.1. Some examples of BV-functions with jumps.

It is natural to try to extend the above (or part of the above) results to the class of maps g in $BV(\Omega; S^1)$, where $\Omega = \partial G$, $G \subset \mathbb{R}^3$ as in the Introduction. Every $g \in BV(\Omega; S^1)$ admits a lifting $\varphi \in BV(\Omega; \mathbb{R})$ (see [GMS2] and also [DI]). Hence, we may define the two quantities $E(g)$ and $E_{\text{rel}}(g)$ as in (1.3) and (1.4), and we always have $E(g) = E_{\text{rel}}(g)$. The difficulty starts when we try to find a simple formula for E as in Theorem 1. To illustrate the heart of the difficulty, it is worthwhile to start, as in Section 2, with the simpler case $BV(S^1; S^1)$.

Clearly, every $g \in BV(S^1; S^1)$ admits a lifting $\varphi \in BV(S^1; \mathbb{R})$. Hence we may define the two quantities $E(g)$ and $E_{\text{rel}}(g)$ as in (2.1) and (2.2), and we always have $E(g) = E_{\text{rel}}(g)$. It is natural to ask for an explicit formula for $E(g)$. For S^1-valued maps, there are two natural ways of defining the BV-norm of g :

$$|g|_{BV} = \int_{S^1} |\dot{g}|$$

and

$$|g|_{BVS^1} = \int_{S^1} \big(|\dot{g}_{ac}| + |\dot{g}_C|\big) + \sum_n d_{S^1}(g(a_n+), g(a_n-)),$$

where d_{S^1} denotes the geodesic distance on S^1. It is easy to see that

$$|g|_{BV} = \operatorname{Inf}\left\{\liminf_{n\to\infty} \int_{S^1} |\dot{g}_n| \ ; \ g_n \in C^\infty(S^1; \mathbb{R}^2) \text{ and } g_n \to g \text{ a.e.}\right\},$$

$$|g|_{BVS^1} = \operatorname{Inf}\left\{\liminf_{n\to\infty} \int_{S^1} |\dot{g}_n| \ ; \ g_n \in C^\infty(S^1; S^1) \text{ and } g_n \to g \text{ a.e.}\right\}.$$

We also have, for every $g \in BV(S^1; S^1)$,

$$E(g) \geq |g|_{BVS^1} \geq |g|_{BV}.$$

Moreover $E(g) - |g|_{BV} = 0 \iff g \in C^0$ and $\deg g = 0$. R. Ignat [I] has recently obtained an explicit formula for $E(g) - |g|_{BVS^1}$ involving the jumps of $g \in BV$ and a kind of degree in the sense of Definition 2 below.

An interesting estimate for $E(g)$ when $g \in BV$ is the following

Theorem 12. *For every $g \in BV(S^1; S^1)$, we have*

$$E(g) \leq 2|g|_{BV}. \tag{5.1}$$

The above result is a variant of a nice theorem of [DI] which asserts that if $u \in BV(U; S^1)$, where U is a domain in $\mathbb{R}^N$, then $u = \mathrm{e}^{i\varphi}$ for some $\varphi \in BV(U; \mathbb{R})$ with $|\varphi|_{BV} \leq 2|g|_{BV}$. The proof of Theorem 12 is a straightforward adaptation of the ingenious method in [DI]. Surprisingly, the natural proof of (5.1) — via the explicit formula [I] for $E(g)$ — turns out to be quite involved (see [I]) !

As we have already pointed out in Remark 2, the constant 2 in Theorem 12 is optimal in $W^{1,1}$. A less intuitive fact is that the constant 2 is also optimal for piecewise constant functions. Here is an example :

Example 2. Fix an integer $k \geq 1$ and set

$$g(\theta) = \mathrm{e}^{i2\pi j/k} \quad \text{for } \frac{2\pi j}{k} < \theta < \frac{2\pi(j+1)}{k}, \quad j = 0, 1, \ldots, k-1.$$

Then

$$|g|_{BV} = 2k \sin\frac{\pi}{k} \quad \text{and} \quad E(g) = 4\pi - \frac{4\pi}{k}.$$

The inequality

$$E(g) \leq 4\pi - \frac{4\pi}{k}$$

is straightforward ; however, the reverse inequality is more delicate and relies on the following lemma whose proof is left to the reader

Lemma 6. *For every choice of $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}$ with $\sum_j \alpha_j = 1$, we have*

$$\sum_{j=1}^{k} \left| \frac{1}{k} - \alpha_j \right| \geq 2 - \frac{2}{k}.$$

A striking difference with formula (2.3) is that neither $\frac{1}{2\pi}(E(g) - |g|_{BV})$ nor $\frac{1}{2\pi}(E(g) - |g|_{BVS^1})$ is necessarily an integer. Here is an example :

Example 3. Let

$$g(\theta) = \begin{cases} 1, & \text{for } 0 < \theta < 2\pi/3 \\ \mathrm{e}^{i2\pi/3}, & \text{for } 2\pi/3 < \theta < 4\pi/3 \\ \mathrm{e}^{i4\pi/3}, & \text{for } 4\pi/3 < \theta < 2\pi \end{cases}.$$

An easy computation shows that

$$E(g) = \frac{8\pi}{3}, \quad |g|_{BV} = 3\sqrt{3} \quad \text{and} \quad |g|_{BVS^1} = 2\pi.$$

In fact, it is hopeless (?) to have an analog of Theorem 6 since there is no reasonable notion of degree for maps in $BV(S^1; S^1)$. This is a consequence of

Theorem 13. *The space $BV(S^1;S^1)$ is path-connected.*

Proof. Let $\varphi \in BV(S^1;\mathbb{R})$ be such that $g = \mathrm{e}^{i\varphi}$. We claim that the map

$$F : t \in [0,1] \longmapsto \mathrm{e}^{it\varphi} \in BV(S^1;S^1) \tag{5.2}$$

is strongly continuous ; this implies that every map in $BV(S^1;S^1)$ can be connected to 1.

The continuity of F in (5.2) follows from

Lemma 7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that :*

(i) $t \mapsto f(t,x)$ is continuous, $\forall\ x \in \mathbb{R}$;

(ii) f_x is continuous and bounded.

Then, for every $\varphi \in BV(\Omega;\mathbb{R})$, the map

$$t \mapsto f(t,\varphi) \in BV(\Omega;\mathbb{R})$$

is continuous.

Proof. It suffices to establish continuity at $t = 0$. Set $F(t) = f(t,\varphi)$. For every t, we have $F(t) \in BV(\Omega;\mathbb{R})$. Let $C > 0$ be such that $|f_x(t,x)| \le C$, $\forall\ t$, $\forall\ x$.

Since
$$\big|f(t,x)\big| \le \big|f(t,0)\big| + C|x|,$$
we find that $F(t) \to F(0)$ in $L^1(\Omega)$ as $t \to 0$. Therefore, it suffices to prove that $DF(t) \to DF(0)$ in $\mathcal{M}(\Omega)$. By the chain rule, we have

$$DF(t) = f_x\big(t,\varphi(x)\big)(D\varphi)_d + \frac{f(t,\varphi^+) - f(t,\varphi^-)}{\varphi^+ - \varphi^-}(D\varphi)_J.$$

Thus, $|DF(t)| \le C|D\varphi|$, $\forall\ t$. On the other hand, $f_x\big(t,\varphi(x)\big) \to f_x\big(0,\varphi(x)\big)$ a.e. with respect to $(D\varphi)_d$. Moreover,

$$\frac{f(t,\varphi^+) - f(t,\varphi^-)}{\varphi^+ - \varphi^-} \to \frac{f(0,\varphi^+) - f(0,\varphi^-)}{\varphi^+ - \varphi^-}$$

a.e. with respect to $(D\varphi)_J$. Therefore,

$$\big|D\varphi(t) - D\varphi(0)\big|_{\mathcal{M}} \to 0 \quad \text{as } t \to 0,$$

by dominated convergence.

There is however an interesting concept of multivalued degree which associates to every $g \in BV(S^1;S^1)$ a bounded subset of $\mathbb{Z}$. The starting point is the following

Definition 1. *Let $g \in BV(I; S^1)$, where I is an interval. A canonical lifting of g is any map $\varphi \in BV(I; \mathbb{R})$ such that*

$$g = \mathrm{e}^{i\varphi} \quad \textit{and} \quad E(g) = |D\varphi|_{\mathcal{M}(I)}.$$

The structure of canonical liftings is quite rigid. In fact, the following holds :

Theorem 14. *If φ_1 and φ_2 are two canonical liftings of the same map g, then*

$$\dot{\varphi}_1 - \dot{\varphi}_2 = \pi \sum_{\text{finite}} \pm\delta_{a_i}.$$

Moreover, if $g \in BV \cap C^0$, then the canonical lifting is uniquely determined modulo 2π and coincides with a continuous lifting.

Using canonical liftings, we may define a multivalued degree for all maps in $BV(S^1; S^1)$:

Definition 2. *Let $g \in BV(S^1; S^1)$. Assume g is continuous at $z \in S^1$. We let*

$$\operatorname{Deg}_1 g = \left\{ \frac{\varphi(z-) - \varphi(z+)}{2\pi} \ ; \ \varphi \textit{ is a canonical lifting of } g \textit{ in } S^1 \backslash \{z\} \right\}.$$

Since, clearly, for each canonical lifting we have

$$\left| \frac{\varphi(z-) - \varphi(z+)}{2\pi} \right| \leq \frac{1}{2\pi} \int_{S^1} |\dot{\varphi}|,$$

the set $\operatorname{Deg}_1 g$ is bounded. It follows from the second part of Theorem 14 that $\operatorname{Deg}_1 g = \{\deg g\}$ if $g \in BV \cap C^0$. As another example, let

$$g(\theta) = \begin{cases} 1, & \text{if } 0 < \theta < \pi, \\ -1, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

Then it is easy to see that $\operatorname{Deg}_1 g = \{-1, 0, 1\}$.

We collect below some properties of Deg_1 :

Theorem 15. *Assume $g \in BV(S^1; S^1)$. Then,*

(a) $\operatorname{Deg}_1 g$ *is a finite set of successive integers ;*

(b) $\operatorname{Deg}_1 g$ *is independent of the choice of z.*

Another possible definition of a multivalued degree is the following

Definition 3. *Given $g \in BV(S^1; S^1)$, we set*

$$\mathrm{Deg}_2\, g = \left\{ d \ ; \ \exists\ (g_n) \subset C^\infty(S^1; S^1) \text{ such that } g_n \to g \text{ a.e.}, \int |\dot g_n| \to \int |\dot g|,\ \deg g_n = d \right\}.$$

Actually, both definitions yield the same degree :

Theorem 16. *We have*

$$\mathrm{Deg} := \mathrm{Deg}_1 = \mathrm{Deg}_2\,.$$

Moreover, the function $g \mapsto \mathrm{Deg}\, g$ is continuous in the multivalued sense.

A final interesting property of Deg is that it is "almost always" single-valued :

Theorem 17. *Let*

$$\mathcal{U} = \left\{ g \in BV(S^1; S^1) \ ; \ \mathrm{Deg}\, g \text{ is single-valued} \right\}.$$

Then $\mathcal{U}$ is a dense open subset of $BV(S^1; S^1)$.

We omit the proofs of Theorems 14-17 and we refer the reader to [BMP] for details.

5.2. Some analogs of Theorems 1, 3, and 5 for bounded domains in $\mathbb{R}^2$.

Most of the above results admit counterparts in the case where the 2-d manifold Ω is replaced by a bounded, simply connected domain in $\mathbb{R}^2$ with smooth boundary. To illustrate this, we state the analogs of the main results ; namely, Theorems 1, 3 and 5.

Let $g \in W^{1,1}(\Omega; S^1)$ and consider the distribution

$$\langle T(g), \zeta \rangle = \int_\Omega (g \wedge \nabla g) \cdot \nabla^\perp \zeta, \quad \forall\ \zeta \in W_0^{1,\infty}(\Omega; S^1).$$

A natural (semi-) metric on $\overline{\Omega}$ is given by

$$d_\Omega(x, y) = \mathrm{Min} \left\{ |x - y| \ , \ d(x, \partial\Omega) + d(y, \partial\Omega) \right\}.$$

Note that, if $\zeta \in W_0^{1,\infty}(\Omega)$, then

$$\big|\zeta(x) - \zeta(y)\big| \le \|\nabla \zeta\|_{L^\infty} d_\Omega(x, y), \quad \forall\ x, y \in \overline{\Omega}.$$

We also set

$$L(g) = \frac{1}{2\pi} \operatorname*{Max}_{\substack{\zeta \in W_0^{\infty}(\Omega) \\ \|\nabla\zeta\|_{L^\infty} \le 1}} \langle T(g), \zeta \rangle.$$

We then have the following

Theorem 3′. *There exist sequences* $(P_i), (N_i)$ *in* $\overline{\Omega}$ *such that* $\sum_i d_\Omega(P_i, N_i) < \infty$ *and*

$$T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \big[W_0^{1,\infty}(\Omega)\big]^*.$$

Moreover,

$$L(g) = \operatorname{Inf} \sum_i d_\Omega(P_i, N_i),$$

where the infimum is taken over all possible representations of $T(g)$.

With $E(g)$ defined exactly as in (1.3), and $E_{\mathrm{rel}}(g)$ as in (1.4) (where Ω is replaced by $\overline{\Omega}$), we have

Theorem 1′. *For every* $g \in W^{1,1}(\Omega; S^1)$,

$$E(g) = E_{\mathrm{rel}}(g) = \int_\Omega |\nabla g| + 2\pi L(g).$$

Similarly, defining $TV(g)$ as in (1.14) (with Ω replaced by $\overline{\Omega}$), we also have

Theorem 5′. *Let* $g \in W^{1,1}(\Omega; S^1)$. *Then*

$$TV(g) < \infty \qquad \Longleftrightarrow \qquad \mathrm{Det}(\nabla g) \in \mathcal{M}(\Omega) = \big[C_0(\overline{\Omega})\big]^*.$$

In this case, there exist a finite number of points $a_i \in \Omega$ *and integers* $d_i \in \mathbb{Z}\backslash\{0\}$ *such that*

$$\mathrm{Det}\,(\nabla g) = \pi \sum_{i=1}^k d_i \delta_{a_i} \quad \text{in } \big[W_0^{1,\infty}(\Omega)\big]^*$$

and

$$TV(g) = |\,\mathrm{Det}\,(\nabla g)|_{\mathcal{M}} = \pi \sum_{i=1}^k |d_i|.$$

Theorems 1′, 3′ and 5′ are established in [BMP].

5.3. Extensions of Theorems 1, 2, and 3 to higher dimensions.
Let $G \subset \mathbb{R}^{N+1}$, $N \geq 2$, be a smooth bounded domain and $\Omega = \partial G$. Given $u \in W^{1,N-1}(\Omega; S^{N-1})$, we define the L^1-vector field

$$D(u) = (D_1, \dots, D_N),$$

where

$$D_j = \det\,(u_{x_1}, \dots, u_{x_{j-1}}, u, u_{x_{j+1}}, \dots, u_{x_N})$$

and det refers to the determinant of an $N \times N$ matrix (u is viewed as a vector in $\mathbb{R}^N$).

We then associate to the map u the distribution

$$T(u) = \operatorname{div} D(u) = N \operatorname{Det}(\nabla u).$$

Set

$$L(u) = \frac{1}{\sigma_N} \operatorname*{Max}_{\|\nabla\zeta\|_{L^\infty} \le 1} \langle T(u), \zeta \rangle,$$

where $\sigma_N = |S^{N-1}|$. The relaxed energy is defined by

$$E_{\text{rel}}(u) = \operatorname{Inf} \left\{ \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{N-1} \ ; \ u_n \in C^\infty(\Omega; S^{N-1}) \text{ and } u_n \to u \text{ a.e.} \right\},$$

where $| \ |$ denotes the Euclidean norm.

We then have the following analogs of Theorems 1–3 :

Theorem 1″. *For every $u \in W^{1,N-1}(\Omega; S^{N-1})$,*

$$E_{\text{rel}}(u) = \int_\Omega |\nabla u|^{N-1} + (N-1)^{\frac{N-1}{2}} \sigma_N L(u).$$

Theorem 2″. *For every $u \in W^{1,N-1}(\Omega; S^{N-1})$,*

$$\operatorname*{Inf}_{v \in C^\infty(\Omega; S^{N-1})} \int_\Omega |D(u) - D(v)| = \sigma_N L(u).$$

Theorem 3″. *For every $u \in W^{1,N-1}(\Omega; S^{N-1})$, there exist sequences (P_i), (N_i) in Ω such that $\sum_i |P_i - N_i| < \infty$ and*

$$T(u) = \sigma_N \sum_i (\delta_{P_i} - \delta_{N_i}).$$

For the proofs, we refer to [BMP].

5.4. Extension of TV to higher dimensions and to fractional Sobolev spaces.

Let Ω and u be as in Section 5.3. Set, for $u \in W^{1,N-1}(\Omega; S^{N-1})$,

(5.3)

$$TV(u) = \operatorname{Inf} \left\{ \liminf_{n\to\infty} \int_\Omega |\det \nabla u_n| \ ; \ u_n \in C^\infty(\Omega; \mathbb{R}^N) \text{ and } u_n \to u \text{ in } W^{1,N-1} \right\}.$$

The analog of Theorem 5 becomes

Theorem 5$''$. *Let $u \in W^{1,N-1}(\Omega; S^{N-1})$. Then,*

$$TV(u) < \infty \quad \Longleftrightarrow \quad \operatorname{Det}(\nabla u) \quad \textit{is a measure}$$

In this case, we have

$$\operatorname{Det}(\nabla u) = \frac{\sigma_N}{N} \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i})$$

and

$$TV(u) = |\operatorname{Det}(\nabla u)|_{\mathcal{M}}.$$

Remark 14. In the definition (5.3), one cannot replace the strong convergence in $W^{1,N-1}$ by weak convergence when $N \geq 3$. Indeed, every $u \in W^{1,N-1}(\Omega; S^{N-1})$ is a weak limit in $W^{1,N-1}$ of a sequence $(u_n) \subset C^\infty(\Omega; S^{N-1})$, when $N \geq 3$. However, one can replace in (5.3) the strong convergence of u_n in $W^{1,N-1}$ by the weak convergence of u_n in $W^{1,N-1}$ **and** the equi-integrability of $|\nabla u_n|^{N-1}$ (see [BMP]).

We may even go one step further. Let $N - 1 < p < \infty$. In [BBM3] we have defined the distribution $\operatorname{Det}(\nabla u)$ for maps $u \in W^{(N-1)/p,p}(\Omega; S^{N-1})$. By analogy with the above definitions of TV, set

$$TV(u) = \operatorname{Inf}\left\{\liminf_{n\to\infty} \int_\Omega |\det \nabla u_n| \ ; \ u_n \in C^\infty(\Omega; \mathbb{R}^N), \ u_n \to u \text{ in } W^{(N-1)/p,p}\right\}.$$

We have the following

Theorem 5$'''$. *Let $N - 1 < p \leq N$ and $u \in W^{(N-1)/p,p}(\Omega; S^{N-1})$. Then,*

$$TV(u) < \infty \quad \Longleftrightarrow \quad \operatorname{Det}(\nabla u) \quad \textit{is a measure}$$

and the conclusions of Theorem 5$''$ hold.

We refer to [BMP] for the proofs of Theorems 5$''$ and 5$'''$.

Open Problem 6. Does the assertion of Theorem 5$'''$ hold when $p > N$?

Another topic to explore is the following:

Open Direction 7. Very likely, all the results of Sections 3 and 4 extend to maps $g \in W^{1,1}(S^N; S^1)$, $N \geq 3$. For example, when $N = 3$, point singularities are replaced by curves ; the analog of $L(g)$ is the area of a minimal surface spanned by these curves and the analog of $TV(g)$ is their total length. Some useful tools may be found in [ABO].

5.5. Extension of Theorem 3 to maps with values into a curve.

Let $G \subset \mathbb{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. Assume $\Gamma \subset \mathbb{R}^2$ is a smooth curve, with finitely many self-intersections. We then define

$$W^{1,1}(\Omega;\Gamma) = \Big\{ g \in W^{1,1}(\Omega;\mathbb{R}^2) \ ; \ g(x) \in \Gamma \text{ for a.e. } x \in \Omega \Big\}.$$

Given a map $g \in W^{1,1}(\Omega;\Gamma)$, we define the distribution $T(g)$ exactly as in (1.8). We denote by $A_1, \dots, A_k$ the bounded connected components of $\mathbb{R}^2 \backslash \Gamma$. We then have (see [BMP]) :

Theorem 3''''. *Given $g \in W^{1,1}(\Omega;\Gamma)$, there exist sequences $(P_{i,j})$, $(N_{i,j})$ in Ω, with $j = 1, \dots, k$, such that $\sum_{i,j} |A_j|\, d(P_{i,j}, N_{i,j}) < \infty$ and*

$$T(g) = 2 \sum_{j=1}^{k} |A_j| \sum_i (\delta_{P_{i,j}} - \delta_{N_{i,j}}). \tag{5.4}$$

There are many open directions here :

1) Does Theorem 3'''' remain valid for any smooth (or even rectifiable) curve, without assuming that the number of self-intersections of Γ is finite ?

2) What are the counterparts of Theorems 1, 2, and 5 in this general setting ?

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