COMPLEMENTS TO THE PAPER "$W^{1,1}$-MAPS WITH VALUES INTO S^1"

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The purpose of these notes is to complement some of our results in [BMP]. We also establish some of the claims we stated there without proof.

A. Extending Theorem 10 to other seminorms in $W^{1,1}(\Omega; S^1)$.

In view of Theorem 10, it is natural to introduce the following quantity

$$\rho(P, N) = \frac{1}{2\pi} \operatorname{Inf} \Big\{ [g]_{W^{1,1}} \ ; \ g \in W^{1,1}(\Omega; S^1),\ T(g) = 2\pi(\delta_P - \delta_N) \Big\}.$$

Here, $[\ ]_{W^{1,1}}$ is a general given seminorm on $W^{1,1}(\Omega; \mathbb{R}^2)$ equivalent to $|\ |_{W^{1,1}}$. We require from $[\ ]_{W^{1,1}}$ some structural properties :

(P1) $[\alpha g]_{W^{1,1}} = [g]_{W^{1,1}},\ \forall g \in W^{1,1}(\Omega; \mathbb{R}^2),\ \forall \alpha \in S^1$;

(P2) $[\bar{g}]_{W^{1,1}} = [g]_{W^{1,1}},\ \forall g \in W^{1,1}(\Omega; \mathbb{R}^2)$;

(P3) $[gh]_{W^{1,1}} \leq \|g\|_{L^\infty} [h]_{W^{1,1}} + \|h\|_{L^\infty} [g]_{W^{1,1}},\ \forall g, h \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$.

It follows from (P3) that ρ is a distance.

Alternatively, we may define ρ starting from maps in $\mathcal{R}$:

Lemma A1. *We have*

$$\rho(P, N) = \frac{1}{2\pi} \operatorname{Inf} \left\{ [g]_{W^{1,1}} \ \middle| \ \begin{array}{c} g \in C^\infty(\Omega \setminus \{P, N\}; S^1) \cap W^{1,1}, \\ \deg(g, P) = +1,\ \deg(g, N) = -1 \end{array} \right\}.$$

Proof. It suffices to prove that, for $g \in W^{1,1}(\Omega; S^1)$ such that $T(g) = 2\pi(\delta_P - \delta_N)$, we may find a sequence $(g_n) \subset \mathcal{R}$ such that

$$T(g_n) = 2\pi(\delta_P - \delta_N) \quad \text{and} \quad g_n \to g \quad \text{in } W^{1,1}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Note that the desired conclusion is invariant with respect to orientation-preserving diffeomorphisms of Ω. Therefore, we may assume that $\Omega = S^2$ and that P, N are respectively the North and the South pole of S^2. Let $h(x,y,z) = \frac{(x,y)}{|(x,y)|}$. Then

$$h \in \mathcal{R} \quad \text{and} \quad T(h) = 2\pi(\delta_P - \delta_N).$$

It follows that $k = g\bar{h} \in Y$. Thus, we may find a sequence $(k_n) \subset C^\infty(\Omega; S^1)$ such that $k_n \to k$ in $W^{1,1}$. Set $g_n = hk_n$. Then

$$g_n \in \mathcal{R}, \quad T(g_n) = 2\pi(\delta_P - \delta_N) \quad \text{and} \quad g_n \to g \quad \text{in } W^{1,1}.$$

Another simple property of ρ is

$$C_1 d(P,N) \le \rho(P,N) \le C_2 d(P,N) \tag{A1}$$

for some $C_1, C_2 > 0$ independent of P, N. This follows from the equivalence of $|\ |_{W^{1,1}}$ and $[\ ]_{W^{1,1}}$.

Part of Theorem 10 holds in this more general setting :

Theorem A1. *Let $P_i, N_i \in \Omega$ be such that*

$$\sum \rho(P_i, N_i) < \infty \quad \textit{(or, equivalently, } \sum d(P_i, N_i) < \infty\textit{)}.$$

Set

$$\widetilde{L} = \operatorname{Inf}\Big\{\sum \rho(\widetilde{P}_j, \widetilde{N}_j)\ ;\ \sum(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = \sum(\delta_{P_i} - \delta_{N_i})\Big\}.$$

Then

$$\frac{1}{2\pi}\operatorname{Inf}\Big\{[g]_{W^{1,1}}\ ;\ g \in W^{1,1}(\Omega; S^1),\ T(g) = 2\pi\sum(\delta_{P_i} - \delta_{N_i})\Big\} \le \widetilde{L}.$$

Proof. Let $\varepsilon > 0$ and let $\widetilde{P}_j, \widetilde{N}_j$ be such that

$$\sum(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = \sum(\delta_{P_i} - \delta_{N_i}) \quad \text{and} \quad \sum \rho(\widetilde{P}_j, \widetilde{N}_j) \le \widetilde{L} + \varepsilon.$$

By definition, for each j we may find some $g_j \in W^{1,1}(\Omega; S^1)$ such that

$$T(g_j) = 2\pi(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j})$$

and

$$[g_j]_{W^{1,1}} \le 2\pi\rho(\widetilde{P}_j, \widetilde{N}_j) + \frac{\varepsilon}{2^j}.$$

We claim that there is a sequence $k_n \to \infty$ such that

$$\prod_{j=1}^{k_n} g_j \to g \quad \text{in } W^{1,1},$$

for some $g \in W^{1,1}(\Omega; S^1)$. By Lemma 1, this implies that

$$T(g) = 2\pi \sum_j (\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}).$$

Using (P3), we will also have $[g]_{W^{1,1}} \leq 2\pi \widetilde{L} + \varepsilon$. Therefore, the conclusion of Theorem A1 follows if we prove the existence of the sequence (k_n). We adapt below an argument used in [BBM2]. Set

$$H = \sum_{j \geq 1} |\nabla g_j|$$

By the equivalence between $|\ |_{W^{1,1}}$ and $[\ ]_{W^{1,1}}$, we have $H \in L^1$. Since

$$\left| \nabla \left(\prod_{j=1}^{k} g_j \right) \right| \leq H, \quad \forall k,$$

we may find a sequence $k_n \to \infty$ and a map $g \in BV(\Omega; S^1)$ such that

$$h_n = \prod_{j=1}^{k_n} g_j \to g \quad \text{a.e.}$$

Then, for $m > n$, we have

$$\begin{aligned}
|h_m - h_n|_{W^{1,1}} &= |h_n(h_m \bar{h}_n - 1)|_{W^{1,1}} \\
&\leq |h_m \bar{h}_n|_{W^{1,1}} + \|(1 - h_m \bar{h}_n)\nabla h_n\|_{L^1} \\
&\leq \sum_{j=k_n+1}^{k_m} \|\nabla g_j\|_{L^1} + \|(1 - h_m \bar{h}_n) H\|_{L^1} = A_{m,n} + B_{m,n}.
\end{aligned}$$

Let $0 < \delta < 1$. Then, clearly, $A_{m,n} < \delta$ provided m, n are sufficiently large. On the other hand,

$$B_{m,n} \leq \delta \|H\|_{L^1} + 2 \int_{\{x : |1 - h_m(x)\bar{h}_n(x)| \geq \delta\}} |H|.$$

Note that

$$\Big\{x; |1-h_m(x)\bar{h}_n(x)| \geq \delta\Big\} \subset \Big\{x; |g(x)-h_m(x)| \geq \frac{\delta}{3}\Big\} \cup \Big\{x; |g(x)-h_n(x)| \geq \frac{\delta}{3}\Big\}.$$

Since $h_n \to g$ a.e., we find that $B_{m,n} \leq \delta(\|H\|_{L^1}+1)$, provided m, n are sufficiently large. Therefore, (h_n) is a Cauchy sequence in $W^{1,1}$ and converges to the above g in $W^{1,1}$.

It is not clear whether the reverse inequality in Theorem A1 is valid in general :

Open Problem 8. Let $P_i, N_i \in \Omega$ be such that $T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i})$. Is it true that

$$[g]_{W^{1,1}} \geq 2\pi \widetilde{L}\ ?$$

Note that, by definition, the answer is yes if $T(g) = 2\pi(\delta_P - \delta_N)$.

B. Proof of Theorems 1′, 3′, and 5′.

Proof of Theorem 3′. Let us first assume that $g \in C^\infty(\overline{\Omega}\backslash\{a_1, \dots, a_k\}; S^1) \cap W^{1,1}$. It is then easy to see that

$$\langle T(g), \zeta \rangle = 2\pi \sum_{j=1}^{k} d_j \zeta(a_j) + \int_{\partial\Omega} (g \wedge g_\tau)\zeta, \quad \forall \zeta \in \mathrm{Lip}\,(\Omega; \mathbb{R}),$$

where d_j denotes the topological degree of g with respect to any small circle centered at a_j. In particular,

$$\langle T(g), \zeta \rangle = 2\pi \sum_{j=1}^{k} d_j \zeta(a_j), \quad \forall \zeta \in W_0^{1,\infty}(\Omega). \tag{B1}$$

Note that, in general, $\sum_j d_j \neq 0$. This means that we do not necessarily have the same number of positive and negative points as before. In order to compensate this, we insert points from $\partial\Omega$ into (B1). Since $\zeta = 0$ on $\partial\Omega$, equality in (B1) remains true. We can then relabel the points a_j as $P_1, \dots, P_\ell$, $N_1, \dots, N_\ell$, according to their multiplicity d_j, so that (B1) becomes

$$T(g) = 2\pi \sum_{j=1}^{\ell} (\delta_{P_j} - \delta_{N_j}) \quad \text{in } W_0^{1,\infty}(\Omega).$$

For a general $g \in W^{1,1}(\Omega; S^1)$, we argue by density using Lemma 2 to conclude that

$$T(g) = 2\pi \sum (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \big[W_0^{1,\infty}(\Omega)\big]^*.$$

We observe that d_Ω induces a metric on the space $\overline{\Omega}/\partial\Omega$, where $\partial\Omega$ is identified with a single point. Moreover, Lipschitz functions ζ on $\overline{\Omega}/\partial\Omega$ with $|\zeta|_{\mathrm{Lip}} \leq 1$ and $\zeta(\partial\Omega) = 0$ correspond to elements in $W_0^{1,\infty}(\Omega)$ such that $\|\nabla\zeta\|_{L^\infty} \leq 1$. Applying Lemma 12′ in [BBM2] to $\overline{\Omega}/\partial\Omega$, we obtain

$$L(g) = \operatorname{Inf} \sum_i d_\Omega(P_i, N_i).$$

Remark B1. The main new feature when Ω is a bounded domain in $\mathbb{R}^2$ is that a minimal connection is made of segments from a positive singularity P_i to some negative N_j, but we can also have line segments joining the singularities P_i, N_i to the boundary $\partial\Omega$. This is the analog of Example 3 in [BCL].

Proof of Theorem 1′. The proof of $E(g) = E_{\mathrm{rel}}(g)$ is exactly the same as in Proposition 2 and we shall omit it.

We are left to show that

$$E(g) = \int_\Omega |\nabla g| + 2\pi L(g). \tag{B2}$$

Let $\varphi \in BV(\Omega;\mathbb{R})$ be such that $g = \mathrm{e}^{i\varphi}$. Using Vol'pert's chain rule as in the proof of Lemma 5, we have

$$|D\varphi|_{\mathcal{M}(\Omega)} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}. \tag{B3}$$

We claim that

$$|g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \geq 2\pi L(g). \tag{B4}$$

In fact, for every $\zeta \in C_0^\infty(\Omega)$ such that $\|\nabla\zeta\|_{L^\infty} \leq 1$,

$$|g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \geq \int_\Omega (g \wedge \nabla g) \cdot \nabla^\perp\zeta - \int_\Omega D\varphi \cdot \nabla^\perp\zeta = \langle T(g), \zeta\rangle.$$

Taking the supremum with respect to ζ, we conclude that (B4) holds. Inequality "$\geq$" in (B2) follows immediately from (B3) and (B4).

We now establish "$\leq$" in (B2). Let us assume for the moment that g is smooth outside finitely many points $a_1, \dots, a_k$, and that g has topological degree ±1 at each one of those points. Let $\mathcal{C}$ be a minimal connection between those points with respect to the distance d_Ω. Note that on any closed curve contained in $\Omega\backslash\mathcal{C}$, g has zero topological degree. We conclude that g has a smooth lifting φ on $\Omega\backslash\mathcal{C}$.

Moreover, as we cross any one of the line segments of $\mathcal{C}$, φ jumps by 2π. Thus, $\varphi \in BV(\Omega;\mathbb{R})$ and

$$\int_\Omega |D\varphi| = \int_\Omega |\nabla g| + 2\pi|\mathcal{C}| = \int_\Omega |\nabla g| + 2\pi L(g).$$

We can now argue by density, using Lemma 2, to conclude that for any $g \in W^{1,1}(\Omega;S^1)$ there exists $\varphi \in BV(\Omega;\mathbb{R})$ such that $g = \mathrm{e}^{i\varphi}$ a.e. in Ω and

$$\int_\Omega |D\varphi| \leq \int_\Omega |\nabla g| + 2\pi L(g). \tag{B5}$$

This concludes the proof of the theorem.

Proof of Theorem 5′. Using exactly the same argument as in the proof of Proposition 3, we have

$$|\langle \mathrm{Det}\,(\nabla g), \zeta\rangle| \leq TV(g)\|\zeta\|_{L^\infty}, \quad \forall \zeta \in C_0^\infty(\Omega). \tag{B6}$$

Thus, if $TV(g) < \infty$, then $\mathrm{Det}\,(\nabla g) \in \mathcal{M}(\Omega)$. We now apply Proposition 3.2 in [S] (see also [P]) to the quotient space $\overline{\Omega}/\partial\Omega$. We conclude that there exist distinct points $a_1, \dots, a_k \in \Omega$ and nonzero integers $d_1, \dots, d_k$ such that

$$\mathrm{Det}\,(\nabla g) = \pi \sum_{j=1}^k d_j \delta_{a_j}. \tag{B7}$$

We now define

$$h(x) := \left(\frac{x-a_1}{|x-a_1|}\right)^{-d_1} \cdots \left(\frac{x-a_k}{|x-a_k|}\right)^{-d_k} g(x) \quad \text{for a.e. } x \in \Omega.$$

Clearly, $\mathrm{Det}\,(\nabla h) = 0$ in $\mathcal{D}'(\Omega)$. It follows from the analog of Theorem 7 for domains in $\mathbb{R}^2$ (see also [D]) that h has a lifting in $W^{1,1}$. In other words, we can find $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ such that $h = \mathrm{e}^{i\varphi}$ a.e. in Ω. We then conclude that

$$g(x) = \left(\frac{x-a_1}{|x-a_1|}\right)^{d_1} \cdots \left(\frac{x-a_k}{|x-a_k|}\right)^{d_k} \mathrm{e}^{i\varphi(x)} \quad \text{for a.e. } x \in \Omega.$$

Arguing as in the proof of Theorem 5, this implies that

$$TV(g) \leq \pi \sum_{j=1}^k |d_j|. \tag{B8}$$

The reverse inequality already follows from (B6). We then conclude that (6.8) holds. Conversely, if $\mathrm{Det}(\nabla g) \in \mathcal{M}(\Omega)$, then (B7) holds. The above argument then shows that $TV(g) < \infty$ and

$$TV(g) = |\,\mathrm{Det}\,(\nabla g)|_{\mathcal{M}} = \pi \sum_{i=1}^k |d_i|.$$

C. Proof of Theorem 3''''.

Theorem 3'''' follows immediately from Theorem 3 and the next

Lemma C1. *Given $g \in W^{1,1}(S^2;\Gamma)$, we define*

$$g_j := \frac{g - a_j}{|g - a_j|} \in W^{1,1}(S^2; S^1),$$

where a_j is any given point in A_j, $\forall j$. Then,

$$\text{Det}\,(\nabla g) = \frac{1}{\pi} \sum_j |A_j|\,\text{Det}\,(\nabla g_j) \quad \text{in } \mathcal{D}'(S^2). \tag{C1}$$

The proof of Lemma C1 relies on the following

Lemma C2. *For every $u \in W^{1,1}(S^1;\Gamma)$, we have*

$$\frac{1}{2}\int_{S^1} u \wedge u_\tau = \sum_j |A_j| \deg \frac{u - a_j}{|u - a_j|}. \tag{C2}$$

Proof of Lemma C2.

Step 1. Γ is a simple curve.

It is well-known that (C2) holds if $u \in C^1(S^1;\Gamma)$ (see, e.g., [N]). By approximation, we conclude that (C2) is also true for any $u \in W^{1,1}(S^1;\Gamma)$.

Step 2. Assume Γ has finitely many self-intersections, say $q_1, \dots, q_k$.

Since u is continuous, the set

$$S^1 \backslash u^{-1}(\{q_1, \dots, q_k\})$$

is open and can be written as a countable union of open arcs in S^1. Let α_1 be such an arc. It is easy to see that we can select disjoint arcs $\alpha_2, \dots, \alpha_j$ (oriented anticlockwise) such that u at the positive endpoint of α_i coincides with the value of u at the negative endpoint of α_{i+1} for $i = 1, \dots, j$, with the convention that $\alpha_{j+1} = \alpha_1$. By removing arcs from this list if necessary, we can assume that each point q_i appears only twice in the list

$$\big\{u(\partial\alpha_1), \dots, u(\partial\alpha_j)\big\}.$$

This construction induces a function $\tilde{u} \in W^{1,1}(S^1;\Gamma)$ such that

(*a*) $\tilde{u} = u$ on $\alpha_1 \cup \dots \cup \alpha_j$;

(*b*) $\tilde{u}$ is locally constant on $S^1 \backslash \alpha_1 \cup \dots \cup \alpha_j$.

By construction, $\tilde{u}(S^1)$ is a subset of a Jordan curve $\tilde{\Gamma}$ contained in Γ. Let $A_{t_1}, \dots, A_{t_\ell}$ be the components of $\mathbb{R}^2 \backslash \Gamma$ enclosed by $\tilde{\Gamma}$. By our first step, we have

$$\text{(C3)} \qquad \frac{1}{2}\int_{S^1} \tilde{u} \wedge \tilde{u}_\tau = \big|A_{t_1} \cup \cdots \cup A_{t_\ell}\big| \deg \frac{\tilde{u}-a}{|\tilde{u}-a|}.$$

for some point a inside $\tilde{\Gamma}$. Note, however, that

$$\deg \frac{\tilde{u}-a}{|\tilde{u}-a|} = \deg \frac{\tilde{u}-a_{t_s}}{|\tilde{u}-a_{t_s}|}, \quad \forall s = 1, \dots, \ell \ ;$$

moreover,

$$\deg \frac{\tilde{u}-a_j}{|\tilde{u}-a_j|} = 0, \quad \text{if } j \notin \{t_1, \dots, t_\ell\}.$$

We can thus rewrite (C3) as

$$\text{(C4)} \qquad \frac{1}{2}\int_{S^1} \tilde{u} \wedge \tilde{u}_\tau = \sum_{s=1}^{\ell} |A_{t_s}| \deg \frac{\tilde{u}-a_{t_s}}{|\tilde{u}-a_{t_s}|} = \sum_j |A_j| \deg \frac{\tilde{u}-a_j}{|\tilde{u}-a_j|}.$$

We can proceed with the construction of $\tilde{u}$ and "decompose" $u \in W^{1,1}(S^1;\Gamma)$ as $\tilde{u}_1, \tilde{u}_2, \dots$ so that

(a) $\tilde{u}_i \in W^{1,1}(S^1;\Gamma)$ and $\tilde{u}_i(S^1)$ is contained in some Jordan curve for every i ;

(b) $u_\tau = \tilde{u}_{1\tau} + \tilde{u}_{2\tau} + \cdots$ in S^1 ;

(c) $\tilde{u}_i$ coincides with u on finitely many arcs in $S^1 \backslash u^{-1}(\{q_1, \dots, q_k\})$ and $\tilde{u}_i$ is locally constant outside those arcs.

By (C4), we have

$$\frac{1}{2}\int_{S^1} \tilde{u}_i \wedge \tilde{u}_{i\tau} = \sum_j |A_j| \deg \frac{\tilde{u}_i-a_j}{|\tilde{u}_i-a_j|}, \quad \forall i.$$

Note that, by (b) and (c),

$$\int_{S^1} u \wedge u_\tau = \sum_i \int_{S^1} \tilde{u}_i \wedge \tilde{u}_{i\tau}.$$

For the same reason,

$$\deg \frac{u-a_j}{|u-a_j|} = \sum_i \deg \frac{\tilde{u}_i-a_j}{|\tilde{u}_i-a_j|}, \quad \forall j.$$

We conclude that (C2) holds.

Proof of Lemma C1. Let $g \in W^{1,1}(S^2;\Gamma)$. By the coarea formula (see [BBM2]), we have

$$\langle \mathrm{Det}\,(\nabla g), \zeta\rangle = \int_{\mathbb{R}} \left(\int_{\Sigma_\lambda} g \wedge g_\tau \right) d\lambda, \tag{C5}$$

where $\zeta \in C^\infty(S^2)$, and $\Sigma_\lambda = \left\{x \in S^2 \ ; \ \zeta(x) = \lambda\right\}$ is equipped with the appropriate orientation, whenever λ is a regular value of ζ. Recall that, for a.e. $\lambda \in \mathbb{R}$, $g|_{\Sigma_\lambda}$ belongs to $W^{1,1}$. Applying Lemma C2 to $g|_{\Sigma_\lambda}$ for such λs we get

$$\int_{\Sigma_\lambda} g \wedge g_\tau = 2 \sum_j |A_j| \deg g_j = \frac{1}{\pi} \sum_j |A_j| \int_{\Sigma_\lambda} g_j \wedge g_{j\tau}.$$

Integrate both sides of the identity above with respect to λ. Using (C5), we conclude that

$$\langle \mathrm{Det}\,(\nabla g), \zeta\rangle = \frac{1}{\pi} \sum_j |A_j| \langle \mathrm{Det}\,(\nabla g_j), \zeta\rangle, \quad \forall \zeta \in C^\infty(S^2).$$

This establishes (C1).

We also call the attention of the reader to the following analog of Lemma 12′ in [BBM2] :

Proposition C1. *Let X be a metric space. Given two sequences (P_i), (N_i) in X and nonnegative numbers α_i such that $\sum_i \alpha_i d(P_i, N_i) < \infty$, let*

$$T = \sum_i \alpha_i (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \left[\,\mathrm{Lip}(X)\right]^*. \tag{C6}$$

Define

$$L = \operatorname*{Sup}_{\substack{\zeta \in \mathrm{Lip}(X) \\ |\zeta|_{\mathrm{Lip}} \le 1}} \langle T, \zeta\rangle.$$

Then,

$$L = \mathrm{Inf} \sum_i \alpha_i d(P_i, N_i), \tag{C7}$$

where the infimum is taken over all sequences (P_i), (N_i) in X and numbers $\alpha_i \ge 0$ such that (C6) *holds.*

Proof. Let us denote by $\tilde{L}$ the infimum in (C7). Clearly, $L \leq \tilde{L}$. We now establish the reverse inequality.

Let $\varepsilon > 0$. We take $k \geq 1$ sufficiently large so that

$$\sum_{i>k} \alpha_i d(P_i, N_i) < \varepsilon.$$

Without loss of generality, we can assume that each α_i is rational for $i = 1, \dots, k$. We then choose an integer $J \geq 1$ sufficiently large so that $J\alpha_i$ is an integer for every $i = 1, \dots, k$. Write the points P_i, N_i as $p_1, p_2, \dots$ and $n_1, n_2, \dots$, with multiplicity $J\alpha_i$. It follows from Lemma 4.2 in [BCL] that we can find $\zeta_0 \in \mathrm{Lip}(X)$, with $|\zeta_0|_{\mathrm{Lip}} \leq 1$, such that, after relabeling the points n_j if necessary, we have

$$\sum_{i=1}^{k} J\alpha_i \big[\zeta_0(P_i) - \zeta_0(N_i)\big] = \sum_j \big[\zeta_0(p_j) - \zeta_0(n_j)\big] = \sum_j d(p_j, n_j).$$

Thus,

$$\begin{aligned}
L &\geq \sum_{i=1}^{\infty} \alpha_i \big[\zeta_0(P_i) - \zeta_0(N_i)\big] \\
&\geq \sum_{i=1}^{k} \alpha_i \big[\zeta_0(P_i) - \zeta_0(N_i)\big] - \varepsilon \\
&= \frac{1}{J} \sum_j d(p_j, n_j) - \varepsilon \geq \sum_j \frac{1}{J} d(p_j, n_j) + \sum_{i>k} \alpha_i d(P_i, N_i) - 2\varepsilon.
\end{aligned}$$

Note that

$$T = \sum_j \frac{1}{J}(\delta_{p_j} - \delta_{n_j}) + \sum_{i>k} \alpha_i (\delta_{P_i} - \delta_{N_i}).$$

We conclude that $L \geq \tilde{L} - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies $L \geq \tilde{L}$. Thus, $L = \tilde{L}$ as claimed.

D. Proof of Theorems 1″ and 2″.

We begin with a few preliminary results:

Lemma D1. *Given $\varepsilon > 0$, let $\Phi_\varepsilon : \bar{B}^{N-1} \to \bar{B}^{N-1}$ be defined as*

$$\Phi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| \leq \varepsilon, \\ \dfrac{|x| - \varepsilon}{1 - \varepsilon} \dfrac{x}{|x|} & \text{if } \varepsilon < |x| \leq 1. \end{cases}$$

Then, for every $f \in C^\infty(\bar{B}^{N-1}; \mathbb{R}^M)$, we have

$$f \circ \Phi_\varepsilon \to f \quad \text{in } W^{1,N-1}(B^{N-1}) \quad \text{as } \varepsilon \to 0.$$

Proof. Given $x \in B^{N-1}$, $|x| \geq \varepsilon$, let $r = |x|$ and $\psi_\varepsilon(r) = \frac{r-\varepsilon}{(1-\varepsilon)r}$. Using this notation, we have

$$\Phi_\varepsilon'(x) = \psi_\varepsilon(r)\,\mathrm{Id} + \Big(\frac{x_i x_j}{r}\psi_\varepsilon'(r)\Big).$$

Since $\psi_\varepsilon'(r) = \frac{\varepsilon}{(1-\varepsilon)r^2}$,

$$\Big|\frac{x_i x_j}{r}\psi_\varepsilon'(r)\Big| \leq \frac{C\varepsilon}{r}.$$

Moreover,

$$|\Phi_\varepsilon(x) - x| \leq C\varepsilon \quad \text{and} \quad |\psi_\varepsilon(r) - 1| \leq \frac{C\varepsilon}{r}.$$

We then have

$$\begin{aligned} \big|\nabla f_\varepsilon(x) - \nabla f(x)\big| &= \big|{}^t\Phi_\varepsilon'(x)\,\nabla f(\Phi_\varepsilon(x)) - \nabla f(x)\big| \\ &\leq \big|\nabla f(\Phi_\varepsilon(x)) - \nabla f(x)\big| + \big|\nabla f(\Phi_\varepsilon(x))\big|\,\big|\mathrm{Id} - {}^t\Phi_\varepsilon'(x)\big| \\ &\leq C|\Phi_\varepsilon(x) - x| + C|\psi_\varepsilon(r) - 1| + \frac{C\varepsilon}{r} \leq \frac{C\varepsilon}{r} \end{aligned}$$

for $|x| \geq \varepsilon$. Therefore,

$$\int_{B^{N-1}} |\nabla f_\varepsilon - \nabla f|^{N-1} \leq C\varepsilon^{N-1} \int\limits_{\varepsilon \leq |x| \leq 1} \frac{dx}{|x|^{N-1}} + \int\limits_{|x|<\varepsilon} |\nabla f|^{N-1} \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, we establish the following

Lemma D2. *Given $f \in C^\infty(\bar{B}^{N-1} \times [0,1]; \mathbb{R}^M)$, let*

$$f_\varepsilon(x,t) = f(\Phi_\varepsilon(x), t).$$

Then

$$f_\varepsilon \to f \quad \text{in } W^{1,N-1}(B^{N-1} \times [0,1]) \quad \text{as } \varepsilon \to 0.$$

Proof. Note that

$$\frac{\partial f_\varepsilon}{\partial t}(x,t) - \frac{\partial f}{\partial t}(x,t) = \begin{cases} \dfrac{\partial f}{\partial t}(0,t) - \dfrac{\partial f}{\partial t}(x,t) & \text{if } |x| < \varepsilon, \\ \dfrac{\partial f}{\partial t}(\Phi_\varepsilon(x),t) - \dfrac{\partial f}{\partial t}(x,t) & \text{if } |x| \geq \varepsilon. \end{cases}$$

Thus, since $|\Phi_\varepsilon(x) - x| \le C\varepsilon$, we have

$$\Big|\frac{\partial f_\varepsilon}{\partial t} - \frac{\partial f}{\partial t}\Big| \le C\varepsilon.$$

The result now follows from Lemma D1.

As a consequence of Lemma D2 above, any map $f \in C^\infty(\bar{B}^{N-1} \times [0,1]; S^k)$ can be approximated in $W^{1,N-1}$ by maps f_ε such that

$$f_\varepsilon = f \quad \text{on } S^{N-2} \times [0,1] \quad \text{and} \quad f_\varepsilon(x,t) = g_\varepsilon(t) \quad \text{if } |x| \le \varepsilon.$$

Lemma D3. *Given $g \in C^\infty([0,1]; S^k)$, $k \ge 2$, let*

$$f(x,t) = g(t) \quad \forall (x,t) \in \bar{B}^{N-1} \times [0,1].$$

Then, there exists a sequence (f_ε) in $W^{1,N-1}(B^{N-1} \times [0,1]; S^k)$ such that

$$f_\varepsilon(x,t) = \mathit{const} \quad \text{if } |x| \le \varepsilon, \quad f_\varepsilon(x,t) = g(t) \quad \text{if } |x| = 1,$$

and

$$f_\varepsilon \to f \quad \text{in } W^{1,N-1}(B^{N-1} \times [0,1]) \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\zeta_\varepsilon : \bar{B}^{N-1} \to \mathbb{R}$ be given by

$$\zeta_\varepsilon(x) = \begin{cases} \dfrac{\log \frac{|x|}{\varepsilon}}{\log \frac{1}{\varepsilon}} & \text{if } \varepsilon \le |x| \le 1, \\ 0 & \text{if } |x| < \varepsilon. \end{cases}$$

It is easy to see that

$$\int_{B^{N-1}} |\nabla \zeta_\varepsilon|^{N-1} \to 0 \quad \text{and} \quad \int_{B^{N-1}} |\zeta_\varepsilon - 1|^{N-1} \to 0.$$

Since $k \ge 2$, there exists $Q \in S^k$ such that $Q \notin g([0,1])$. Let $\Psi : S^k \setminus \{Q\} \to \mathbb{R}^k$ denote the stereographic projection. Set $F = \Psi \circ f$. Clearly, in order to establish the lemma, it suffices to approximate F in $W^{1,N-1}$ by a sequence F_ε such that $|F_\varepsilon| \le C$,

$$F_\varepsilon(x,t) = \text{const} \quad \text{if } |x| \le \varepsilon \quad \text{and} \quad F_\varepsilon(x,t) = F(x,t) \quad \text{if } |x| = 1.$$

Set $G(t) = \Psi \circ g(t)$ and $F_\varepsilon(x,t) = \zeta_\varepsilon(x) G(t)$. We then have

$$\iint |\nabla_x F_\varepsilon - \nabla F|^{N-1}\, dx\, dt = \int |\nabla \zeta_\varepsilon|^{N-1} \int |G|^{N-1} \to 0$$

and

$$\iint |\partial_t F_\varepsilon - \partial_t F|^{N-1}\, dx\, dt = \int |1 - \zeta_\varepsilon|^{N-1} \int |G'|^{N-1} \to 0.$$

The proof of Lemma D3 is complete.

We conclude from Lemma D3 that, given any $u \in W^{1,N-1}(S^N; S^{N-1})$, $N \geq 3$, there exists a sequence (u_n) such that $u_n \to u$ in $W^{1,N-1}$, where each u_n satisfies the following properties :

(i) u_n has a finite number of point singularities P_i, N_i ;

(ii) u_n is homogeneous of degree 0 in a neighborhood of each singularity ;

(iii) $u_n \equiv Q$ in some conic neighborhood of a geodesic joining P_i and N_i.

In fact, (i) follows from [BZ]. Next, (ii) holds since every topological singularity may be approximated by homogeneous singularities (see Lemma E5 below). We then apply Lemmas D2 and D3 to obtain property (iii).

We shall say that a map is *good* if it satisfies properties (i)–(iii).

Proof of Theorem 2″. Our goal is to show that, for every $u \in W^{1,N-1}(S^N; S^{N-1})$,

$$\text{(D1)} \qquad \operatorname*{Inf}_{v \in C^\infty(S^N; S^{N-1})} \int_{S^N} |D(u) - D(v)| = \sigma_N L(u).$$

Proof of "$\geq$". For every $\zeta \in \operatorname{Lip}(S^N)$ such that $\|\nabla \zeta\|_{L^\infty} \leq 1$, we have

$$\int_{S^N} |D(u) - D(v)| \geq -\int_{S^N} \big[D(u) - D(v)\big] \cdot \nabla \zeta = -\int_{S^N} D(u) \cdot \nabla \zeta = \langle T(u), \zeta \rangle.$$

Taking the supremum with respect to ζ, we get

$$\int_{S^N} |D(u) - D(v)| \geq \sigma_N L(u) \quad \forall v \in C^\infty(S^N; S^{N-1}).$$

Proof of "$\leq$". Assume u is a good map. We shall suppose for simplicity that u has a single dipole P, N. Given $\varepsilon > 0$, let $U_\varepsilon : S^{N-1} \to S^{N-1}$ be a smooth map such that $\deg U_\varepsilon = -1$,

$$U_\varepsilon \equiv Q \quad \text{if } |x - P| \geq \frac{\varepsilon}{2} \quad \text{and} \quad \int_{S^{N-1}} |\nabla_T U_\varepsilon|^{N-1} \leq (N-1)^{\frac{N-1}{2}} \sigma_N + \varepsilon.$$

The existence of such map is established in [BCL, Section VIII].

Let W denote the ε-conic neighborhood of the geodesic segment joining P and N. We decompose W as $W = W_1 \cup W_2$, where W_1 is the cylindric part of W, and W_2 is the union of the two conic caps. We then define u_ε as

$$u_\varepsilon(x) = u(x) \quad \text{if } x \notin W, \quad u_\varepsilon(x) = U_\varepsilon(\theta) \quad \text{if } x = (r,\theta) \in W_1$$

and u_ε is extended by homogeneity of degree 0 in W_2. We then have

$$\int_{S^N} |D(u) - D(u_\varepsilon)| = \int_{W_1} |D(u_\varepsilon)| + o(1).$$

In [BCL, Section VIII], it is proved that

$$|D(u_\varepsilon)| = |\nabla_T U_\varepsilon| \leq \frac{1}{(N-1)^{\frac{N-1}{2}}} |\nabla_T U_\varepsilon|^{N-1}.$$

We conclude that

$$\lim_{\varepsilon \to 0} \int_{S^N} |D(u) - D(u_\varepsilon)| \leq \sigma_N L(u). \tag{D2}$$

On the other hand, since u_ε has only singularities of degree 0, Hopf's theorem implies that $u_\varepsilon \in \overline{C^\infty(S^N; S^{N-1})}^{W^{1,N-1}}$. It then follows from (D2) that inequality "$\leq$" holds in (D1), at least when u is a good map.

We now establish "$\leq$" in (D1) for any map $u \in W^{1,N-1}(S^N; S^{N-1})$. Let (u_n) be a sequence of good maps such that $u_n \to u$ in $W^{1,N-1}$. For each $n \geq 1$, we have just shown that there exists $v_n : S^N \to S^{N-1}$ smooth such that

$$\int_{S^N} |D(u_n) - D(v_n)| \leq \sigma_N L(u_n) + \frac{1}{n}.$$

Thus

$$\int_{S^N} |D(u) - D(v_n)| \leq \int_{S^N} |D(u) - D(u_n)| + \sigma_N L(u_n) + \frac{1}{n} = \sigma_N L(u) + o(1),$$

which gives the desired result.

Proof of Theorem 1''. We want to show that

$$E_{\text{rel}}(u) = \int_{S^N} |\nabla u|^{N-1} + (N-1)^{\frac{N-1}{2}} \sigma_N L(u). \tag{D3}$$

Proof of "≤". It suffices to establish the result for good maps. In fact, if $u_n \to u$ in $W^{1,N-1}$, then

$$E_{\rm rel}(u) \le \liminf_{n\to\infty} E_{\rm rel}(u_n),$$

while the right-hand side of (D3) is continuous with respect to the strong topology in $W^{1,N-1}$. Thus, we may assume that u is good and we can proceed exactly as in the proof of Theorem 2$''$. We shall leave the details to the reader.

Proof of "≥". As in [BCL, Section VIII], we have

$$\text{(D4)} \qquad \left|(w\cdot z_2\wedge\cdots\wedge z_N, z_1\cdot w\wedge z_3\wedge\cdots\wedge z_N,\dots)\right| \le \frac{|w|\left(\sum_j |z_j|^2\right)^{\frac{N-1}{2}}}{(N-1)^{\frac{N-1}{2}}}$$

for every $w, z_1, \dots, z_N \in \mathbb{R}^N$.

On the other hand, given a sequence $(u_n) \subset C^\infty(S^N; S^{N-1})$ such that (u_n) is bounded in $W^{1,N-1}$ and $u_n \to u$ a.e., we have

$$\int_{S^N} |\nabla u_n|^{N-1} = \int_{S^N} |\nabla u|^{N-1} + \int_{S^N} |\nabla u_n - \nabla u|^{N-1} + o(1).$$

Let $\zeta \in \operatorname{Lip}(S^N)$ be such that $\|\nabla\zeta\|_{L^\infty} \le 1$. Applying (D4) to $w = u_n$ and $z_i = (u_n - u)_{x_i}$, we get, with $v_n = u_n - u$,

$$\text{(D.5)} \qquad \frac{\displaystyle\int_{S^N} |\nabla u_n - \nabla u|^{N-1}}{(N-1)^{\frac{N-1}{2}}} \ge$$
$$\ge (-1)^N \sum_{i=1}^N \int_{S^N} \det\left(v_{nx_1},\dots,v_{nx_{i-1}}, u_n, v_{nx_{i+1}},\dots,v_{nx_N}\right)\zeta_{x_i}.$$

We write the right-hand side of (D.5) as

$$-\int_{S^N} D(u)\cdot\nabla\zeta + R_n(\zeta),$$

so that (D.5) becomes

$$\text{(D6)} \qquad \int_{S^N} |\nabla u_n - \nabla u|^{N-1} \ge (N-1)^{\frac{N-1}{2}}\left\{-\int_{S^N} D(u)\cdot\nabla\zeta + R_n(\zeta)\right\}.$$

Assume for the moment that $R_n(\zeta) \to 0$. It follows from (D5) and (D6) that

$$\liminf_{n\to\infty} \int_{S^N} |\nabla u_n|^{N-1} \ge \int_{S^N} |\nabla u|^{N-1} + (N-1)^{\frac{N-1}{2}}\langle T(u), \zeta\rangle.$$

Taking the supremum over ζ, we obtain "$\geq$" in (D3). Thus, in order to conclude the proof of Theorem $1''$, we need to show that $R_n(\zeta) \to 0$. Since each u_n is smooth,

$$\int_{S^N} D(u_n) \cdot \nabla\zeta = 0.$$

It then follows that

$$R_n(\zeta) = S_n(\zeta) + o(1),$$

where $S_n(\zeta)$ is a sum of integrals of the form

$$I_n = \pm \int_{S^N} u_n \cdot v_{nx_{i_1}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t},$$

where $k + l = N - 1$. It is important to notice that $k \geq 1$ and $l \geq 1$. Since

$$(u_n - u) \cdot u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \to 0 \quad \text{strongly in } L^{\frac{N-1}{l}},$$

we may replace I_n by

$$\pm \int_{S^N} u \cdot v_{nx_{i_1}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t}.$$

We can now formally integrate by parts to write $S_n(\zeta)$ as a sum of integrals of the form

$$\mp \int_{S^N} u_{x_{i_1}} \cdot v_n \wedge v_{nx_{i_2}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t}.$$

Such integration by parts can be easily justified by approximation ; note that all the second derivatives are canceled by symmetry. Since

$$v_n \cdot u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \to 0 \quad \text{strongly in } L^{\frac{N-1}{l+1}},$$

we conclude that

$$R_n(\zeta) = S_n(\zeta) + o(1) = o(1).$$

The proof of Theorem $1'$ is complete.

E. Proof of Theorem $5''$.

Throughout this section, we assume that $\Omega = \partial G$, where G is a domain in $\mathbb{R}^{N+1}$; more generally, Ω could be any smooth domain (with boundary) in ∂G. We start with

Lemma E1. *Let $0 < \sigma < \infty$, $1 \le q < \infty$. Then*

$$\overline{C^\infty(\Omega; S^{N-1}) \cap W^{\sigma,q}}^{W^{\sigma,q}} = \overline{C^\infty(\overline{\Omega}; S^{N-1})}^{W^{\sigma,q}}.$$

In other words, for each map $u \in C^\infty(\Omega; S^{N-1}) \cap W^{\sigma,q}$, there is a sequence $(u_n) \subset C^\infty(\overline{\Omega}; S^{N-1})$ such that $u_n \to u$ in $W^{\sigma,q}$.

Proof. Let, for $t > 0$ sufficiently small, $\Omega_t = \{x \in \Omega \; ; \; d(x, \partial\Omega) > t\}$. Consider, for any such t, a diffeomorphism $\Phi_t : \Omega \to \Omega_t$ such that

$$\|D^k \Phi_t - D^k \mathrm{Id}\|_{L^\infty} \le C_k t, \quad k = 0, 1, 2 \dots$$

and set, for $u \in C^\infty(\Omega; S^{N-1})$, $u^t = u \circ \Phi_t$. Then

$$u^t \in C^\infty(\overline{\Omega}; S^{N-1}) \qquad \text{and} \qquad u^t \to u \quad \text{in } W^{\sigma,q}.$$

Lemma E2. *Let $u \in W^{1,N-1}(\Omega; S^{N-1})$. Then*

$$L(u) = 0 \qquad \Longleftrightarrow \qquad u \in \overline{C^\infty(\Omega; S^{N-1}) \cap W^{1,N-1}}^{W^{1,N-1}}.$$

Here,

$$\langle T(u), \zeta \rangle = -\int_\Omega D(u) \cdot \nabla \zeta,$$

where $\zeta \in W^{1,\infty}(\Omega)$ and ζ is constant on each connected component of $\partial\Omega$; $L(u)$ is computed accordingly. When $N = 3$, this result is due to Bethuel [B1] ; the same proof yields Lemma E2.

Lemma E3. *Let $1 \le p < N$. For $g \in W^{1,p}(S^{N-1}; S^{N-1})$, set $\tilde{g}(x) = g\left(\frac{x}{|x|}\right)$ for $x \in B^N$. Then $\tilde{g} \in W^{1,p}$ and the map*

$$g \in W^{1,p}(S^{N-1}; S^{N-1}) \quad \longmapsto \quad \tilde{g} \in W^{1,p}(B^N; S^{N-1})$$

is continuous and verifies $|\tilde{g}|_{W^{1,p}} \le C |g|_{W^{1,p}}$.

Proof. Trivial computation.

Lemma E4. *In the definition of TV, we may replace $C^\infty(\overline{\Omega}; \mathbb{R}^N)$-maps by maps in* $\mathrm{Lip}\,(\overline{\Omega}; \mathbb{R}^N)$.

Proof. Clear, by approximation.

Lemma E5. *Let $N \geq 2$, $N-1 \leq p < N$, and let $u \in W^{1,p}(\Omega; S^{N-1})$. Fix $a_1, \dots, a_k \in \Omega$ and define, for $\rho > 0$ sufficiently small,*

$$u_\rho(x) = \begin{cases} u(x) & \text{if } d(x, \{a_1, \dots, a_k\}) \geq \rho \\ u(y) & \text{if, for some } j,\ d(x, a_j) < \rho \text{ and} \\ & x \text{ lies on the geodesic segment from} \\ & a_j \text{ to } y, \text{ where } d(a_j, y) = \rho. \end{cases}$$

Then $u_{\rho_n} \to u$ for some sequence $\rho_n \to 0$.

Here, d denotes the geodesic distance in S^N.

Proof. For simplicity, we may assume Ω is flat near each a_j. Then the definition of u_ρ becomes

$$u_\rho(x) = \begin{cases} u(x) & \text{if } \ d(x, \{a_1, \dots, a_k\}) \geq \rho, \\ u\left(\rho \dfrac{x - a_j}{|x - a_j|}\right) & \text{if } \ |x - a_j| < \rho. \end{cases}$$

Alternatively, denoting by $u_j^\rho(y) = u(a_j + \rho y)$, $y \in S^{N-1}$, then

$$u_\rho(x) = u_j^\rho \left(\frac{x - a_j}{|x - a_j|} \right) \quad \text{if } |x - a_j| < \rho.$$

Assume, for simplicity, that there is only one singularity, say $a_1 = 0$. Let, for $n \geq 1$, ρ_n be such that $\dfrac{1}{2n} \leq \rho_n \leq \dfrac{1}{n}$ and

$$\frac{1}{2n} \int_{S_{\rho_n}} |\nabla u|^p \leq \int_{\frac{1}{2n}}^{\frac{1}{n}} \left(\int_{|x|=\rho} |\nabla u|^p \right) d\rho = \int_{\frac{1}{2n} \leq |x| \leq \frac{1}{n}} |\nabla u|^p.$$

Then

$$\int_{B_{\rho_n}} |\nabla u_{\rho_n}|^p \leq C_{N,p} \rho_n \int_{S_{\rho_n}} |\nabla u|^p \leq C \int_{\frac{1}{2n} \leq |x| \leq \frac{1}{n}} |\nabla u|^p \overset{n \to \infty}{\longrightarrow} 0.$$

Thus

$$\begin{aligned} \int_\Omega |\nabla u_{\rho_n} - \nabla u|^p &= \int_{B_{\rho_n}} |\nabla u_{\rho_n} - \nabla u|^p \\ &\leq C \left(\int_{B_{\rho_n}} |\nabla u_{\rho_n}|^p + \int_{B_{\rho_n}} |\nabla u|^p \right) \overset{n \to \infty}{\longrightarrow} 0. \end{aligned}$$

Proof of Theorem 5″. The equivalence

$$TV(u) < \infty \quad \Longleftrightarrow \quad \operatorname{Det}(\nabla u) \quad \text{is a measure}$$

is established as in the proof of Theorem 5. As already noted, $TV(u) < \infty$ implies

$$\operatorname{Det}(\nabla u) = \frac{\sigma_N}{N} \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}).$$

Let $a_1, \dots, a_k$ be the collection of points P_i, N_i. Given $n \geq 1$, let

$$\Omega^n = \left\{ x \in \Omega \; ; \; d(x, \{a_1, \dots, a_k\}) > \frac{1}{n} \right\}$$

and

$$A^n = \left\{ x \in \Omega \; ; \; \frac{1}{n} \leq d(x, \{a_1, \dots, a_k\}) \leq \frac{2}{n} \right\}.$$

Clearly, $L(u|_{\Omega_n}) = 0$. Consider a sequence $(u_n^m) \subset C^\infty(\overline{\Omega^n}; S^{N-1})$ such that

$$u_n^m \to u \quad \text{in } W^{1,N-1} \quad \text{as } m \to \infty.$$

There is some $\rho_n \in \left(\frac{1}{n}, \frac{2}{n}\right)$ such that, up to a subsequence in m,

(i) $u_n^m|_{\Sigma_{\rho_n}} \to u|_{\Sigma_{\rho_n}}$ in $W^{1,N-1}(\Sigma_{\rho_n})$ as $m \to \infty$,

(ii) $\displaystyle\int_{\Sigma_{\rho_n}} |\nabla u|^{N-1} \leq Cn \int_{A_n} |\nabla u|^{N-1}$.

Here, $\Sigma_\rho = \big\{ x \; ; \; d(x, \{a_1, \dots, a_k\}) = \rho \big\}$.

Extend u_n^m to Ω as in Lemma 5 (by homogeneity of degree 0) ; let $(u_n^m)_{\rho_n} = \tilde{u}_n^m$ be this extension. By (i) and Lemma E5, we have

$$\tilde{u}_n^m \to u_{\rho_n} \quad \text{in } W^{1,N-1}(\Omega) \quad \text{as } m \to \infty.$$

By (ii) and Lemma E3, we have $u_{\rho_n} \to u$ in $W^{1,N-1}(\Omega)$. Thus we may find a sequence $(v_n) \subset \operatorname{Lip}_{\text{loc}}(\Omega \setminus \{a_1, \dots, a_k\}; S^{N-1})$ such that

(a) v_n is homogeneous of degree 0 near each a_j ;

(b) $v_n \to u$ in $W^{1,N-1}(\Omega)$;

(c) near each a_j, the degree of v_n is d_j.

Assertion (c) follows from (i), the continuity of degree of maps from S^{N-1} into S^{N-1} for $W^{1,N-1}$-convergence and

$$T(v_n) = \sigma_N \sum_j \deg(v_n, a_j)\delta_{a_j} \rightharpoonup T(u) = \sigma_N \sum_j d_j \delta_{a_j}.$$

For the remaining part of the proof, assume for simplicity that there is only one singularity $a = 0$ of degree $d > 0$ and that Ω is flat near a.

Let ρ_n be such that v_n is homogeneous of degree 0 in $B_{\rho_n}(0)$. Here, $B_\rho(0)$ is a ball in Ω centered at $a = 0$. Fix d distinct points $p_1, \dots, p_d$ in B_1. Let $\varepsilon > 0$ be sufficiently small. For $w \in \mathrm{Lip}\,(\partial B_1; S^{N-1})$ (B_1 is the unit ball in $\mathbb{R}^N$), with $\deg w = d$, let $\tilde{w} : B_1 \setminus \bigcup_j B_\varepsilon(p_j) \to S^{N-1}$ be a Lipschitz function such that

$$\tilde{w}|_{\partial B_1} = w \quad \text{and} \quad \tilde{w}(x) = \frac{x - p_j}{|x - p_j|} \quad \text{if } |x - p_j| = \varepsilon, \quad \forall j = 1, \dots, d.$$

(Such a map exists, since $\deg w = d$.) We then extend $\tilde{w}$ to B_1 by setting

$$\tilde{w}(x) = \frac{x - p_j}{\varepsilon} \quad \text{if } |x - p_j| < \varepsilon, \quad \forall j = 1, \dots, d.$$

Thus $\tilde{w}$ is still Lipschitz. Define, for $0 < \rho < \rho_n$,

$$v_{n,\rho}(x) = \begin{cases} v_n(x) & \text{if } d(x,0) > \rho \\ \tilde{w}_n\left(\frac{x}{\rho}\right) & \text{if } d(x,0) \leq \rho. \end{cases}$$

Here, $w_n(x) = v_n(\rho_n x)$, if $|x| = 1$. Lemma E5 yields

$$v_{n,\rho} \to v_n \quad \text{in } W^{1,N-1} \quad \text{as } \rho \to 0.$$

Clearly, by the definition of $v_{n,\rho}$, we have

$$\int_\Omega |\det \nabla v_{n,\rho}| = \omega_N d.$$

Considering now the case of several singularities, we obtain by a diagonal procedure a sequence $(w_n) \subset \mathrm{Lip}\,(\Omega; \mathbb{R}^N)$ such that

$$w_n \to u \quad \text{in } W^{1,N-1} \quad \text{and} \quad \int_\Omega |\det \nabla w_n| = \omega_N \sum_j |d_j|.$$

F. Proof of Theorem $5'''$.

We start with the following well-known

Lemma F1. *Let $1 < p < \infty$ and $1/p < s < 1$. Given $u \in W_0^{s,p}(\mathbb{R}_+^N)$, let*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_N > 0, \\ 0 & \text{if } x_N \le 0. \end{cases}$$

Then, $u \mapsto \tilde{u}$ is a continuous mapping from $W_0^{s,p}(\mathbb{R}_+^N)$ into $W^{s,p}(\mathbb{R}^N)$.

Proof. By density, it suffices to deal with $u \in C_0^\infty(\mathbb{R}_+^N)$. Using the Besov seminorm, we have

$$|\tilde{u}|^p_{W^{s,p}} \sim \|u\|^p_{W^{s,p}} + \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|\tilde{u}(x', t+\tau) - \tilde{u}(x', t)|^p}{\tau^{1+sp}}\, dx'\, dt\, d\tau. \tag{F1}$$

Denote by I the last term in the right-hand side of (F1). Clearly, it suffices to estimate I :

(F2)
$$I \sim \int_{\mathbb{R}^{N-1}} \int_0^\infty \int_0^\infty \frac{|u(x',t) - u(x',\sigma)|^p}{|t-\sigma|^{1+sp}}\, d\sigma\, dt\, dx' + \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{|u(x',t)|^p}{t^{sp}}\, dt\, dx'.$$

It then suffices to estimate the last term in (F2). By Fubini, we only need to consider the 1-dimensional integral

$$\int_0^\infty \frac{|v(t)|^p}{t^{sp}}\, dt, \quad v \in C_0^\infty(0,\infty).$$

This integral is controlled via the following

Lemma F2. *Let $1 < p < \infty$ and $1/p < s < 1$. Given $v \in C_0^\infty(0,\infty)$, we have*

$$\int_0^\infty \frac{|v(t)|^p}{t^{sp}}\, dt \le C \int_0^\infty \int_0^\infty \frac{|v(x) - v(y)|^p}{|x-y|^{1+sp}}\, dx\, dy. \tag{F3}$$

Proof. We first point out that both integrals are finite. Given $0 < \alpha < \beta < 1$, we have

$$|v(t)| \le |v(x)| + |v(t) - v(x)| \quad \forall x \in [\alpha t, \beta t] =: I_t.$$

Thus,

$$|v(t)|^p \le 2^p |v(x)|^p + C|v(t) - v(x)|^p \quad \forall x \in I_t.$$

Integrating over $x \in I_t$, we get

$$|v(t)|^p \leq \frac{2^p}{(\beta-\alpha)t}\int_0^\infty |v(x)|^p\,dx + \frac{C}{(\beta-\alpha)t}\int_{I_t} |v(t)-v(x)|^p\,dx. \tag{F4}$$

Since $\alpha, \beta < 1$, we have $|t-x| \sim t$ for every $x \in I_t$. Dividing both sides of (F4) by t^{sp} and integrating with respect to t we then have

$$\begin{aligned}\int_0^\infty \frac{|v(t)|^p}{t^{sp}}\,dt &\leq \frac{2^p}{\beta-\alpha}\int_0^\infty\int_{I_t} \frac{|v(x)|^p}{t^{sp}}\,dx\,dt + \frac{C}{\beta-\alpha}\int_0^\infty\int_0^\infty \frac{|v(t)-v(x)|^p}{|t-x|^{1+sp}}\,dx\,dt\\ &= \frac{2^p(\beta^{sp}-\alpha^{sp})}{sp(\beta-\alpha)}\int_0^\infty \frac{|v(x)|^p}{x^{sp}}\,dx + \frac{C}{\beta-\alpha}\int_0^\infty\int_0^\infty \frac{|v(t)-v(x)|^p}{|t-x|^{1+sp}}\,dx\,dt. \end{aligned}\tag{F5}$$

Note that $\frac{\beta^{sp}-\alpha^{sp}}{sp(\beta-\alpha)} = \gamma^{sp-1}$ for some $\gamma \in [\alpha,\beta]$. Thus, by taking $\beta > 0$ sufficiently small, we have

$$\frac{2^p(\beta^{sp}-\alpha^{sp})}{sp(\beta-\alpha)} \leq \frac{1}{2}.$$

With a such choice, (F3) trivially follows from (F5).

Lemma F3. *Let ω_1, Ω be two smooth domains, $\omega_1 \subset\subset \Omega$. Set $\omega_2 = \Omega\backslash\overline{\omega}_1$. Assume that $u_n \to u$ in $W^{s,p}(\omega_1)$ and $v_n \to v$ in $W^{s,p}(\omega_2)$, with $\operatorname{tr} u_n = \operatorname{tr} v_n$ on $\partial\omega_1$. Let*

$$w_n = \begin{cases} u_n & \text{in } \omega_1\\ v_n & \text{in } \omega_2 \end{cases} \quad \text{and} \quad w = \begin{cases} u & \text{in } \omega_1\\ v & \text{in } \omega_2 \end{cases}.$$

Then

$$w_n \to w \quad \text{in } W^{s,p}(\Omega).$$

Proof. It suffices to show that

$$\|w\|_{W^{s,p}(\Omega)} \leq C\big(\|u\|_{W^{s,p}(\omega_1)} + \|v\|_{W^{s,p}(\omega_2)}\big).$$

Let $\eta = \operatorname{tr} u = \operatorname{tr} v$. By the standard trace theory,

(i) $\|\eta\|_{W^{s-1/p,p}} \leq C\|u\|_{W^{s,p}(\omega_1)}$;

(ii) $\|\eta\|_{W^{s-1/p,p}} \leq C\|v\|_{W^{s,p}(\omega_2)}$;

(iii) there exists an extension $g \in W^{s,p}(\Omega)$ of η to ω_1 and ω_2 such that

$$\|g\|_{W^{s,p}(\Omega)} \leq C\|\eta\|_{W^{s-1/p,p}}.$$

Let

$$\tilde{w} = \begin{cases} u-g & \text{in } \omega_1,\\ v-g & \text{in } \omega_2. \end{cases}$$

By Lemma F1, $\tilde{w} \in W^{s,p}(\Omega)$ and

$$\|\tilde{w}\|_{W^{s,p}(\Omega)} \leq C\big(\|u\|_{W^{s,p}(\omega_1)} + \|v\|_{W^{s,p}(\omega_2)}\big).$$

Since $w = \tilde{w} + g$ a.e. in Ω, Lemma F.3 follows.

In the sequel, we shall denote by C the cube $(-1,1)^N$. Let $\|x\|_\infty = \operatorname{Max}_i \{|x_i|\}$.

Lemma F4 (Brezis-Mironescu [BM1]). *Let $0 < s < 1$ and $1 < p < \infty$, with $sp < N$. Given $f \in W^{s,p}(\partial C)$, set $\tilde{f}(x) = f\big(x/\|x\|_\infty\big)$, $x \in C$. Then, $\tilde{f} \in W^{s,p}(C)$ and the mapping*

$$f \mapsto \tilde{f}$$

is continuous from $W^{s,p}(\partial C)$ into $W^{s,p}(C)$.

We refer the reader to [BM1, Lemma D.1] for a proof of Lemma F4.

We next denote by

$$C_\varepsilon(Q) = Q + (-\varepsilon, \varepsilon)^N \quad \text{and} \quad C_\varepsilon = C_\varepsilon(0).$$

The following lemma is a variant of the approximation procedure in [BM2]:

Lemma F5. *Let $0 < s < 1$ and $1 < p < \infty$, with $sp < N$. Let $f \in W^{s,p}(\mathbb{R}^N)$. Given $\varepsilon > 0$ and $Q \in C_\varepsilon$, set*

$$f_{\varepsilon,Q}(x) = \begin{cases} f(x) & \text{if } x \notin C_\varepsilon(Q), \\ f(\pi_{\varepsilon,Q}(x)) & \text{if } x \in C_\varepsilon(Q), \end{cases}$$

where $\pi_{\varepsilon,Q}(x) = Q + \varepsilon \frac{x-Q}{\|x-Q\|_\infty}$ is the projection of x to $\partial C_\varepsilon(Q)$, with respect to Q. Then, there exist $\varepsilon_n \to 0$ and $Q_n \in C_{\varepsilon_n/2}$ such that

$$f_{\varepsilon_n,Q_n} \to f \quad \text{in } W^{s,p}(\mathbb{R}^N).$$

Proof. Set $g_{\varepsilon,Q} = f_{\varepsilon,Q} - f$. We have

$$\begin{aligned} |g_{\varepsilon,Q}|^p_{W^{s,p}} &= 2\int_{\mathbb{R}^N \setminus C_\varepsilon(Q)} dy \int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x) - f(x)|^p}{|x-y|^{N+sp}}\, dx + \\ &\qquad + \int_{C_\varepsilon(Q)} \int_{C_\varepsilon(Q)} \frac{|g_{\varepsilon,Q}(x) - g_{\varepsilon,Q}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \\ &\sim \int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x) - f(x)|^p}{d(x, \partial C_\varepsilon(Q))^{sp}}\, dx + \\ &\qquad + \int_{C_\varepsilon(Q)} \int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x) - f_{\varepsilon,Q}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + o(1) \\ &=: I_{\varepsilon,Q} + J_{\varepsilon,Q} + o(1). \end{aligned}$$

It suffices to show that

$$\int_0^1 \frac{d\varepsilon}{\varepsilon} \left\{ \fint_{C_\varepsilon} (I_{\varepsilon,Q} + J_{\varepsilon,Q})\, dQ \right\} < \infty.$$

The proof of

$$\int_0^1 \frac{d\varepsilon}{\varepsilon} \fint_{C_\varepsilon} J_{\varepsilon,Q}\, dQ < \infty$$

may be found in [BM2, Appendix A]. Next,

$$\int_0^1 \frac{d\varepsilon}{\varepsilon} \fint_{C_\varepsilon} I_{\varepsilon,Q}\, dQ \leq \int_0^1 \frac{d\varepsilon}{\varepsilon} \int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon(x)} \frac{|f_{\varepsilon,Q}(x) - f(x)|^p}{d(x, \partial C_\varepsilon(Q))^{sp}}\, dQ.$$

We now make the change of variables $Q = x - y$, $y \in C_\varepsilon$. We get

$$\int_0^1 \frac{d\varepsilon}{\varepsilon} \fint_{C_\varepsilon} I_{\varepsilon,Q}\, dQ \leq \int_0^1 \frac{d\varepsilon}{\varepsilon} \int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon} \frac{|f(x) - f(x - y + \varepsilon \frac{y}{\|y\|})|^p}{|y - \varepsilon \frac{y}{\|y\|}|^{sp}}\, dy.$$

Let $z = -y + \varepsilon \frac{y}{\|y\|}$, whose Jacobian is $O\big(\frac{\varepsilon^{N-1}}{\|z\|^{N-1}}\big)$ (see [BM2]). We then have

$$\begin{aligned}
\int_0^1 \frac{d\varepsilon}{\varepsilon} \fint_{C_\varepsilon} I_{\varepsilon,Q}\, dQ &\leq C \int_0^1 \frac{d\varepsilon}{\varepsilon} \int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon} \frac{|f(x) - f(x+z)|^p}{|z|^{N+sp-1}} \varepsilon^{N-1}\, dz \\
&\leq C \int_{C_2} dx \int_{C_1} \frac{|f(x) - f(x+z)|^p}{|z|^{N+sp-1}}\, dz \int_{|z|}^1 \frac{d\varepsilon}{\varepsilon^2} \\
&\leq \iint \frac{|f(x) - f(x+z)|^p}{|z|^{N+sp}}\, dx\, dz < \infty.
\end{aligned}$$

This concludes the proof of Lemma F5.

Given $N - 1 < p \leq N$, any map $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$ has a harmonic extension $U \in W^{N/p,p}(B^{N+1}; \mathbb{R}^N) \subset W^{1,N}$. We then define $T(u)$ as

$$\langle T(u), \zeta \rangle = -\sum_{j=1}^N \int_{B^{N+1}} \det\,(U_{x_1}, \ldots, U_{x_{j-1}}, U, U_{x_{j+1}}, \ldots, U_{x_{N+1}})\, \xi_{x_j},$$

where $\zeta \in \operatorname{Lip}(S^N)$ and ξ is any extension of ζ to B^{N+1}. One can see that this definition is independent of the extension ξ (see [BBM2] for the case $N = p = 2$). Let

$$L(u) = \frac{1}{\sigma_N} \operatorname*{Max}_{\|\nabla\zeta\|_{L^\infty} \leq 1} \langle T(u), \zeta \rangle.$$

We have the following

Lemma F6. *Assume $N \geq 2$. Let $1 < p < \infty$ and $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$. If $T(u) = 0$, then $u \in \overline{C^\infty(S^N; S^{N-1})}^{W^{(N-1)/p,p}}$.*

Proof.

Case 1. Proof of the lemma if $N \geq 3$.

Note that good maps are dense in $W^{1,N-1}(S^N; S^{N-1})$ and, by interpolation, in $W^{(N-1)/p,p}(S^N; S^{N-1})$ (see the definition of a good map in Section D above). Thus, it suffices to show that if u is a good map, then there exists $v_n \in C^\infty(S^N; S^{N+1})$ such that

$$\|u - v_n\|_{W^{1,N-1}} \leq CL(u) + \frac{1}{n}, \quad \forall n \geq 1,$$

which can be done by a dipole construction. By interpolation, we obtain Lemma F6.

Case 2. Proof of the lemma if $N = 2$.

The interpolation argument does not work in this case. However, for any map $u \in \mathcal{R}$, the dipole construction in [BBM2] gives a sequence (v_n) such that

$$T(v_n) = T(u), \quad |v_n|^p_{W^{1/p,p}} \leq CL(u), \quad \text{and} \quad v_n \to 1 \quad \text{a.e.}$$

Clearly, $u\bar{v}_n \in \overline{C^\infty(S^N; S^{N-1})}^{W^{1/p,p}}$ and

$$\begin{aligned} |u - u\bar{v}_n|^p_{W^{1/p,p}} &\leq 2^p |1 - \bar{v}_n|^p_{W^{1/p,p}} + 2^p \iint |1 - \bar{v}_n(x)|^p \frac{|u(x) - u(y)|^p}{|x - y|^2}\, dx\, dy \\ &\leq 2^p CL(u) + o(1). \end{aligned}$$

Using the density of $\mathcal{R}$ in $W^{1/p,p}(S^2; S^1)$, we obtain the desired result.

We also point out the following extension of Lemma F6 whose proof is left to the reader :

Lemma F7. *Assume $N - 1 < p \leq N$. Let Ω be a smooth subdomain of S^N. For any $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$, $T(u|_\Omega)$ is well-defined when computed against Lipschitz functions which are constant on each connected component of $\partial\Omega$. If $T(u|_\Omega) = 0$, then $u|_\Omega \in \overline{C^\infty(\overline{\Omega}; S^{N-1})}^{W^{(N-1)/p,p}}$.*

A key ingredient in the proof of Theorem 5‴ is the following

Proposition F1. *Assume $N - 1 < p \leq N$. Let $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$ be such that*

$$T(u) = \sigma_N \sum_{\text{finite}} d_i \delta_{M_i}.$$

Then, there exist $(u_n) \subset W^{(N-1)/p,p}(S^N; S^{N-1})$, $\varepsilon_n \to 0$, *and* $M_i^n \to M_i$ *such that*

(i) $u_n \in C^\infty(S^N \setminus \bigcup_i C_{\varepsilon_n}(M_i^n); S^{N-1})$;

(ii) u_n *is homogeneous of degree* 0 *on each cube* $C_{\varepsilon_n}(M_i^n)$;

(iii) $T(u_n) = \sigma_N \sum\limits_{\text{finite}} d_i \delta_{M_i^n}$;

(iv) $u_n \to u$ *in* $W^{(N-1)/p,p}$.

Proof. We first observe that the set of good pairs (ε, Q), in the sense of Lemma F5, is "fat". More precisely, there exists a sequence $\varepsilon_n \to 0$ such that

$$\frac{\left|\left\{Q \in C_{\varepsilon_n/4} \; ; \; |f_{\varepsilon_n,Q} - f|_{W^{(N-1)/p,p}} < 1/n\right\}\right|}{|C_{\varepsilon_n/4}|} \geq \frac{1}{2}.$$

In particular, the set

$$\left\{Q \in C_{\varepsilon_n/4} \; ; \; |f_{\varepsilon_n,Q} - f|_{W^{(N-1)/p,p}} < 1/n\right\}$$

intersects the complement of any null set of $C_{\varepsilon_n/4}$. For $n \geq 1$ fixed, consider

$$v_n = u|_{S^N \setminus \bigcup_i C_{\varepsilon_n/4}(M_i)}.$$

Then, $T(v_n) = 0$, so that there exists a sequence $(v_n^k)_k$,

$$v_n^k \in C^\infty\big(S^N \setminus \bigcup_i C_{\varepsilon_n/4}(M_i); S^{N-1}\big) \quad \forall k \geq 1,$$

such that

$$v_n^k \to v_n \quad \text{in } W^{(N-1)/p,p}\big(S^N \setminus \bigcup_i C_{\varepsilon_n/4}(M_i)\big) \quad \text{as } k \to \infty.$$

By Fubini, for a.e. $Q \in C_{\varepsilon_n/4}$, we have

$$v_n^k|_{\bigcup_i \partial C_{\varepsilon_n}(Q+M_i)} \to u|_{\bigcup_i \partial C_{\varepsilon_n}(Q+M_i)} \quad \text{in } W^{(N-1)/p,p} \quad \text{as } k \to \infty.$$

By Lemmas F3 and F4, for any such Q we have

$$\tilde{v}_n^k \to \tilde{u}_n \quad \text{in } W^{(N-1)/p,p} \quad \text{as } k \to \infty,$$

where $\tilde{v}_n^k$ (resp., $\tilde{u}_n$) is v_n^k (resp., u) extended by homogeneity of degree 0 on each cube $C_{\varepsilon_n}(Q + M_i)$. By Lemma F5, we can choose $Q = Q_n$ such that

$$\tilde{u}_n \to u \quad \text{in } W^{(N-1)/p,p}.$$

Applying a diagonalization argument, u_n may be taken among $(\tilde{u}_n^k)$. We only need to show that (iii) of Proposition F.1 holds for the sequence (u_n). Note that $\tilde{v}_n^k$ is locally Lipschitz on $S^N \backslash \bigcup_i M_i^n$, where $M_i^n = Q_n + M_i$. Thus,

$$T(\tilde{v}_n^k) = \sigma_N \sum_{\text{finite}} \tilde{d}_i \delta_{M_i^n}.$$

Since $T(u_n) \rightharpoonup T(u)$, for n large enough we have (iii). This concludes the proof of Proposition F1.

We may now present the

Proof of Theorem 5‴.
Step 1. If $TV(u) < \infty$, then $\operatorname{Det}(\nabla u)$ is a measure and

$$|\operatorname{Det}(\nabla u)|_{\mathcal{M}} \le TV(u). \tag{F6}$$

Let $(u_n) \subset C^\infty(\Omega; \mathbb{R}^N)$ be such that $u_n \to u$ in $W^{(N-1)/p,p}$ and

$$\int_\Omega |\det \nabla u_n| \to TV(u).$$

Clearly, we may replace u_n in the definition of $TV(u)$ by

$$\tilde{u}_n = \begin{cases} u_n & \text{if } |u_n| \le 1, \\ \dfrac{u_n}{|u_n|} & \text{if } |u_n| > 1. \end{cases}$$

We may thus assume that $|u_n| \le 1$. Since $u_n \to u$ in $W^{(N-1)/p,p}$ and $|u_n| \le 1$, we have

$$\langle T(u_n), \zeta \rangle \to \langle T(u), \zeta \rangle$$

for every $\zeta \in \operatorname{Lip}(S^N; S^{N-1})$. In addition,

$$\frac{1}{N} \langle T(u_n), \zeta \rangle = \int_{S^{N-1}} (\det \nabla u_n)\, \zeta \le TV(u) \|\zeta\|_{L^\infty} + o(1).$$

Thus, $T(u)$ is a measure and (F6) holds.

Step 2. If $\mathrm{Det}\,(\nabla u)$ is a measure, then $TV(u) < \infty$ and

$$\frac{1}{\omega_n} TV(u) = \text{number of topological singularities of } u. \tag{F7}$$

By Proposition F1, it suffices to compute $TV(u)$ when u is smooth outside finitely many (disjoint) cubes and u is homogeneous of degree 0 inside each one of these cubes. By (F6), we have "$\geq$" in (F7). It then suffices to show the reverse inequality.

Note that $u \in W^{1,q}(S^N; S^{N-1})$ for every $q < N$. As in the proof of the case $W^{1,N-1}(S^N; S^{N-1})$, we can find $(u_n) \subset C^\infty$, $u_n \to u$ in $W^{1,q}$, with

$$\frac{1}{\omega_N} \int_{S^{N-1}} |\det \nabla u_n| = \text{number of topological singularities of } u.$$

For $N - 1 < q < N$, we have $W^{1,q} \cap L^\infty \subset W^{(N-1)/p,p}$, so that

$$u_n \to u \quad \text{in } W^{(N-1)/p,p}.$$

We conclude that (F7) holds.

G. Proofs of Theorems 14–17.

We start by establishing the precise value of $E(g)$:

Lemma G1. *Let $g \in BV(I; S^1)$ and let A be the set of jump points of g. Then*

$$E(g) = |\dot{g}_d| + \sum_{a \in A} d_{S^1}(g(a+), g(a-)). \tag{G1}$$

Proof. Let $\varphi \in BV(I; \mathbb{R})$ be any lifting of g. We claim that

$$|\dot{\varphi}_d|_{\mathcal{M}(I)} = |\dot{g}_d|_{\mathcal{M}(I)}. \tag{G2}$$

Indeed, recall that, by the chain rule, we have

$$\dot{\varphi}_d = -i\bar{g}\dot{g}_d.$$

Set $\nu = \dot{g}_d$ and $\mu = |\nu|$. Then there is some $k \in L^\infty((I, d\mu); S^1)$ such that $\nu = k\mu$. Since ν is diffuse and $-i\bar{g} \in BV$, we have $-ig \in L^\infty((I, d\mu); S^1)$, and thus $\dot{\varphi}_d = \ell\mu$, where $\ell = -i\bar{g}k \in L^\infty((I, d\mu); S^1)$. It follows that

$$|\dot{\varphi}_d|_{\mathcal{M}(I)} = \operatorname*{Sup}_{\substack{\zeta \in C_0(I;\mathbb{C}) \\ |\zeta| \leq 1}} \langle \dot{\varphi}_d, \zeta \rangle = \operatorname*{Sup}_{\substack{\zeta \in C_0(I;\mathbb{C}) \\ |\zeta| \leq 1}} \langle \mu, \ell\zeta \rangle = \langle \mu, |\ell| \rangle = |\mu|_{\mathcal{M}(I)} = |\dot{g}_d|_{\mathcal{M}(I)}.$$

Let now B denote the set of the jump points of φ; clearly, $B \supset A$. For each $a \in A$, we have

$$g(a+) = \mathrm{e}^{i\varphi(a+)} \quad \text{and} \quad g(a-) = \mathrm{e}^{i\varphi(a-)},$$

so that

$$|\varphi(a+) - \varphi(a-)| \geq d_{S^1}(g(a+), g(a-)).$$

If $a \in B \setminus A$, then $\mathrm{e}^{i\varphi(a+)} = \mathrm{e}^{i\varphi(a-)}$; thus

$$|\varphi(a+) - \varphi(a-)| \geq 2\pi.$$

Consequently,

$$\begin{aligned} |\dot\varphi|_{\mathcal{M}(I)} &= |\dot\varphi_d|_{\mathcal{M}(I)} + \sum_{a\in A} |\varphi(a+) - \varphi(a-)| + \sum_{a \in B\setminus A} |\varphi(a+) - \varphi(a-)| \\ &\geq |\dot g_d| + \sum_{a\in A} d_{S^1}(g(a+), g(a-)) + 2\pi \operatorname{card}(B \setminus A) \qquad \text{(G3)} \\ &\geq |\dot g_d| + \sum_{a\in A} d_{S^1}(g(a+), g(a-)). \end{aligned}$$

This proves "$\geq$" in (G1). Note that equality holds in (G3) if and only if

$$|\varphi(a+) - \varphi(a-)| = d_{S^1}(g(a+), g(a-)) \quad \forall a \in A \quad \text{and} \quad B = A.$$

In order to prove "$\leq$", we split A as $A = A_1 \cup A_2$, where

$$A_1 = \Big\{ a \in A\,;\, |g(a+) - g(a-)| = 2 \Big\} \quad \text{and} \quad A_2 = \Big\{ a \in A\,;\, |g(a+) - g(a-)| < 2 \Big\}.$$

If $a \in A_2$ we may define a *signed* distance

$$\delta_{S^1}(g(a+), g(a-)) = \arg\left(\frac{g(a+)}{g(a-)}\right).$$

Here, arg stands for the argument in $(-\pi, \pi)$. Set

$$\mu = -i\bar g \dot g_d + \pi \sum_{a\in A_1} \delta_a + \sum_{a\in A_2} \delta_{S^1}(g(a+), g(a-))\, \delta_a. \tag{G4}$$

We claim that μ is a measure. Indeed, A_1 is finite, since $g \in BV$. On the other hand,

$$\operatorname*{Sup}_{a\in A_2} |g(a+) - g(a-)| = d < 2$$

(again since $g \in BV$). Thus there exists some $C > 0$ such that

$$\left|\delta_{S^1}(g(a+), g(a-))\right| \le C|g(a+) - g(a-)|.$$

It follows that

$$\sum_{a \in A_2} \left|\delta_{S^1}(g(a+), g(a-))\right| \le C \sum_{a \in A_2} |g(a+) - g(a-)| < \infty.$$

Assume that $I = (0, \alpha)$ for some $\alpha > 0$ and set $\varphi_0(x) = \mu((0, x))$, $x \in I$. We claim that, up to a constant, φ_0 is a lifting of g and that $|\dot{\varphi}_0|_{\mathcal{M}(I)} = E(g)$. Indeed, using the chain rule for a product we have

$$\text{(G5)} \qquad \dot{\overline{g\mathrm{e}^{-i\varphi_0}}} = \mathrm{e}^{-i\varphi_0}\dot{g}_d - ig\,\mathrm{e}^{-i\varphi_0}(\dot{\varphi}_0)_d + \sum_{a \in A} \left(g\mathrm{e}^{-i\varphi_0}(a+) - g\mathrm{e}^{-i\varphi_0}(a-)\right)\delta_a.$$

Here, we have used the fact that φ is continuous outside A. For $a \in A$, we have

$$\varphi_0(a+) = \varphi_0(a-) + \mu(\{a\}),$$

so that

$$\mathrm{e}^{-i\varphi_0(a+)} = \mathrm{e}^{-i\varphi_0(a-)}\frac{g(a-)}{g(a+)},$$

by our definition of μ. Thus the last term in (G5) vanishes. On the other hand,

$$(\dot{\varphi}_0)_d = -i\bar{g}\dot{g}_d,$$

so that

$$\mathrm{e}^{-i\varphi_0}\dot{g}_d - ig\mathrm{e}^{-i\varphi_0}(\dot{\varphi}_0)_d = \mathrm{e}^{-i\varphi_0}(\dot{g}_d - \dot{g}_d) = 0.$$

Thus, there exists some $C \in \mathbb{C}$ such that $\varphi = \varphi_0 + C$ is a lifting of g. On the other hand,

$$\begin{aligned}|\dot{\varphi}|_{\mathcal{M}(I)} = |\dot{\varphi}_0|_{\mathcal{M}(I)} &= |-i\bar{g}\dot{g}_d|_{\mathcal{M}(I)} + \pi\,\mathrm{card}\,(A_1) + \sum_{a \in A_2} \left|\delta_{S^1}(g(a+), g(a-))\right| \\ &= |\dot{g}_d|_{\mathcal{M}(I)} + \sum_{a \in A} d_{S^1}(g(a+), g(a-)).\end{aligned}$$

The proof of Lemma G1 is complete.

Proof of Theorem 14. We shall prove a slightly stronger assertion, which implies all the properties stated in the theorem. Namely, a lifting $\varphi \in BV(I;\mathbb{R})$ of g is a canonical lifting if and only if

(i) φ and g have the same jump sets ;

(ii) for $a \in A_1$, we have $\varphi(a+) - \varphi(a-) = \pm\pi$;

(iii) for $a \in A_2$, we have $\varphi(a+) - \varphi(a-) = \delta_{S^1}(g(a+), g(a-))$.

(The sets A_1, A_2 are defined in the proof of Lemma G1.)

Property (i) was seen to be necessary for optimality in the proof of Lemma G1. Recall that, in addition to (i), equality in (G3) amounts to

$$|\varphi(a+) - \varphi(a-)| = d_{S^1}(g(a+), g(a-)) \quad \forall a \in A. \tag{G6}$$

If $a \in A_1$, then

$$|\varphi(a+) - \varphi(a-)| = \pi,$$

so that (ii) holds. Assume $a \in A_2$. Since $\mathrm{e}^{i\varphi(a+)} = g(a+)$ and $\mathrm{e}^{i\varphi(a-)} = g(a-)$, then by (G6) we have

$$\varphi(a+) - \varphi(a-) = \arg\left(\frac{g(a+)}{g(a-)}\right),$$

which gives (iii). Conversely, it is easy to see that, if (i)–(iii) are fulfilled, then equality holds in (G3).

Proof of Theorem 15. We identify $S^1 \setminus \{z\}$ with an interval I. Let A, A_1, A_2 be defined as in the proof of Lemma G1. We claim that, for *each* choice of integers $\varepsilon_a \in \{-1, 1\}$, $a \in A_1$, there is a canonical lifting φ of g on I such that

$$\varphi(a+) - \varphi(a-) = \varepsilon_a \pi \quad \forall a \in A_1.$$

This φ is obtained, as in the proof of Lemma G1, as $\varphi = \mu((0,x)) + C$, $x \in I$. One simply has to modify the definition of μ by taking

$$\mu = -i\bar{g}\dot{g}_d + \sum_{a \in A_1} \varepsilon_a \pi \delta_a + \sum_{a \in A_2} \delta_{S^1}(g(a+), g(a-))\, \delta_a.$$

Moreover, the proof of Theorem 14 shows that, by this procedure, we obtain *all* canonical liftings. We claim that if φ is the canonical lifting corresponding to the choice ε_a, $a \in A_1$, and $\tilde{\varphi}$ the one corresponding to $\tilde{\varepsilon}_a$, $a \in A_1$, then

$$\frac{\tilde{\varphi}(z-) - \tilde{\varphi}(z+)}{2\pi} = \frac{\varphi(z-) - \varphi(z+)}{2\pi} + \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a) \in \mathbb{Z}.$$

If we identify $S^1 \setminus \{z\}$ with $I = (0, \alpha)$, $\alpha > 0$, this amounts to proving that

$$\frac{\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0+)}{2\pi} = \frac{\varphi(\alpha-) - \varphi(0+)}{2\pi} + \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a).$$

We have $\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0+) = \tilde{\mu}(I)$ (where $\tilde{\mu}$ is the corresponding measure); a similar assertion holds for φ. Thus

$$\frac{\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0)}{2\pi} - \frac{\varphi(\alpha-) - \varphi(0+)}{2\pi} = \frac{(\tilde{\mu} - \mu)(I)}{2\pi} = \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a).$$

Let φ be the canonical lifting corresponding to the choice $\varepsilon_a = -1$, $a \in A_1$. Then it is clear that, with $d = \frac{\varphi(z-) - \varphi(z+)}{2\pi}$ and $k = \operatorname{card} A_1$, we have

$$\operatorname{Deg}_1 g = \{d, d+1, \dots, d+k\}.$$

To see that d is an integer, note that $e^{i\varphi(z-)} = e^{i\varphi(z+)}$. Thus,

$$\varphi(z-) - \varphi(z+) \in 2\pi\mathbb{Z}.$$

It remains to establish that $\operatorname{Deg}_1 g$ does not depend on the choice of z. Let w be any other continuity point of g. Let ψ be a canonical lifting of g on $S^1 \setminus \{w\}$. Since g is continuous at w, there is some $k \in \mathbb{Z}$ such that $\psi(w-) = \psi(w+) + 2k\pi$. We set

$$\varphi(\xi) = \begin{cases} \psi(\xi) & \text{if } \xi \in (w+, z-), \\ \psi(\xi) - 2k\pi & \text{if } \xi \in (z+, w-). \end{cases}$$

Clearly, $\varphi \in BV$ and φ is continuous at w. It is obvious that

$$|\dot{\varphi}|_{\mathcal{M}(S^1 \setminus \{z\})} = |\dot{\psi}|_{\mathcal{M}(S^1 \setminus \{z,w\})} = |\dot{\psi}|_{\mathcal{M}(S^1 \setminus \{w\})}.$$

It follows that φ is a canonical lifting of g on $S^1 \setminus \{z\}$. Indeed, by Lemma G1 we have

$$E(g|_{S^1 \setminus \{z\}}) = E(g|_{S^1 \setminus \{w\}}) = |g|_{BV S^1}$$

if z, w are continuity points of g. Since

$$\begin{aligned} \varphi(z-) - \varphi(z+) &= \varphi(z-) - \varphi(w+) + \varphi(w+) - \varphi(w-) + \varphi(w-) - \varphi(z+) \\ &= \psi(w-) - \psi(w+), \end{aligned}$$

we find that the degrees obtained by cutting at z are among the ones obtained by cutting at w. By reversing the roles, we conclude that $\operatorname{Deg}_1 g$ is independent of z.

Erratum. In the definition of Deg_2 (see [BMP, Definition 3]), the convergence

$$\int |\dot{g}_n| \to \int |\dot{g}|,$$

should be replaced by

$$\int |\dot{g}_n| \to |g|_{BVS^1}.$$

Proof of Theorem 16. Let z be a continuity point of g and let φ be a canonical lifting of g in $S^1 \setminus \{z\}$. Assume, e.g., that $z = 1$; we identify $S^1 \setminus \{1\}$ with $(0, 2\pi)$. Consider a sequence $(\varphi_n) \subset C^\infty([0, 2\pi])$ such that

$$\int_0^{2\pi} |\dot{\varphi}_n| \to |\dot{\varphi}|_{\mathcal{M}((0,2\pi))} \quad \text{and} \quad \varphi_n \to \varphi \quad \text{a.e.}$$

We may assume, in addition, that

$$\varphi_n(0) \to \varphi(0+) \quad \text{and} \quad \varphi_n(2\pi) \to \varphi(2\pi-).$$

(This is the case, e.g., if the functions φ_n are obtained from φ by mollification). By replacing φ_n with $\varphi_n + \alpha_n x + \beta_n$, for some appropriate $\alpha_n \to 0$ and $\beta_n \to 0$, we may further assume that

$$\varphi_n(0) = \varphi(0+) \quad \text{and} \quad \varphi_n(2\pi) = \varphi(2\pi-) \quad \forall n \geq 1.$$

Set $g_n = \mathrm{e}^{i\varphi_n}$. Then, clearly,

$$g_n \in C^\infty(S^1 \setminus \{z\}) \cap C^0(S^1) \quad \text{and} \quad \deg g_n = \frac{\varphi(1-) - \varphi(1+)}{2\pi}.$$

By further mollifying g_n, we find a sequence $(h_n) \subset C^\infty(S^1; S^1)$ such that

$$h_n \to g \quad \text{a.e.}, \quad \deg h_n = \frac{\varphi(1-) - \varphi(1+)}{2\pi} \quad \forall n \geq 1,$$

and

$$\int_{S^1} |\dot{h}_n| \to \int_{S^1 \setminus \{z\}} |\dot{\varphi}| = E(g|_{S^1 \setminus \{z\}}).$$

It follows that

$$\mathrm{Deg}_2\, g \supset \mathrm{Deg}_1\, g.$$

Conversely, let $d \in \mathrm{Deg}_2\, g$ and let $(g_n) \subset C^\infty(S^1; S^1)$ be such that

$$g_n \to g \quad \text{a.e.}, \qquad \int_{S^1} |\dot{g}_n| \to |g|_{BVS^1} \quad \text{and} \quad \deg g_n = d \quad \forall n \geq 1.$$

Let $z \in S^1$ be a continuity point of g. Write, in $S^1 \setminus \{z\}$, $g_n = \mathrm{e}^{i\varphi_n}$. Then

$$\int_{S^1} |\dot{\varphi}_n| \to |g|_{BVS^1}.$$

Up to some subsequence and after subtracting a suitable multiple of 2π, we may assume that $\varphi_n \to \varphi$ a.e., where $\varphi \in BV$ is a lifting of g. Since

$$|\dot{\varphi}|_{\mathcal{M}(S^1 \setminus \{z\})} \leq |g|_{BVS^1},$$

we find that φ has to be a canonical lifting of g. Given $\varepsilon > 0$, there exists some $\delta > 0$ such that, if I is the interval of size δ centered at z, then we have $|g|_{BVS^1(I)} < \varepsilon$. We may further assume that g is continuous at the endpoints of I. Then

$$|g|_{BVS^1(S^1)} = |g|_{BVS^1(I)} + |g|_{BVS^1(S^1 \setminus I)}.$$

Arguing as above, we find that

$$\int_{S^1 \setminus I} |\dot{\varphi}_n| \to |g|_{BVS^1(S^1 \setminus I)} \quad \text{and} \quad \int_{I \setminus \{z\}} |\dot{\varphi}_n| \to |g|_{BVS^1(I)}.$$

In particular, for every $n \geq 1$ sufficiently large,

$$\big|(\varphi_n(z-t) - \varphi_n(z+t)) - (\varphi_n(z-) - \varphi_n(z+))\big| < 2\varepsilon, \quad \forall t \in (0, \delta).$$

We pick such t so that, in addition,

$$\varphi_n(z-t) \to \varphi(z-t) \quad \text{and} \quad \varphi_n(z+t) \to \varphi(z+t).$$

We then find

$$\big|\varphi(z-t) - \varphi(z+t) - 2\pi d\big| \leq 2\varepsilon.$$

Letting $\varepsilon \to 0$ and $\delta \to 0$, we obtain

$$\varphi(z-) - \varphi(z+) = 2\pi d,$$

i.e., $d \in \mathrm{Deg}_1\, g$.

We complete the proof of Theorem 16 by proving that $g \mapsto \mathrm{Deg}\, g$ is continuous in the multivalued sense. Since Deg is $\mathbb{Z}$-valued, this amounts to proving that, for each $d \in \mathbb{Z}$, the set

$$\big\{g \in BV(S^1; S^1)\ ;\ d \in \mathrm{Deg}\, g\big\}$$

is open. To this purpose, we start with the following

Lemma G2. *Let $g \in BV(I; S^1)$. Let $\varphi \in BV(I; \mathbb{R})$ be a lifting of g. If φ is not a canonical lifting of g, then*

$$|\dot{\varphi}|_{\mathcal{M}(I)} \geq E(g) + \pi.$$

Proof. Let A, B be the set of jump points of g, φ, respectively. Recall that

$$\begin{aligned}|\dot{\varphi}|_{\mathcal{M}(I)} &\geq |\dot{g}_d|_{\mathcal{M}(I)} + \sum_{a \in A} |\varphi(a+) - \varphi(a-)| + \sum_{a \in B \setminus A} [\varphi(a+) - \varphi(a-)| \\ &\geq E(g) + \sum_{a \in B \setminus A} |\varphi(a+) - \varphi(a-)|.\end{aligned}$$

If $B \neq A$, then

$$|\varphi(a+) - \varphi(a-)| \geq 2\pi \quad \forall a \in B \setminus A,$$

and the conclusion is clear. If $B = A$, then there is some $a \in A$ such that

$$|\varphi(a+) - \varphi(a-)| > d_{S^1}(g(a+), g(a-)),$$

for otherwise φ would be a canonical lifting. For any such a, we have

$$|\varphi(a+) - \varphi(a-)| \equiv d_{S^1}(g(a+), g(a-)) \mod 2\pi.$$

Since $d_{S^1}(g(a+), g(a-)) \leq \pi$, we find that

$$|\varphi(a+) - \varphi(a-)| \geq d_{S^1}(g(a+), g(a-)) + \pi.$$

To finish the proof note that, for any $b \in A \setminus \{a\}$, we have

$$|\varphi(b+) - \varphi(b-)| \geq d_{S^1}(g(b+), g(b-)),$$

and the conclusion follows.

Proof of Theorem 16 completed. If $g, h \in BV(S^1; S^1)$, then

$$|g\bar{h}|_{BV} = |g\bar{h} - 1|_{BV} = |g\,(\bar{h} - \bar{g})|_{BV} \leq |g|_{BV}\|h - g\|_{L^\infty} + |h - g|_{BV}. \tag{G7}$$

Let $g \in BV(S^1; S^1)$ and let $d \in \operatorname{Deg} g$. In view of (G7), there is some $\varepsilon > 0$ such that if

$$h \in BV(S^1; S^1) \quad \text{and} \quad \|g - h\|_{BV} < \varepsilon,$$

then

$$|g\bar{h}|_{BV} < \frac{1}{10}.$$

We claim that $d \in \operatorname{Deg} h$ for any such h. Indeed, let z be a continuity point for both g and h, and let φ be a canonical lifting of g in $S^1 \setminus \{z\}$. Set $k = \bar{g}h$ and let ψ be a canonical lifting of k. Since $|k|_{BV} < \frac{1}{10}$, each jump point a of k is such that $|k(a+) - k(a-)| < \frac{1}{10}$. Thus

$$|\psi(a+) - \psi(a-)| \le 2|k(a+) - k(a-)|$$

for any such a. It follows that

$$|\psi|_{BV} = |\dot{k}_d|_{\mathcal{M}(I)} + \sum_{\substack{\text{jump points}\\ \text{of } k}} |\psi(a+) - \psi(a-)| \le 2|k|_{BV} < \frac{1}{5}.$$

Set $\phi = \varphi + \psi$. Then ϕ is a lifting of h and

$$\big|(\phi(z-) - \phi(z+)) - (\varphi(z-) - \varphi(z+))\big| < \frac{2}{5},$$

so that

$$\phi(z-) - \phi(z+) = \varphi(z-) - \varphi(z+)$$

(since both quantities are multiple of 2π). In order to complete the proof of Theorem 16, it suffices to prove that ϕ is a canonical lifting of h. Indeed, on the one hand we have

$$E(h) \le |\dot{\phi}|_{\mathcal{M}(S^1\setminus\{z\})} \le |\dot{\varphi}|_{\mathcal{M}(S^1\setminus\{z\})} + \frac{1}{5},$$

so that $E(h) \le E(g) + \frac{1}{5}$. By reversing the roles, we obtain on the other hand that $E(g) \le E(h) + \frac{1}{5}$; thus

$$E(h) \le |\dot{\phi}|_{\mathcal{M}(S^1\setminus\{z\})} \le E(h) + \frac{2}{5}.$$

Lemma G2 implies that ϕ is a canonical lifting of h.

Proof of Theorem 17. With the notation we already used, we have

$$\big\{g\ ;\ \operatorname{Deg} g \text{ is single-valued}\big\} = \{g\ ;\ A_1(g) = \phi\} =: \mathcal{U}_1.$$

Thus, we have to prove that $\mathcal{U}_1$ is dense in $BV(S^1; S^1)$.

Let $g \in BV(S^1; S^1)$; then $A_1(g)$ is finite. If $A_1(g) = \phi$, then $g \in \mathcal{U}_1$. Otherwise, we may assume, for simplicity, that A_1 consists of a single point, say $A_1 = \{1\}$; the general case can be treated along the same lines. We have

$$|g(1-) - g(1+)| = 2.$$

Without loss of generality, we may assume that $g(1-) = -1$ and $g(1+) = 1$. Given $\varepsilon > 0$, let $h_\varepsilon : S^1 \to S^1$ be given by

$$h_\varepsilon(\mathrm{e}^{i\theta}) = \begin{cases} \mathrm{e}^{i\theta} & \text{if } 1 \le \theta \le 2\pi - 1, \\ \mathrm{e}^{i(\varepsilon + (1-\varepsilon)\theta)} & \text{if } 0 \le \theta \le 1, \\ \mathrm{e}^{i((2\pi-1)(2\pi+\varepsilon)-(2\pi-1+\varepsilon)\theta)} & \text{if } 2\pi - 1 \le \theta < 2\pi. \end{cases}$$

It is immediate that $h_\varepsilon(1+) = \mathrm{e}^{i\varepsilon}$, $h_\varepsilon(1-) = \mathrm{e}^{-i\varepsilon}$,

$$|h_\varepsilon|_{BV} \to 0 \quad \text{and} \quad h_\varepsilon \to 1 \quad \text{uniformly.}$$

Thus,

$$gh_\varepsilon \to g \quad \text{in } BV \quad \text{as } \varepsilon \to 0.$$

On the other hand, since $h_\varepsilon \in C^0(S^1 \setminus \{1\})$, we have

$$A_1(gh_\varepsilon) \setminus \{1\} = A_1(g) \setminus \{1\}.$$

In particular, $A_1(gh_\varepsilon) \subset \{1\}$. Since, by construction, $1 \notin A_1(gh_\varepsilon)$, we have $gh_\varepsilon \in \mathcal{U}_1$. The proof of Theorem 17 is complete.

References

[B1] F. Bethuel, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 269–286.

[BZ] F. Bethuel and X.M. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.

[BBM2] J. Bourgain, H. Brezis and P. Mironescu, *$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 1–115.

[BCL] H. Brezis, J.-M. Coron and E.H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

[BM1] H. Brezis and P. Mironescu, *On some questions of topology for S^1-valued fractional Sobolev spaces*, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **95** (2001), 121–143.

[BM2] ———, in preparation.

[BMP] H. Brezis, P. Mironescu and A.C. Ponce, *$W^{1,1}$-maps with values into S^1*, Geometric analysis of PDE and several complex variables (S. Chanillo, P. Cordaro, N. Hanges, J. Hounie and A. Meziani, eds.), Contemporary Mathematics, American Mathematical Society, Providence, RI, 2004.

[D] F. Demengel, *Une caractérisation des applications de $W^{1,p}(B^N, S^1)$ qui peuvent être approchées par des fonctions régulières*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), 553–557.

[Mi] V. Millot, *Energy with weight for S^2-valued maps with prescribed singularities*, to appear in Calc. Var. Partial Differential Equations.

[N] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Lecture Notes in Mathematics 6, New York University, New York, 2001.

[P] A.C. Ponce, *On the distributions of the form* $\sum_i(\delta_{p_i} - \delta_{n_i})$, J. Funct. Anal. **210** (2004), 391–435.

[S] D. Smets, *On some infinite sums of integer valued Dirac's masses*, C. R. Math. Acad. Sci. Paris **334** (2002), 371–374.

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