NONLINEAR ELLIPTIC EQUATIONS WITH MEASURES REVISITED

Haïm Brezis[(1),(2)], Moshe Marcus[(3)] and Augusto C. Ponce[(4)]

Abstract. We study the existence of solutions of the nonlinear problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a Radon measure and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $g(0) = 0$. This equation need not have a solution for every measure μ, and we say that μ is a good measure if (P) admits a solution. We show that for every μ there exists a largest good measure $\mu^* \leq \mu$. This *reduced measure* has a number of remarkable properties.

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0. Introduction.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function such that $g(0) = 0$. In this paper we are concerned with the problem

$$(0.1) \qquad \begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a measure. The study of (0.1) when $\mu \in L^1(\Omega)$ was initiated by Brezis-Strauss [BS]; their main result asserts that for *every* $\mu \in L^1$ and *every* g as above, problem (0.1) admits a unique weak solution (see Theorem B.2 in Appendix B below). The right concept of weak solution is the following:

$$(0.2) \qquad \begin{cases} u \in L^1(\Omega),\ g(u) \in L^1(\Omega) \text{ and} \\ -\displaystyle\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ on } \partial\Omega. \end{cases}$$

It will be convenient to write

$$C_0(\overline{\Omega}) = \big\{ \zeta \in C(\overline{\Omega})\ ;\ \zeta = 0 \text{ on } \partial\Omega \big\}$$

and

$$C_0^2(\overline{\Omega}) = \big\{ \zeta \in C^2(\overline{\Omega})\ ;\ \zeta = 0 \text{ on } \partial\Omega \big\},$$

and to say that (0.1) holds in the sense of $(C_0^2)^*$. We will often omit the word "weak" and simply say that u is a solution of (0.1), meaning (0.2). It follows from standard (linear) regularity theory that a weak solution u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ (see, e.g., [S] and Theorem B.1 below).

The case where μ is a measure turns out to be much more subtle than one might expect. It was observed in 1975 by Ph. Bénilan and H. Brezis (see [B1], [B2], [B3], [B4], [BB] and Theorem B.6 below) that if $N \geq 3$ and $g(t) = |t|^{p-1}t$ with $p \geq \frac{N}{N-2}$, then (0.1) has *no solution* when $\mu = \delta_a$, a Dirac mass at a point $a \in \Omega$. On the other hand, it was also proved (see Theorem B.5 below) that if $g(t) = |t|^{p-1}t$ with $p < \frac{N}{N-2}$ (and $N \geq 2$), then (0.1) has a solution for any measure μ. Later Baras-Pierre [BP] (see also [GM]) *characterized* all measures μ for which (0.1) admits a solution. Their necessary and sufficient condition for the existence of a solution when $p \geq \frac{N}{N-2}$ can be expressed in two equivalent ways:

$$(0.3) \qquad \begin{cases} \mu \text{ admits a decomposition } \mu = f_0 - \Delta v_0 \text{ in the } (C_0^2)^*\text{-sense}, \\ \text{with } f_0 \in L^1 \text{ and } v_0 \in L^p, \end{cases}$$

or

$$|\mu|(A) = 0 \quad \text{for every Borel set } A \subset \Omega \text{ with } \operatorname{cap}_{2,p'}(A) = 0, \tag{0.4}$$

where $\operatorname{cap}_{2,p'}$ denotes the capacity associated to $W^{2,p'}$.

Our goal in this paper is to analyze the nonexistence mechanism and to describe what happens if one "forces" (0.1) to have a solution in cases where the equation "refuses" to possess one. The natural approach is to introduce an approximation scheme. For example, μ is kept fixed and g is truncated. Alternatively, g is kept fixed and μ is approximated, e.g., via convolution. It was originally observed by one of us (see [B4]) that if $N \geq 3$, $g(t) = |t|^{p-1}t$, with $p \geq \frac{N}{N-2}$, and $\mu = \delta_a$, with $a \in \Omega$, then all "natural" approximations (u_n) of (0.1) converge to $u \equiv 0$. And, of course, $u \equiv 0$ is *not* a solution of (0.1) corresponding to $\mu = \delta_a$! It is this kind of phenomenon that we propose to explore in full generality. We are led to study the convergence of the approximate solutions (u_n) under various assumptions on the sequence of data.

Concerning the function g we will assume *throughout the rest of the paper* (except in Section 7) that $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and that

$$g(t) = 0 \quad \forall t \leq 0. \tag{0.5}$$

Remark 1. Assumption (0.5) is harmless when the data μ is nonnegative, since the corresponding solution u is nonnegative by the maximum principle and it is only the restriction of g to $[0, \infty)$ which is relevant. However when μ is a signed measure it is worthwhile to remove assumption (0.5) and this is done in Section 7 below.

By a *measure* μ we mean a continuous linear functional on $C_0(\overline{\Omega})$, or equivalently a finite measure on $\overline{\Omega}$ such that $|\mu|(\partial\Omega) = 0$ (see Appendix C below). The space of measures is denoted by $\mathcal{M}(\Omega)$ and is equipped with the standard norm

$$\|\mu\|_{\mathcal{M}} = \sup\left\{ \int_\Omega \varphi \, d\mu \, ; \, \varphi \in C_0(\overline{\Omega}) \text{ and } \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

By a (weak) *solution* u of (0.1) we mean that (0.2) holds. A (weak) *subsolution* u of (0.1) is a function u satisfying

$$\begin{cases} u \in L^1(\Omega), \ g(u) \in L^1(\Omega) \text{ and} \\ -\displaystyle\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta \leq \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \ \zeta \geq 0 \text{ in } \Omega. \end{cases} \tag{0.6}$$

We will say that $\mu \in \mathcal{M}(\Omega)$ is a *good measure* if (0.1) admits a solution. If μ is a good measure, then equation (0.1) has exactly one solution u (see Corollary B.1 in Appendix B). We denote by $\mathcal{G}$ the set of good measures (relative to g).

Remark 2. In many places throughout this paper, the quantity $\int_\Omega \zeta \, d\mu$, with $\zeta \in C_0^2(\overline{\Omega})$, plays an important role. Such an expression makes sense even for measures μ which are not bounded but merely locally bounded in Ω, and such that $\int_\Omega \rho_0 \, d|\mu| < \infty$, where $\rho_0(x) = d(x, \partial\Omega)$. Many of our results remain valid for such measures provided some of the statements (and the proofs) are slightly modified. In this case, the condition $g(u) \in L^1(\Omega)$ in (0.2) (and also in (0.6)) must be replaced by $g(u)\rho_0 \in L^1(\Omega)$. Since we have not pursued this direction, we shall leave the details to the reader.

In Section 1 we will introduce the first approximation method, namely μ is fixed and g is "truncated". In the sequel we denote by (g_n) a sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ which are continuous, nondecreasing and satisfy the following conditions:

$$0 \le g_1(t) \le g_2(t) \le \ldots \le g(t) \quad \forall t \in \mathbb{R}, \tag{0.7}$$

$$g_n(t) \to g(t) \quad \forall t \in \mathbb{R}. \tag{0.8}$$

(Recall that, by Dini's lemma, conditions (0.7) and (0.8) imply that $g_n \to g$ uniformly on compact subsets of $\mathbb{R}$).
If $N \ge 2$, we assume in addition that each g_n has subcritical growth, i.e., that there exist $C > 0$ and $p < \frac{N}{N-2}$ (possibly depending on n) such that

$$g_n(t) \le C(|t|^p + 1) \quad \forall t \in \mathbb{R}. \tag{0.9}$$

A good example to keep in mind is $g_n(t) = \min\{g(t), n\}$, $\forall t \in \mathbb{R}$.

Our first result is

Proposition 1. *Given any measure $\mu \in \mathcal{M}(\Omega)$, let u_n be the unique solution of*

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.10}$$

Then $u_n \downarrow u^*$ in Ω as $n \uparrow \infty$, where u^* is the largest subsolution of (0.1)*. Moreover we have*

$$\Big| \int_\Omega u^* \Delta\zeta \Big| \le 2\|\mu\|_{\mathcal{M}} \|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}) \tag{0.11}$$

and

$$\int_\Omega g(u^*) \le \|\mu\|_{\mathcal{M}}. \tag{0.12}$$

An important consequence of Proposition 1 is that u^* *does not depend on the choice of the truncating sequence* (g_n). *It is an intrinsic object which will play an important role in the sequel.* In some sense, u^* is the "best one can do" (!) in the absence of a solution.

Remark 3. If μ is a good measure, then u^* coincides with the unique solution u of (0.1); this is an easy consequence of standard comparison arguments (see Corollary B.2 in Appendix B).

We now introduce the basic concept of *reduced measure.* From (0.11), (0.12), and the density of $C_0^2(\overline{\Omega})$ in $C_0(\overline{\Omega})$ (easy to check), we see that there exists a unique measure $\mu^* \in \mathcal{M}(\Omega)$ such that

$$-\int_\Omega u^* \Delta\zeta + \int_\Omega g(u^*)\zeta = \int_\Omega \zeta \, d\mu^* \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{0.13}$$

We call μ^* the reduced measure associated to μ. Clearly, μ^* is always a good measure. Since u^* is a subsolution of (0.1), we have

$$\mu^* \le \mu. \tag{0.14}$$

Even though we have not indicated the dependence on g we emphasize that μ^* *does depend* on g (see Section 8 below).

One of our main results is

Theorem 1. *The reduced measure* μ^* *is the largest good measure* $\le \mu$.

Here is an easy consequence:

Corollary 1. *We have*

$$0 \le \mu - \mu^* \le \mu^+ = \sup\{\mu, 0\}. \tag{0.15}$$

In particular,

$$|\mu^*| \le |\mu| \tag{0.16}$$

and

$$[\mu \ge 0] \quad \Longrightarrow \quad [\mu^* \ge 0]. \tag{0.17}$$

Indeed, every measure $\nu \le 0$ is a good measure since the solution v of

$$\begin{cases} -\Delta v = \nu & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $v \le 0$ in Ω, and therefore by (0.5)

$$-\Delta v + g(v) = \nu \quad \text{in } (C_0^2)^*.$$

In particular, $-\mu^-$ is a good measure (recall that $\mu^- = \sup\{-\mu, 0\}$). Since $-\mu^- \le \mu$, we deduce from Theorem 1 that

$$-\mu^- \le \mu^*,$$

and consequently

$$\mu - \mu^* \le \mu + \mu^- = \mu^+.$$

Our next result asserts that the measure $\mu - \mu^*$ is concentrated on a small set:

Theorem 2. *There exists a Borel set $\Sigma \subset \Omega$ with* $\operatorname{cap}(\Sigma) = 0$ *such that*

$$(\mu - \mu^*)(\Omega \setminus \Sigma) = 0. \tag{0.18}$$

Here and throughout the rest of the paper "cap" denotes the Newtonian (H^1) capacity with respect to Ω.

Remark 4. Theorem 2 is optimal in the following sense. Given any measure $\mu \ge 0$ concentrated on a set of zero capacity, there exists some g such that $\mu^* = 0$ (see Theorem 14 below). In particular, $\mu - \mu^*$ can be *any* nonnegative measure concentrated on a set of zero capacity.

Here is a useful

Definition. *A measure $\mu \in \mathcal{M}(\Omega)$ is called diffuse if $|\mu|(A) = 0$ for every Borel set $A \subset \Omega$ such that* $\operatorname{cap}(A) = 0$.

An immediate consequence of Corollary 1 and Theorem 2 is

Corollary 2. *Every diffuse measure $\mu \in \mathcal{M}(\Omega)$ is a good measure.*

Indeed, let Σ be as in Theorem 2, so that $\operatorname{cap}(\Sigma) = 0$ and

$$(\mu - \mu^*)(\Omega \setminus \Sigma) = 0.$$

On the other hand, (0.15) implies

$$(\mu - \mu^*)(\Sigma) \le \mu^+(\Sigma) = 0,$$

since μ is diffuse. Therefore

$$(\mu - \mu^*)(\Omega) = 0,$$

so that $\mu = \mu^*$ and thus μ is a good measure.

Remark 5. The converse of Corollary 2 is *not* true. In Example 5 (see Section 8 below) the measure $\mu = c\delta_a$, with $0 < c \leq 4\pi$ and $a \in \Omega$, is a good measure, but it is not diffuse — $\operatorname{cap}(\{a\}) = 0$, while $\mu(\{a\}) = c > 0$. See, however, Theorem 5.

Remark 6. Recall that a measure μ is diffuse if and only if $\mu \in L^1 + H^{-1}$; more precisely, there exist $f_0 \in L^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \zeta \, d\mu = \int_\Omega f_0 \zeta - \int_\Omega \nabla v_0 \cdot \nabla \zeta \quad \forall \zeta \in C_0(\overline{\Omega}) \cap H_0^1. \tag{0.19}$$

The implication $[\mu \in L^1 + H^{-1}] \Rightarrow [\mu \text{ diffuse}]$ is due to Grun-Rehomme [GRe]. (In fact he proved only that $[\nu \in H^{-1}] \Rightarrow [\nu \text{ diffuse}]$, but L^1-functions are diffuse measures — since $[\operatorname{cap}(A) = 0] \Rightarrow [|A| = 0]$ — and the sum of two diffuse measures is diffuse). The converse $[\mu \text{ diffuse}] \Rightarrow [\mu \in L^1 + H^{-1}]$ is due to Boccardo-Gallouët-Orsina [BGO1] (and was suggested by earlier results of Baras-Pierre [BP] and Gallouët-Morel [GM]). As a consequence of Corollary 2 we obtain that, for every measure μ of the form (0.19), the problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.20}$$

admits a unique solution. In fact, the same conclusion was already known for any *distribution* in $L^1 + H^{-1}$, not necessarily in $\mathcal{M}(\Omega)$. (The proof, which combines techniques from Brezis-Browder [BBr] and Brezis-Strauss [BS], is sketched in Appendix B below; see Theorem B.4). A very useful sharper version of the [BGO1] decomposition is the following:

Theorem 3. *Assume $\mu \in \mathcal{M}(\Omega)$ is a diffuse measure. Then, there exist $f \in L^1(\Omega)$ and $v \in C_0(\overline{\Omega}) \cap H_0^1$ such that*

$$\int_\Omega \zeta \, d\mu = \int_\Omega f \zeta - \int_\Omega \nabla v \cdot \nabla \zeta \quad \forall \zeta \in C_0(\overline{\Omega}) \cap H_0^1. \tag{0.21}$$

In addition, given any $\delta > 0$, then f and v can be chosen so that

$$\|f\|_{L^1} \leq \|\mu\|_{\mathcal{M}}, \quad \|v\|_{L^\infty} \leq \delta \|\mu\|_{\mathcal{M}} \quad \textit{and} \quad \|v\|_{H^1} \leq \delta^{1/2} \|\mu\|_{\mathcal{M}}. \tag{0.22}$$

The proof of Theorem 3 is presented in Appendix D below.

In Section 2 we present some basic properties of the good measures. Here is a first one:

Theorem 4. *Suppose μ_1 is a good measure. Then any measure $\mu_2 \leq \mu_1$ is also a good measure.*

We now deduce a number of consequences:

Corollary 3. *Let $\mu \in \mathcal{M}(\Omega)$. If μ^+ is diffuse, then μ is a good measure.*

In fact, by Corollary 2, μ^+ diffuse implies that μ^+ is a good measure. Since $\mu \leq \mu^+$, it follows from Theorem 4 that μ is a good measure.

Corollary 4. *If μ_1 and μ_2 are good measures, then so is $\nu = \sup\{\mu_1, \mu_2\}$.*

Indeed, by Theorem 1 we have $\mu_1 \leq \nu^*$ and $\mu_2 \leq \nu^*$. Thus $\nu \leq \nu^* \leq \nu$, and hence $\nu = \nu^*$ is good measure.

Corollary 5. *The set $\mathcal{G}$ of good measures is convex.*

Indeed, let $\mu_1, \mu_2 \in \mathcal{G}$. For any $t \in [0,1]$, we have

$$t\mu_1 + (1-t)\mu_2 \leq \sup\{\mu_1, \mu_2\}.$$

Applying Corollary 4 and Theorem 4, we deduce that $t\mu_1 + (1-t)\mu_2 \in \mathcal{G}$.

Corollary 6. *For every measure $\mu \in \mathcal{M}(\Omega)$ we have*

$$\|\mu - \mu^*\|_{\mathcal{M}} = \min_{\nu \in \mathcal{G}} \|\mu - \nu\|_{\mathcal{M}}. \tag{0.23}$$

Moreover, μ^* is the unique good measure which achieves the minimum.

Proof. Let $\nu \in \mathcal{G}$ and write

$$|\mu - \nu| = (\mu - \nu)^+ + (\mu - \nu)^- \geq (\mu - \nu)^+ = \mu - \inf\{\mu, \nu\}.$$

But $\tilde{\nu} = \inf\{\mu, \nu\} \in \mathcal{G}$ by Theorem 4. Applying Theorem 1 we find $\tilde{\nu} \leq \mu^*$. Hence

$$|\mu - \nu| \geq \mu - \tilde{\nu} \geq \mu - \mu^* \geq 0,$$

and therefore

$$\|\mu - \nu\|_{\mathcal{M}} \geq \|\mu - \mu^*\|_{\mathcal{M}},$$

which gives (0.23). In order to establish uniqueness, assume $\nu \in \mathcal{G}$ attains the minimum in (0.23). Note that $\inf\{\mu, \nu\}$ is a good measure $\leq \mu$ and

$$\big\|\mu - \inf\{\mu, \nu\}\big\|_{\mathcal{M}} \leq \|\mu - \nu\|_{\mathcal{M}}.$$

Thus, $\nu = \inf\{\mu, \nu\} \leq \mu$. By Theorem 1, we deduce that $\nu \leq \mu^* \leq \mu$. Since ν achieves the minimum in (0.23), we must have $\nu = \mu^*$.

As we have already pointed out, the set $\mathcal{G}$ of good measures associated to (0.1) depends on the nonlinearity g. Sometimes, in order to emphasize this dependence, we shall denote $\mathcal{G}$ by $\mathcal{G}(g)$. By Corollary 3, if $\mu \in \mathcal{M}(\Omega)$ and μ^+ is diffuse, then $\mu \in \mathcal{G}(g)$ for every g satisfying (0.5). The converse is also true. More precisely,

Theorem 5. *Let $\mu \in \mathcal{M}(\Omega)$. Then $\mu \in \mathcal{G}(g)$ for every g if and only if μ^+ is diffuse.*

We also have a characterization of good measures in the spirit of the Baras-Pierre result (0.3):

Theorem 6. *A measure $\mu \in \mathcal{M}(\Omega)$ is a good measure if and only if μ admits a decomposition*

$$\mu = f_0 - \Delta v_0 \quad \text{in } \mathcal{D}'(\Omega),$$

with $f_0 \in L^1(\Omega)$, $v_0 \in L^1(\Omega)$ and $g(v_0) \in L^1(\Omega)$.

Corollary 7. *We have*

$$\mathcal{G} + L^1(\Omega) \subset \mathcal{G}.$$

In Section 3 we discuss some properties of the mapping $\mu \mapsto \mu^*$. For example, we show that for every $\mu, \nu \in \mathcal{M}(\Omega)$, we have

$$(\mu^* - \nu^*)^+ \leq (\mu - \nu)^+. \tag{0.24}$$

Inequality (0.24) implies, in particular, that

$$[\mu \leq \nu] \quad \Longrightarrow \quad [\mu^* \leq \nu^*] \tag{0.25}$$

and

$$|\mu^* - \nu^*| \leq |\mu - \nu|. \tag{0.26}$$

In Section 4 we examine another approximation scheme. We now keep g fixed but we smooth μ via convolution. Let $\mu_n = \rho_n * \mu$ and let u_n be the solution of

$$\begin{cases} -\Delta u_n + g(u_n) = \mu_n & \text{in } \Omega, \\ \qquad\qquad\quad u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.27}$$

We prove (assuming in addition g is convex) that $u_n \to u^*$ in $L^1(\Omega)$, where u^* is given by Proposition 1. In Section 5 we discuss other convergence results.

Theorem 5 is established in Section 6. In Section 7 we extend Proposition 1 to deal with the case where $\mu \in \mathcal{M}(\Omega)$ is a signed measure, but assumption (0.5) is no longer satisfied. Finally, in Section 8 we present several examples where the measure μ^* can be explicitly identified and in Section 9 we propose various directions of research.

Part of the results in this paper were announced in [BMP].

1. Construction of u^* and μ^*. Proofs of Proposition 1 and Theorems 1, 2.

We start with the

Proof of Proposition 1. Using Corollary B.2 in Appendix B we see that the sequence (u_n) is non-increasing. Also (see Corollary B.1)

$$\|g_n(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$$

and thus

$$\|\Delta u_n\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}.$$

Consequently,

$$\|u_n\|_{L^1} \leq C\|\mu\|_{\mathcal{M}}.$$

Therefore, (u_n) tends in L^1 to a limit denoted u^*. By Dini's lemma, $g_n \uparrow g$ uniformly on compact sets; thus

$$g_n(u_n) \to g(u^*) \quad \text{a.e.}$$

Hence $g(u^*) \in L^1(\Omega)$, (0.11)–(0.12) hold and, by Fatou's lemma,

$$-\int_\Omega u^*\Delta\zeta + \int_\Omega g(u^*)\zeta \leq \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega.$$

Therefore u^* is a subsolution of (0.1). We claim that u^* is the *largest* subsolution. Indeed let v be any subsolution of (0.1). Then

$$-\Delta v + g_n(v) \leq -\Delta v + g(v) \leq \mu \quad \text{in } (C_0^2)^*.$$

By comparison (see Corollary B.2)

$$v \leq u_n \quad \text{a.e.}$$

and, as $n \to \infty$,

$$v \leq u^* \quad \text{a.e.}$$

Hence u^* is the largest subsolution.

Recall (see [FST], or Appendix A below) that any measure μ on Ω can be uniquely decomposed as a sum of two measures, $\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}}$ ("d" stands for diffuse and "c" for concentrated), satisfying $|\mu_{\mathrm{d}}|(A) = 0$ for every Borel set $A \subset \Omega$ such that $\operatorname{cap}(A) = 0$, and $|\mu_{\mathrm{c}}|(\Omega \setminus F) = 0$ for some Borel set $F \subset \Omega$ such that $\operatorname{cap}(F) = 0$. Note that a measure μ is diffuse if and only if $\mu_{\mathrm{c}} = 0$, i.e., $\mu = \mu_{\mathrm{d}}$.

A key ingredient in the proof of Theorems 1 and 2 is the following version of Kato's inequality (see [K]) due to Brezis-Ponce [BP2].

Theorem 7 (Kato's inequality when Δv is a measure). *Let $v \in L^1(\Omega)$ be such that Δv is a measure on Ω. Then, for every open set $\omega \subset\subset \Omega$, Δv^+ is a measure on ω and the following holds:*

$$(\Delta v^+)_{\mathrm{d}} \geq \chi_{[v\geq 0]}(\Delta v)_{\mathrm{d}} \quad \textit{in } \omega, \tag{1.1}$$

$$(-\Delta v^+)_{\mathrm{c}} = (-\Delta v)_{\mathrm{c}}^+ \qquad \textit{in } \omega. \tag{1.2}$$

Note that the right-hand side of (1.1) is well-defined because the function v is quasi-continuous. More precisely, if $v \in L^1(\Omega)$ and Δv is a measure, then there exists $\tilde{v} : \Omega \to \mathbb{R}$ quasi-continuous such that $v = \tilde{v}$ a.e. in Ω (see [A1] and also [BP1, Lemma 1]). Recall that $\tilde{v}$ is quasi-continuous if and only if, given any $\varepsilon > 0$, one can find an open set $\omega_\varepsilon \subset \Omega$ such that $\operatorname{cap}(\omega_\varepsilon) < \varepsilon$ and $\tilde{v}|_{\Omega\setminus\omega_\varepsilon}$ is continuous. In particular, $\tilde{v}$ is finite q.e. (= quasi-everywhere = outside a set of zero capacity). It is easy to see that $\chi_{[\tilde{v}\geq 0]}$ is integrable with respect to the measure $|(\Delta v)_d|$. When $v \in L^1$ and Δv is a measure, we will systematically replace v by its quasi-continuous representative.

Here are two consequences of Theorem 7 which will be used in the sequel. The first one was originally established by Dupaigne-Ponce [DP] and it is equivalent to (1.2):

Corollary 8 ("Inverse" maximum principle). *Let $v \in L^1(\Omega)$ be such that Δv is a measure. If $v \geq 0$ a.e. in Ω, then*

$$(-\Delta v)_{\mathrm{c}} \geq 0 \quad \textit{in } \Omega.$$

Another corollary is the following

Corollary 9. *Let $u \in L^1(\Omega)$ be such that Δu is a measure. Then,*

$$\Delta T_k(u) \leq \chi_{[u\leq k]}(\Delta u)_{\mathrm{d}} + (\Delta u)_{\mathrm{c}}^+ \quad \textit{in } \mathcal{D}'(\Omega).$$

Here, $T_k(s) = k - (k-s)^+$ for every $s \in \mathbb{R}$.

Proof. Let $\omega \subset\subset \Omega$. Applying (1.1) and (1.2) to $v = k - u$, yields

$$(\Delta T_k(u))_{\mathrm{d}} = -(\Delta v^+)_{\mathrm{d}} \leq -\chi_{[v\geq 0]}(\Delta v)_{\mathrm{d}} = \chi_{[u\leq k]}(\Delta u)_{\mathrm{d}} \quad \text{in } \omega$$

and

$$(\Delta T_k(u))_{\mathrm{c}} = (\Delta u)_{\mathrm{c}}^+ \quad \text{in } \omega.$$

Combining these two facts, we conclude that

$$\Delta T_k(u) \leq \chi_{[u\leq k]}(\Delta u)_{\mathrm{d}} + (\Delta u)_{\mathrm{c}}^+ \quad \text{in } \mathcal{D}'(\omega).$$

Since $\omega \subset\subset \Omega$ was arbitrary, the result follows.

Let u^* be the largest subsolution of (0.1), and define $\mu^* \in \mathcal{M}(\Omega)$ by (0.13). We have the following

Lemma 1. *The reduced measure μ^* satisfies*

$$\mu^* \geq \mu_{\mathrm{d}} - \mu_{\mathrm{c}}^- .$$

Proof. Let (u_n) be the sequence constructed in Proposition 1. By Corollary 9, we have

$$\Delta T_k(u_n) \leq \chi_{[u_n \leq k]} (\Delta u_n)_{\mathrm{d}} + (\Delta u_n)_{\mathrm{c}}^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{1.3}$$

Since u_n satisfies (0.10),

$$(\Delta u_n)_{\mathrm{d}} = g_n(u_n) - \mu_{\mathrm{d}} \quad \text{and} \quad (\Delta u_n)_{\mathrm{c}} = -\mu_{\mathrm{c}}.$$

Inserting into (1.3) gives

$$\begin{aligned} -\Delta T_k(u_n) &\geq \chi_{[u_n \leq k]} \big\{ \mu_{\mathrm{d}} - g_n(u_n) \big\} - \mu_{\mathrm{c}}^- \\ &\geq \chi_{[u_n \leq k]} \mu_{\mathrm{d}} - g_n(T_k(u_n)) - \mu_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega). \end{aligned}$$

For every $n \geq 1$ we have $u^* \leq u_n \leq u_1$, so that

$$[u^* \leq k] \supset [u_n \leq k] \supset [u_1 \leq k]$$

and

$$\chi_{[u_n \leq k]} \mu_{\mathrm{d}} \geq \chi_{[u_1 \leq k]} \mu_{\mathrm{d}}^+ - \chi_{[u^* \leq k]} \mu_{\mathrm{d}}^- .$$

Thus

$$-\Delta T_k(u_n) + g_n(T_k(u_n)) \geq \chi_{[u_1 \leq k]} \mu_{\mathrm{d}}^+ - \chi_{[u^* \leq k]} \mu_{\mathrm{d}}^- - \mu_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega). \tag{1.4}$$

By dominated convergence,

$$g_n(T_k(u_n)) \to g(T_k(u^*)) \quad \text{in } L^1(\Omega), \quad \text{as } n \to \infty.$$

As $n \to \infty$ in (1.4), we get

$$-\Delta T_k(u^*) + g(T_k(u^*)) \geq \chi_{[u_1 \leq k]} \mu_{\mathrm{d}}^+ - \chi_{[u^* \leq k]} \mu_{\mathrm{d}}^- - \mu_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega).$$

Let $k \to \infty$. Since both sets $[u_1 = +\infty]$ and $[u^* = +\infty]$ have zero capacity (recall that u_1 and u^* are quasi-continuous and, in particular, both functions are finite q.e.), we conclude that

$$\mu^* = -\Delta u^* + g(u^*) \geq \mu_{\mathrm{d}}^+ - \mu_{\mathrm{d}}^- - \mu_{\mathrm{c}}^- = \mu_{\mathrm{d}} - \mu_{\mathrm{c}}^- .$$

This establishes the lemma.

Proof of Theorems 1 and 2. It follows from (0.14) and Lemma 1 that

$$\mu_{\mathrm{d}} - \mu_{\mathrm{c}}^{-} \le \mu^* \le \mu.$$

By taking the diffuse parts, we have

$$(\mu^*)_{\mathrm{d}} = \mu_{\mathrm{d}}. \tag{1.5}$$

Thus $\mu - \mu^* = (\mu - \mu^*)_{\mathrm{c}}$, which proves Theorem 2.

We now turn to the proof of Theorem 1. Let λ be a good measure $\le \mu$. We must prove that $\lambda \le \mu^*$. Denote by v the solution of (0.1) corresponding to λ,

$$\begin{cases} -\Delta v + g(v) = \lambda & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By (1.5),

$$\lambda_{\mathrm{d}} \le \mu_{\mathrm{d}} = (\mu^*)_{\mathrm{d}}.$$

Since u^* is the largest subsolution of (0.1), we also have

$$v \le u^* \quad \text{a.e.}$$

By the "inverse" maximum principle,

$$\lambda_{\mathrm{c}} = (-\Delta v)_{\mathrm{c}} \le (-\Delta u^*)_{\mathrm{c}} = (\mu^*)_{\mathrm{c}}.$$

Therefore $\lambda \le \mu^*$. This establishes Theorem 1.

The following lemma will be used later on:

Lemma 2. *Given a measure $\mu \in \mathcal{M}(\Omega)$, let (u_n) be the sequence defined in Proposition 1. Then,*

$$g_n(u_n) \overset{*}{\rightharpoonup} g(u^*) + (\mu - \mu^*) = g(u^*) + (\mu - \mu^*)_{\mathrm{c}} \quad \textit{weak}^* \textit{ in } \mathcal{M}(\Omega).$$

Proof. Let $\zeta \in C_0^2(\overline{\Omega})$. For every $n \ge 1$, we have

$$\int_\Omega g_n(u_n)\zeta = \int_\Omega u_n \Delta\zeta + \int_\Omega \zeta\, d\mu.$$

By Proposition 1, $u_n \to u^*$ in $L^1(\Omega)$. Thus,

$$\lim_{n\to\infty} \int_\Omega g_n(u_n)\zeta = \int_\Omega u^*\Delta\zeta + \int_\Omega \zeta\, d\mu = \int_\Omega g(u^*)\zeta + \int_\Omega \zeta\, d(\mu - \mu^*).$$

In other words,

$$g_n(u_n) \overset{*}{\rightharpoonup} g(u^*) + (\mu - \mu^*) \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega).$$

Since $(\mu^*)_{\mathrm{d}} = \mu_{\mathrm{d}}$, the result follows.

2. Good measures. Proofs of Theorems 4, 6.

We start with

Lemma 3. *If μ is a good measure with solution u, and u_n is given by* (0.10), *then*

$$u_n \to u \quad \text{in } W_0^{1,1}(\Omega) \qquad \text{and} \qquad g_n(u_n) \to g(u) \quad \text{in } L^1(\Omega).$$

Proof. We have

$$-\Delta u_n + g_n(u_n) = \mu \quad \text{and} \quad -\Delta u + g(u) = \mu \quad \text{in } (C_0^2)^*,$$

so that

$$-\Delta(u_n - u) + g_n(u_n) - g(u) = 0 \quad \text{in } (C_0^2)^*.$$

Thus

$$-\Delta(u_n - u) + g_n(u_n) - g_n(u) = g(u) - g_n(u) \quad \text{in } (C_0^2)^*.$$

Hence, by standard estimates (see Proposition B.3),

$$\int_\Omega |g_n(u_n) - g_n(u)| \le \int_\Omega |g(u) - g_n(u)| \to 0.$$

Thus

$$\int_\Omega |g_n(u_n) - g(u)| \le 2 \int_\Omega |g(u) - g_n(u)| \to 0.$$

In other words, $g_n(u_n) \to g(u)$ in $L^1(\Omega)$. This clearly implies that $\Delta(u_n - u) \to 0$ in $L^1(\Omega)$ and thus $u_n \to u$ in $W_0^{1,1}(\Omega)$.

We now turn to the

Proof of Theorem 4. Let $u_{1,n}, u_{2,n} \in L^1(\Omega)$ be such that

$$\begin{cases} -\Delta u_{i,n} + g_n(u_{i,n}) = \mu_i & \text{in } \Omega, \\ \qquad\qquad\quad u_{i,n} = 0 & \text{on } \partial\Omega, \end{cases}$$

for $i = 1, 2$. Since $\mu_2 \le \mu_1$, we have

$$u_{2,n} \le u_{1,n} \quad \text{a.e.}$$

Thus $g_n(u_{2,n}) \le g_n(u_{1,n}) \to g(u_1^*)$ strongly in L^1 by Lemma 3. Hence $g_n(u_{2,n}) \to g(u_2^*)$ strongly in L^1 and we have

$$-\Delta u_2^* + g(u_2^*) = \mu_2 \quad \text{in } (C_0^2)^*,$$

i.e., μ_2 is a good measure.

A simple property of $\mathcal{G}$ is

Proposition 2. *The set $\mathcal{G}$ of good measures is closed with respect to strong convergence in $\mathcal{M}(\Omega)$.*

Proof. Let (μ_k) be a sequence of good measures such that $\mu_k \to \mu$ strongly in $\mathcal{M}(\Omega)$. For each $k \geq 1$, let u_k be such that

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard estimates (see Corollary B.1),

$$\int_\Omega |g(u_{k_1}) - g(u_{k_2})| \leq \|\mu_{k_1} - \mu_{k_2}\|_{\mathcal{M}} \tag{2.1}$$

and

$$\int_\Omega |u_{k_1} - u_{k_2}| \leq C\big\|\Delta(u_{k_1} - u_{k_2})\big\|_{\mathcal{M}} \leq 2C\|\mu_{k_1} - \mu_{k_2}\|_{\mathcal{M}}. \tag{2.2}$$

By (2.1) and (2.2), both (u_k) and $(g(u_k))$ are Cauchy sequences in $L^1(\Omega)$. Thus, there exist $u, v \in L^1(\Omega)$ such that

$$u_k \to u \quad \text{and} \quad g(u_k) \to v \quad \text{in } L^1(\Omega).$$

In particular, $v = g(u)$ a.e. It is then easy to see that

$$-\Delta u + g(u) = \mu \quad \text{in } (C_0^2)^*.$$

Thus μ is a good measure.

We next present a result slightly sharper than Theorem 6:

Theorem 6′. *Let $\mu \in \mathcal{M}(\Omega)$. The following conditions are equivalent:*

(a) *μ is a good measure;*

(b) *μ^+ is a good measure;*

(c) *μ_c is a good measure;*

(d) *$\mu = f_0 - \Delta v_0$ in $\mathcal{D}'(\Omega)$, for some $f_0 \in L^1$ and some $v_0 \in L^1$ with $g(v_0) \in L^1$.*

Proof. (a) $\Rightarrow$ (b). Since μ and 0 are good measures, it follows from Corollary 4 that $\mu^+ = \sup\{\mu, 0\}$ is a good measure.

(b) $\Rightarrow$ (a). Since μ^+ is a good measure and $\mu \leq \mu^+$ in Ω, it follows from Theorem 4 that μ is a good measure.

(b) $\Rightarrow$ (c). Note that we always have

$$\mu_{\mathrm{c}} \leq \mu^+. \tag{2.3}$$

Indeed, $(\mu^+ - \mu_{\mathrm{c}})_{\mathrm{d}} = (\mu^+)_{\mathrm{d}} \geq 0$ and $(\mu^+ - \mu_{\mathrm{c}})_{\mathrm{c}} = \mu_{\mathrm{c}}^+ - \mu_{\mathrm{c}} \geq 0$.
[Here and in the sequel we use the fact that $(\mu^+)_{\mathrm{d}} = (\mu_{\mathrm{d}})^+$ and $(\mu^+)_{\mathrm{c}} = (\mu_{\mathrm{c}})^+$ which will be simply denoted μ_{d}^+ and μ_{c}^+].
Since μ^+ is a good measure, it follows from (2.3) and Theorem 4 that μ_{c} is also a good measure.

(c) $\Rightarrow$ (b). It is easy to see that, for every measure λ,

$$\lambda^+ = \sup\{\lambda_{\mathrm{d}}, \lambda_{\mathrm{c}}\}. \tag{2.4}$$

Assume μ_{c} is a good measure. Since μ_{d} is diffuse, Corollary 2 implies that μ_{d} is also a good measure. By Corollary 4 and (2.4), $\mu^+ = \sup\{\mu_{\mathrm{d}}, \mu_{\mathrm{c}}\}$ is a good measure as well.

(a) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (c). We split the argument into two steps.

Step 1. Proof of (d) $\Rightarrow$ (c) if v_0 has compact support.

Since $\mu = f_0 - \Delta v_0$ in $\mathcal{D}'(\Omega)$ and v_0 has compact support, we have

$$\mu = f_0 - \Delta v_0 \quad \text{in } (C_0^2)^*.$$

Thus, $\mu - f_0 + g(v_0)$ is a good measure. Using the equivalence (a) $\Leftrightarrow$ (c), we conclude that $\mu_{\mathrm{c}} = \big[\mu - f_0 + g(v_0)\big]_{\mathrm{c}}$ is a good measure.

Step 2. Proof of (d) $\Rightarrow$ (c) completed.

By assumption,

$$\mu = f_0 - \Delta v_0 \quad \text{in } \mathcal{D}'(\Omega).$$

In particular, we have $\Delta v_0 \in \mathcal{M}(\Omega)$, so that $v_0 \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\forall p < \frac{N}{N-1}$ (see Theorem B.1 below). Let $(\varphi_n) \subset C_{\mathrm{c}}^\infty(\Omega)$ be such that $0 \leq \varphi_n \leq 1$ in Ω and $\varphi_n(x) = 1$ if $d(x, \partial\Omega) > \frac{1}{n}$. Then

$$\varphi_n \mu = f_n - \Delta(\varphi_n v_0) \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$f_n = \varphi_n f_0 + 2\nabla v_0 \cdot \nabla \varphi_n + v_0 \Delta \varphi_n \in L^1(\Omega).$$

Moreover, since $0 \le g(\varphi_n v_0) \le g(v_0)$ a.e., we have $g(\varphi_n v_0) \in L^1(\Omega)$. Thus, by Step 1,

$$\varphi_n \mu_c = (\varphi_n \mu)_c \in \mathcal{G} \quad \forall n \ge 1.$$

Since $\varphi_n \mu_c \to \mu_c$ strongly in $\mathcal{M}(\Omega)$ and $\mathcal{G}$ is closed with respect to the strong topology in $\mathcal{M}(\Omega)$, we conclude that $\mu \in \mathcal{G}$.

We may now strengthen Corollary 7:

Corollary 7′. *We have*

$$\mathcal{G} + \mathcal{M}_\mathrm{d}(\Omega) \subset \mathcal{G},$$

where $\mathcal{M}_\mathrm{d}(\Omega)$ denotes the space of diffuse measures.

Proof. Let $\mu \in \mathcal{G}$. By Theorem 6′, μ_c is a good measure. Thus, for any $\nu \in \mathcal{M}_\mathrm{d}$, $(\mu + \nu)_c = \mu_c$ is a good measure. It follows from the equivalence (a) $\Leftrightarrow$ (c) in the theorem above that $\mu + \nu \in \mathcal{G}$.

Proposition 3. *Assume*

$$g(2t) \le C(g(t) + 1) \quad \forall t \ge 0. \tag{2.5}$$

Then the set of good measures is a convex cone.

Remark 7. Assumption (2.5) is called in the literature the Δ_2-condition. It holds if $g(t) = t^p$ for $t \ge 0$ (any $p > 1$), but (2.5) *fails* for $g(t) = \mathrm{e}^t - 1$. In this case, the set of good measures is *not* a cone. As we will see in Section 8, Example 5, if $N = 2$, then for any $a \in \Omega$ we have $c\delta_a \in \mathcal{G}$ if $c > 0$ is small, but $c\delta_a \notin \mathcal{G}$ if c is large.

Proof of Proposition 3. Assume $\mu \in \mathcal{G}$. Clearly, it suffices to show that $2\mu \in \mathcal{G}$.

Let u be the solution of

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus,

$$2\mu = -\Delta(2u) + 2g(u) \quad \text{in } \mathcal{D}'(\Omega).$$

By (2.5), $g(2u) \in L^1$. We can now invoke the equivalence (a) $\Leftrightarrow$ (d) in Theorem 6′ to conclude that $2\mu \in \mathcal{G}$.

3. Some properties of the mapping $\mu \mapsto \mu^*$.

We start with an easy result, which asserts that the mapping $\mu \mapsto \mu^*$ is order preserving:

Proposition 4. *Let $\mu, \nu \in \mathcal{M}(\Omega)$. If $\mu \le \nu$, then $\mu^* \le \nu^*$.*

Proof. Since the reduced measure μ^* is a good measure and $\mu^* \le \mu \le \nu$, it follows from Theorem 1 that $\mu^* \le \nu^*$.

Next, we have

Theorem 8. *If $\mu_1, \mu_2 \in \mathcal{M}(\Omega)$ are mutually singular, then*

$$(\mu_1 + \mu_2)^* = (\mu_1)^* + (\mu_2)^*. \tag{3.1}$$

Proof. Since μ_1 and μ_2 are mutually singular, $(\mu_1)^*$ and $(\mu_2)^*$ are also mutually singular (by (0.16)). In particular, we have

$$(\mu_1)^* + (\mu_2)^* \le \big[(\mu_1)^* + (\mu_2)^*\big]^+ = \sup\big\{(\mu_1)^*, (\mu_2)^*\big\}. \tag{3.2}$$

By Corollary 4, the right-hand side of (3.2) is a good measure. It follows from Theorem 4 that $(\mu_1)^* + (\mu_2)^*$ is also a good measure. Since

$$(\mu_1)^* + (\mu_2)^* \le \mu_1 + \mu_2,$$

we conclude from Theorem 1 that

$$(\mu_1)^* + (\mu_2)^* \le (\mu_1 + \mu_2)^*. \tag{3.3}$$

We now establish the reverse inequality. Assume λ is a good measure $\le (\mu_1 + \mu_2)$. By Radon-Nikodym, we may decompose λ in terms of three measures:

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2,$$

where λ_0 is singular with respect to $|\mu_1| + |\mu_2|$, and, for $i = 1, 2$, λ_i is absolutely continuous with respect to $|\mu_i|$. Since $\lambda_0, \lambda_1, \lambda_2 \le \lambda^+$, each λ_j, $j = 0, 1, 2$, is a good measure. Moreover, $\lambda \le \mu_1 + \mu_2$ implies

$$\lambda_0 \le 0, \quad \lambda_1 \le \mu_1 \quad \text{and} \quad \lambda_2 \le \mu_2.$$

Thus, in particular, $\lambda_i \le (\mu_i)^*$ for $i = 1, 2$. Therefore,

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 \le (\mu_1)^* + (\mu_2)^*.$$

Since λ was arbitrary, we have

$$(\mu_1 + \mu_2)^* \le (\mu_1)^* + (\mu_2)^*. \tag{3.4}$$

Combining (3.3) and (3.4), the result follows.

Here are some consequences of Theorem 8:

Corollary 10. *For every* $\mu \in \mathcal{M}(\Omega)$*, we have*

$$(\mu^*)_{\mathrm{d}} = (\mu_{\mathrm{d}})^* = \mu_{\mathrm{d}} \quad \textit{and} \quad (\mu^*)_{\mathrm{c}} = (\mu_{\mathrm{c}})^*. \tag{3.5}$$

Also,

$$(\mu^*)^+ = (\mu^+)^* \quad \textit{and} \quad (\mu^*)^- = \mu^-. \tag{3.6}$$

Proof. Since μ_{d} is a good measure (see Corollary 2), we have $(\mu_{\mathrm{d}})^* = \mu_{\mathrm{d}}$. By Theorem 8,

$$\mu^* = (\mu_{\mathrm{d}} + \mu_{\mathrm{c}})^* = (\mu_{\mathrm{d}})^* + (\mu_{\mathrm{c}})^*.$$

Comparison between the diffuse and concentrated parts gives (3.5). Similarly,

$$\mu^* = (\mu^+ - \mu^-)^* = (\mu^+)^* + (-\mu^-)^* = (\mu^+)^* - \mu^-,$$

since every nonpositive measure is good. This identity yields (3.6).

More generally, the same argument shows the following:

Corollary 11. *Let* $\mu \in \mathcal{M}(\Omega)$*. For every Borel set* $E \subset \Omega$*, we have*

$$(\mu\lfloor_E)^* = \mu^*\lfloor_E. \tag{3.7}$$

Here $\mu\lfloor_E$ denotes the measure defined by $\mu\lfloor_E(A) = \mu(A \cap E)$ for every Borel set $A \subset \Omega$.

For simplicity, from now on we shall write $\mu_{\mathrm{d}}^* = (\mu^*)_{\mathrm{d}}$ and $\mu_{\mathrm{c}}^* = (\mu^*)_{\mathrm{c}}$.

The following result extends Corollary 7′:

Corollary 12. *For every* $\mu \in \mathcal{M}(\Omega)$ *and* $\nu \in \mathcal{M}_{\mathrm{d}}(\Omega)$*,*

$$(\mu + \nu)^* = \mu^* + \nu.$$

Proof. By Theorem 8 and Corollary 2, we have

$$(\mu + \nu)^* = \mu_{\mathrm{c}}^* + (\mu_{\mathrm{d}} + \nu)^* = \mu_{\mathrm{c}}^* + \mu_{\mathrm{d}} + \nu = (\mu_{\mathrm{c}}^* + \mu_{\mathrm{d}}^*) + \nu = \mu^* + \nu.$$

Next, we have

Theorem 9. *Given $\mu, \nu \in \mathcal{M}(\Omega)$, we have*

$$\big[\inf\{\mu,\nu\}\big]^* = \inf\{\mu^*,\nu^*\}, \tag{3.8}$$

$$\big[\sup\{\mu,\nu\}\big]^* = \sup\{\mu^*,\nu^*\}. \tag{3.9}$$

Proof.
Step 1. Proof of (3.8).

Clearly,

$$\inf\{\mu^*,\nu^*\} \leq \big[\inf\{\mu,\nu\}\big]^*.$$

Assume λ is a good measure $\leq \inf\{\mu,\nu\}$. By Theorem 1, $\lambda \leq \mu^*$ and $\lambda \leq \nu^*$. Thus, $\lambda \leq \inf\{\mu^*,\nu^*\}$, whence

$$\big[\inf\{\mu,\nu\}\big]^* \leq \inf\{\mu^*,\nu^*\}.$$

Step 2. Proof of (3.9).

Applying the Hahn decomposition to $\mu - \nu$, we may write Ω in terms of two disjoint Borel sets $E_1, E_2 \subset \Omega$, $\Omega = E_1 \cup E_2$, so that

$$\mu \geq \nu \quad \text{in } E_1 \quad \text{and} \quad \nu \geq \mu \quad \text{in } E_2.$$

By Proposition 4 and Corollary 11,

$$\mu^*\lfloor_{E_1} = (\mu\lfloor_{E_1})^* \geq (\nu\lfloor_{E_1})^* = \nu^*\lfloor_{E_1}.$$

Thus, $\mu^* \geq \nu^*$ on E_1. Similarly, $\nu^* \geq \mu^*$ on E_2. We then have

$$\sup\{\mu,\nu\} = \mu\lfloor_{E_1} + \nu\lfloor_{E_2} \quad \text{and} \quad \sup\{\mu^*,\nu^*\} = \mu^*\lfloor_{E_1} + \nu^*\lfloor_{E_2}. \tag{3.10}$$

On the other hand, by Theorem 8 and Corollary 11,

$$(\mu\lfloor_{E_1} + \nu\lfloor_{E_2})^* = (\mu\lfloor_{E_1})^* + (\nu\lfloor_{E_2})^* = \mu^*\lfloor_{E_1} + \nu^*\lfloor_{E_2}. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain (3.9).

We now show that $\mu \mapsto \mu^*$ is non-expansive:

Theorem 10. *Given $\mu, \nu \in \mathcal{M}(\Omega)$, we have*

$$|\mu^* - \nu^*| \leq |\mu - \nu|. \tag{3.12}$$

More generally,

$$(\mu^* - \nu^*)^+ \leq (\mu - \nu)^+. \tag{3.13}$$

Proof. Clearly, it suffices to show that (3.13) holds. We split the proof into two steps.
Step 1. Assume $\nu \leq \mu$. Then we claim that

$$\mu^* - \nu^* \leq \mu - \nu. \tag{3.14}$$

Indeed, let v_n be the solution of (0.10) corresponding to the measure ν. Since $\nu \leq \mu$, we have

$$v_n \leq u_n \quad \text{a.e.,} \quad \forall n \geq 1.$$

Recall that g_n is nondecreasing; thus,

$$g_n(v_n) \leq g_n(u_n) \quad \text{a.e.}$$

Let $n \to \infty$. According to Lemma 2, we have

$$g(v^*) + (\nu - \nu^*)_{\rm c} \leq g(u^*) + (\mu - \mu^*)_{\rm c}.$$

Taking the concentrated part on both sides of this inequality yields

$$(\nu - \nu^*)_{\rm c} \leq (\mu - \mu^*)_{\rm c}.$$

Since $\nu_{\rm d} = \nu^*_{\rm d}$ and $\mu_{\rm d} = \mu^*_{\rm d}$ (by Corollary 2), we have

$$\nu - \nu^* \leq \mu - \mu^*,$$

which is (3.14).
Step 2. Proof of (3.1) completed.

Recall that

$$\sup\{\mu, \nu\} = \nu + (\mu - \nu)^+. \tag{3.15}$$

Applying the previous step to the measures ν and $\sup\{\mu, \nu\}$, we have

$$\big[\sup\{\mu, \nu\}\big]^* - \nu^* \leq \sup\{\mu, \nu\} - \nu = (\mu - \nu)^+. \tag{3.16}$$

By (3.9), (3.15) and (3.16),

$$(\mu - \nu)^+ \geq \big[\sup\{\mu, \nu\}\big]^* - \nu^* = \sup\{\mu^*, \nu^*\} - \nu^* = (\mu^* - \nu^*)^+.$$

Therefore, (3.13) holds.

4. Approximation of μ by $\rho_n * \mu$.

Let (ρ_n) be a sequence of mollifiers in $\mathbb{R}^N$ such that $\operatorname{supp}\rho_n \subset B_{1/n}$ for every $n \geq 1$. Given $\mu \in \mathcal{M}(\Omega)$, set

$$\mu_n = \rho_n * \mu,$$

that is,

$$\mu_n(x) = \int_\Omega \rho_n(x-y)\, d\mu(y) \quad \forall x \in \mathbb{R}^N. \tag{4.1}$$

[The integral in (4.1) is well-defined in view of Proposition C.1 in Appendix C below. Here, we identify μ with $\tilde{\mu} \in \big[C(\overline{\Omega})\big]^*$ defined there].

Let u_n be the solution of

$$\begin{cases} -\Delta u_n + g(u_n) = \mu_n & \text{in } \Omega,\\ \qquad\qquad\quad u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Theorem 11. *Assume in addition that g is convex. Then $u_n \to u^*$ in $L^1(\Omega)$, where u^* is given by Proposition 1.*

Proof.
Step 1. The conclusion holds if μ is a good measure.

In this case, there exists $u = u^*$ such that

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega,\\ \qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Let $\omega \subset\subset \Omega$. For $n \geq 1$ sufficiently large, we have

$$-\Delta(\rho_n * u) + \rho_n * g(u) = \mu_n \quad \text{in } \omega.$$

Thus, using the convexity of g,

$$\Delta(\rho_n * u - u_n) = \rho_n * g(u) - g(u_n) \geq g(\rho_n * u) - g(u_n) \quad \text{in } \omega.$$

By the standard version of Kato's inequality (see [K]),

$$\Delta(\rho_n * u - u_n)^+ \geq \big\{g(\rho_n * u) - g(u_n)\big\}^+ \geq 0 \quad \text{in } \mathcal{D}'(\omega). \tag{4.4}$$

Since

$$\int_\Omega |\Delta u_n| \leq 2\|\mu_n\|_{\mathcal{M}} \leq C \quad \forall n \geq 1,$$

we can extract a subsequence (u_{n_k}) such that

$$u_{n_k} \to v \quad \text{in } L^1(\Omega),$$

for some $v \in W_0^{1,1}(\Omega)$. As $n_k \to \infty$ in (4.4), we have

$$-\Delta(u-v)^+ \leq 0 \quad \text{in } \mathcal{D}'(\omega).$$

Since $\omega \subset\subset \Omega$ was arbitrary,

$$-\Delta(u-v)^+ \leq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{4.5}$$

On the other hand,

$$(u-v)^+ \in W_0^{1,1}(\Omega). \tag{4.6}$$

From (4.5), (4.6) and the weak form of the maximum principle (see Proposition B.1) we deduce that

$$(u-v)^+ \leq 0 \quad \text{a.e.}$$

Therefore,

$$v \geq u \quad \text{a.e.}$$

By Fatou's lemma, v is a subsolution of (0.1); comparison with (4.3) yields,

$$v \leq u \quad \text{a.e.}$$

We conclude that

$$v = u \quad \text{a.e.}$$

Since v is independent of the subsequence (u_{n_k}), we must have

$$u_n \to u = u^* \quad \text{in } L^1(\Omega).$$

Step 2. Proof of Theorem 11 completed.

Without loss of generality, we may assume that

$$u_n \to v \quad \text{in } L^1(\Omega).$$

By Fatou, once more, v is a subsolution of (0.1). Proposition 1 yields

$$v \leq u^* \quad \text{a.e.}$$

Let u_n^* denote the solution of

$$\begin{cases} -\Delta u_n^* + g(u_n^*) = \rho_n * \mu^* & \text{in } \Omega, \\ u_n^* = 0 & \text{on } \partial\Omega. \end{cases}$$

By the previous step,

$$u_n^* \to u^* \quad \text{in } L^1(\Omega).$$

On the other hand, we know from the maximum principle that

$$u_n^* \le u_n \quad \text{a.e.}$$

Thus, as $n \to \infty$,

$$u^* \le v \quad \text{a.e.}$$

Since $v \le u^*$ a.e., the result follows.

Open problem 1. Does the conclusion of Theorem 11 remain valid without the convexity assumption on g ?

5. Further convergence results.

We start with the following

Theorem 12. *Let* $(f_n) \subset L^1(\Omega)$ *and* $f \in L^1(\Omega)$*. Assume*

$$f_n \rightharpoonup f \quad \text{weakly in } L^1. \tag{5.1}$$

Let u_n *(resp.* u*) be the solution of* (0.1) *associated with* f_n *(resp.* f*). Then* $u_n \to u$ *in* $L^1(\Omega)$*.*

Proof. By definition,

$$-\Delta u_n + g(u_n) = f_n \quad \text{and} \quad -\Delta u + g(u) = f \quad \text{in } (C_0^2)^*.$$

Using a device introduced by Gallouët-Morel [GM] (see also Proposition B.2 below), we have, for every $M > 0$,

$$\int_{[|u_n| \ge M]} |g(u_n)| \le \int_{[|u_n| \ge M]} |f_n|.$$

Thus

$$\int_E |g(u_n)| = \int_{\substack{E \\ [|u_n| \ge M]}} + \int_{\substack{E \\ [|u_n| < M]}} \le \int_{[|u_n| \ge M]} |f_n| + g(M)|E|. \tag{5.2}$$

On the other hand, $\|\Delta u_n\|_{L^1} \le C$ implies $\|u_n\|_{L^1} \le C$, and thus

$$\operatorname{meas}\big[|u_n| \ge M\big] \le \frac{C}{M}.$$

From (5.1) and a theorem of Dunford-Pettis (see, e.g., [DS, Corollary IV.8.11]) we infer that (f_n) is equi-integrable. Given $\delta > 0$, *fix* $M > 0$ such that

$$\int\limits_{[|u_n| \ge M]} |f_n| \le \delta \quad \forall n \ge 1. \tag{5.3}$$

With this fixed M, choose $|E|$ so small that

$$g(M)|E| < \delta. \tag{5.4}$$

We deduce from (5.2)–(5.4) that $g(u_n)$ is equi-integrable.
Passing to a subsequence, we may assume that $u_{n_k} \to v$ in $L^1(\Omega)$ and a.e., for some $v \in L^1(\Omega)$. Then $g(u_{n_k}) \to g(v)$ a.e. By Egorov's lemma, $g(u_{n_k}) \to g(v)$ in $L^1(\Omega)$. It follows that v is a solution of (0.1) associated to f. By the uniqueness of the limit, we must have $u_n \to u$ in $L^1(\Omega)$.

Remark 8. Theorem 12 is no longer true if one replaces the weak convergence $f_n \rightharpoonup f$ in L^1, by the weak* convergence in the sense of measures. Here is an example:

Example 1. Assume $N \ge 3$ and let $g(t) = (t^+)^q$ with $q \ge \frac{N}{N-2}$. Let $f \equiv 1$ in Ω. We will construct a sequence (f_k) in $C_c^\infty(\Omega)$ such that

$$f_k \overset{*}{\rightharpoonup} f \quad \text{in } \mathcal{M}(\Omega), \tag{5.5}$$

and such that the solutions u_k of (0.1) corresponding to f_k converge to 0 in $L^1(\Omega)$.
Let (μ_k) be any sequence in $\mathcal{M}(\Omega)$ converging weak* to f, as $k \to \infty$, and such that each measure μ_k is a linear combination of Dirac masses. (For example, each μ_k can be of the form $|\Omega| M^{-1} \sum \delta_{a_i}$, where the M points a_i are uniformly distributed in Ω). Recall that for $\mu = \delta_a$, the corresponding u^* in Proposition 1 is $\equiv 0$ (see [B4] or Theorem B.6 below). Similarly, for each μ_k, the corresponding u^* is $\equiv 0$. Set $h_{n,k} = \rho_n * \mu_k$, with the same notation as in Section 4. Let $u_{n,k}$ denote the solution of (0.1) relative to $h_{n,k}$. For each *fixed* k we know, by Theorem 11, that $u_{n,k} \to 0$ strongly in $L^1(\Omega)$ as $n \to \infty$. For each k, choose $N_k > k$ sufficiently large so that $\|u_{N_k,k}\|_{L^1} < 1/k$. Set $f_k = h_{N_k,k}$, so that $u_k = u_{N_k,k}$ is the corresponding solution of (0.1). It is easy to check that, as $k \to \infty$,

$$f_k \overset{*}{\rightharpoonup} f \equiv 1 \quad \text{in } \mathcal{M}(\Omega), \quad \text{but} \quad u_k \to 0 \quad \text{in } L^1(\Omega).$$

Our next result is a refinement of Theorem 12 in the spirit of Theorem 6. Let $\mu \in \mathcal{M}(\Omega)$ and let (μ_n) be a sequence in $\mathcal{M}(\Omega)$. Assume that

$$\mu = f - \Delta v \quad \text{in } (C_0^2)^*, \tag{5.6}$$

$$\mu_n = f_n - \Delta v_n \quad \text{in } (C_0^2)^*, \tag{5.7}$$

where $f \in L^1$, $f_n \in L^1$, $v \in L^1$, $v_n \in L^1$, $g(v) \in L^1$, and $g(v_n) \in L^1$.

By Theorem 6 we know that there exist u and u_n solutions of

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{5.8}$$

$$-\Delta u_n + g(u_n) = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega. \tag{5.9}$$

Theorem 13. *Assume* (5.6)–(5.9) *and moreover*

$$\|\mu_n\|_{\mathcal{M}} \le C, \tag{5.10}$$

$$f_n \rightharpoonup f \quad \textit{weakly in } L^1, \tag{5.11}$$

$$v_n \to v \quad \textit{in } L^1 \quad \textit{and} \quad g(v_n) \to g(v) \quad \textit{in } L^1. \tag{5.12}$$

Then $u_n \to u$ in $L^1(\Omega)$.

Proof. We divide the proof into two steps.

Step 1. Fix $0 < \alpha < 1$ and let $u(\alpha)$, $u_n(\alpha)$ be the solutions of

$$-\Delta u(\alpha) + g(u(\alpha)) = \alpha\mu \quad \text{in } \Omega, \quad u(\alpha) = 0 \quad \text{on } \partial\Omega, \tag{5.13}$$

$$-\Delta u_n(\alpha) + g(u_n(\alpha)) = \alpha\mu_n \quad \text{in } \Omega, \quad u_n(\alpha) = 0 \quad \text{on } \partial\Omega. \tag{5.14}$$

Then $u_n(\alpha) \to u(\alpha)$ in $L^1(\Omega)$.

Note that $u(\alpha)$ and $u_n(\alpha)$ exist since $\alpha\mu = \alpha f - \Delta(\alpha v)$ and $g(\alpha v) \le g(v)$, so that $g(\alpha v) \in L^1$, and similarly for $\alpha\mu_n$. We may then apply Theorem 6 once more. For simplicity we will omit the dependence in α and we will write $\tilde{u}$, $\tilde{u}_n$ instead of $u(\alpha)$, $u_n(\alpha)$ (recall that in this step α is *fixed*). Since

$$\|\Delta \tilde{u}_n\|_{\mathcal{M}} \le 2\alpha \|\mu_n\|_{\mathcal{M}} \le C,$$

we can extract a subsequence of $(\tilde{u}_n)$ converging strongly in $L^1(\Omega)$ and a.e. Let $w \in W_0^{1,1}(\Omega)$ be such that $\tilde{u}_{n_k} \to w$ in $L^1(\Omega)$ and a.e. We will prove that w satisfies (5.13), and therefore, by uniqueness, $w = \tilde{u}$. Since w is independent of the subsequence, we will infer that $(\tilde{u}_n)$ converges to $\tilde{u}$, which is the desired conclusion.

We claim that

$$g(\tilde{u}_n) \quad \text{is equi-integrable.} \tag{5.15}$$

To establish (5.15) we argue as in the proof of Theorem 12. From (5.7) and (5.14) we see that

$$-\Delta(\tilde{u}_n - \alpha v_n) + [g(\tilde{u}_n) - g(\alpha v_n)] = h_n \quad \text{in } (C_0^2)^*, \tag{5.16}$$

with

$$h_n = \alpha f_n - g(\alpha v_n). \tag{5.17}$$

Using (5.11) and (5.12) we see that

$$(h_n) \quad \text{is equi-integrable.} \tag{5.18}$$

From (5.16) and Proposition B.2 we obtain (as in the proof of Theorem 12) that, for every $M > 0$,

$$\int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |g(\tilde{u}_n) - g(\alpha v_n)| \leq \int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |h_n|. \tag{5.19}$$

On the other hand, for any Borel set E of Ω, we have

$$\int_E g(\tilde{u}_n) = \int_{A_n} g(\tilde{u}_n) + \int_{B_n} g(\tilde{u}_n) + \int_{C_n} g(\tilde{u}_n), \tag{5.20}$$

where

$$\begin{aligned} A_n &= [\tilde{u}_n \geq v_n] \cap [|\tilde{u}_n - \alpha v_n| \geq M] \cap E, \\ B_n &= [\tilde{u}_n \geq v_n] \cap [|\tilde{u}_n - \alpha v_n| < M] \cap E, \\ C_n &= [\tilde{u}_n < v_n] \cap E. \end{aligned}$$

To handle the integral on A_n, write

$$\int_{A_n} g(\tilde{u}_n) \leq \int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |g(\tilde{u}_n) - g(\alpha v_n)| + \int_E g(v_n).$$

Thus, by (5.19),

$$\int_{A_n} g(\tilde{u}_n) \leq \int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |h_n| + \int_E g(v_n). \tag{5.21}$$

Next, on B_n, we have

$$\tilde{u}_n < M + \alpha v_n \leq M + \alpha \tilde{u}_n$$

and thus

$$\tilde{u}_n < \frac{M}{1-\alpha}.$$

Therefore

$$\int_{B_n} g(\tilde{u}_n) \leq g\Big(\frac{M}{1-\alpha}\Big)|E|. \tag{5.22}$$

Finally we have

$$\int_{C_n} g(\tilde{u}_n) \leq \int_E g(v_n). \tag{5.23}$$

Combining (5.20)–(5.23) yields

$$\int_E g(\tilde{u}_n) \leq \int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |h_n| + 2\int_E g(v_n) + g\Big(\frac{M}{1-\alpha}\Big)|E|. \tag{5.24}$$

But $\|\tilde{u}_n - \alpha v_n\|_{L^1} \leq C$ and therefore

$$\operatorname{meas}\,[|\tilde{u}_n - \alpha v_n| \geq M] \leq \frac{C}{M}. \tag{5.25}$$

Given $\delta > 0$, *fix* $M > 0$ sufficiently large such that

$$\int\limits_{[|\tilde{u}_n - \alpha v_n| \geq M]} |h_n| \leq \delta \quad \forall n \geq 1$$

(here we use (5.18) and (5.25)). With this fixed M, choose $|E|$ so small that

$$2\int_E g(v_n) + g\Big(\frac{M}{1-\alpha}\Big)|E| \leq \delta \quad \forall n \geq 1.$$

This finishes the proof of (5.15).

Since $g(\tilde{u}_n) \to g(w)$ a.e., we deduce from (5.15) and Egorov's lemma that $g(\tilde{u}_n) \to g(w)$ in L^1. We are now able to pass to the limit in (5.14) and conclude that w satisfies (5.13), which was the goal of Step 1.

Step 2. Proof of the theorem completed.

Here the dependence on α is important and we return to the notation $u(\alpha)$ and $u_n(\alpha)$. From (5.8) and (5.13) we deduce that

$$\|\Delta(u(\alpha)-u)\|_{\mathcal{M}} \le 2(1-\alpha)\|\mu\|_{\mathcal{M}} \tag{5.26}$$

and similarly, from (5.9) and (5.14), we have

$$\|\Delta(u_n(\alpha)-u_n)\|_{\mathcal{M}} \le 2(1-\alpha)\|\mu_n\|_{\mathcal{M}} \le C(1-\alpha). \tag{5.27}$$

Estimates (5.26) and (5.27) yield

$$\|u(\alpha)-u\|_{L^1} + \|u_n(\alpha)-u_n\|_{L^1} \le C(1-\alpha), \tag{5.28}$$

with C independent of n and α. Finally we write

$$\|u_n-u\|_{L^1} \le \|u(\alpha)-u\|_{L^1} + \|u_n(\alpha)-u_n\|_{L^1} + \|u_n(\alpha)-u(\alpha)\|_{L^1}. \tag{5.29}$$

Given $\varepsilon > 0$, fix $\alpha < 1$ so small that

$$C(1-\alpha) < \varepsilon \tag{5.30}$$

and then apply Step 1 to assert that

$$\|u_n(\alpha)-u(\alpha)\|_{L^1} < \varepsilon \quad \forall n \ge N, \tag{5.31}$$

provided N is sufficiently large. Combining (5.28)–(5.31) yields

$$\|u_n-u\|_{L^1} \le 2\varepsilon \quad \forall n \ge N,$$

which is the desired conclusion.

6. Nonnegative measures which are good for every g must be diffuse.

Let $h : [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function with $h(0) = 0$. Given a compact set $K \subset \Omega$, let

$$\operatorname{cap}_{\Delta,h}(K) = \inf\left\{\int_\Omega h(|\Delta\varphi|)\ ;\ \varphi \in C_c^\infty(\Omega),\ 0 \le \varphi \le 1,\ \text{and } \varphi = 1 \text{ on } K\right\},$$

where, as usual, $C_c^\infty(\Omega)$ denotes the set of C^∞-functions with compact support in Ω.

We start with

Proposition 5. *Assume*

$$\lim_{t\to\infty} \frac{g(t)}{t} = +\infty \quad \text{and} \quad g^*(s) > 0 \quad \text{for } s > 0. \tag{6.1}$$

If μ is a good measure, then $\mu^+(K) = 0$ for every compact set $K \subset \Omega$ such that $\mathrm{cap}_{\Delta,g^*}(K) = 0$.

Here, g^* denotes the convex conjugate of g, which is finite in view of the coercivity of g. Note that if $g'(0) = 0$, then $g^*(s) > 0$ for every $s > 0$.

Proof. Since μ is a good measure, μ^+ is also a good measure. Thus,

$$\mu^+ = -\Delta v + g(v) \quad \text{in } (C_0^2)^*$$

for some $v \in L^1(\Omega)$, $v \geq 0$ a.e., such that $g(v) \in L^1(\Omega)$.
Let $\varphi_n \in C_c^\infty(\Omega)$ be such that $0 \leq \varphi_n \leq 1$ in Ω, $\varphi_n = 1$ on K, and

$$\int_\Omega g^*(|\Delta\varphi_n|) \to 0.$$

Passing to a subsequence if necessary, we may assume that

$$g^*(|\Delta\varphi_n|) \to 0 \quad \text{a.e.} \quad \text{and} \quad g^*(|\Delta\varphi_n|) \leq G \in L^1(\Omega) \quad \forall n \geq 1.$$

Since $g^*(s) > 0$ if $s > 0$, we also have

$$\varphi_n,\ |\Delta\varphi_n| \to 0 \quad \text{a.e.}$$

For every $n \geq 1$,

$$\mu^+(K) \leq \int_\Omega \varphi_n \, d\mu^+ = \int_\Omega \big[g(v)\varphi_n - v\Delta\varphi_n\big]. \tag{6.2}$$

Note that

$$\big|g(v)\varphi_n - v\Delta\varphi_n\big| \to 0 \quad \text{a.e.}$$

and

$$\big|g(v)\varphi_n - v\Delta\varphi_n\big| \leq 2g(v) + g^*(|\Delta\varphi_n|) \leq 2g(v) + G \in L^1(\Omega).$$

By dominated convergence, the right-hand side of (6.2) converges to 0 as $n \to \infty$. We then conclude that $\mu^+(K) = 0$.

As a consequence of Proposition 5 we have

Theorem 14. *Given a Borel set $\Sigma \subset \Omega$ with zero H^1-capacity, there exists g such that*

$$\mu^* = -\mu^- \quad \text{for every measure } \mu \text{ concentrated on } \Sigma.$$

In particular, for every nonnegative $\mu \in \mathcal{M}(\Omega)$ concentrated on a set of zero H^1-capacity, there exists some g such that $\mu^* = 0$.

Proof. Let $\Sigma \subset \Omega$ be a Borel set of zero H^1-capacity. Let (K_n) be an increasing sequence of compact sets in Σ such that

$$\mu^+\Big(\Sigma \setminus \bigcup_n K_n\Big) = 0.$$

For each $n \geq 1$, K_n has zero H^1-capacity. By Lemma E.1, one can find $\psi_n \in C_c^\infty(\Omega)$ such that $0 \leq \psi_n \leq 1$ in Ω, $\psi_n = 1$ in some neighborhood of K_n, and

$$\int_\Omega |\Delta \psi_n| \leq \frac{1}{n} \quad \forall n \geq 1.$$

In particular, $\Delta \psi_n \to 0$ in $L^1(\Omega)$. Passing to a subsequence if necessary, we may assume that

$$\Delta \psi_n \to 0 \quad \text{a.e.} \quad \text{and} \quad |\Delta \psi_n| \leq G \in L^1(\Omega) \quad \forall n \geq 1.$$

According to a theorem of De La Vallée-Poussin (see [DVP, Remarque 23] or [DM, Théorème II.22]), there exists a convex function $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, $h(s) > 0$ for $s > 0$,

$$\lim_{t\to\infty} \frac{h(t)}{t} = +\infty, \quad \text{and} \quad h(G) \in L^1(\Omega).$$

By dominated convergence, we then have $h(|\Delta\psi_n|) \to 0$ in $L^1(\Omega)$. Thus,

$$\operatorname{cap}_{\Delta,h}(K_n) = 0 \quad \forall n \geq 1. \tag{6.3}$$

Let $g(t) = h^*(t)$ if $t \geq 0$, and $g(t) = 0$ if $t < 0$. By duality, $h = g^*$ on $[0,\infty)$.
Let $\mu \in \mathcal{M}(\Omega)$ be any measure concentrated on Σ. By Proposition 5, (6.3) yields

$$(\mu^*)^+(K_n) = 0 \quad \forall n \geq 1,$$

where the reduced measure μ^* is computed with respect to g. Thus, $(\mu^*)^+(\Sigma) = 0$. Since μ is concentrated on Σ, we have $(\mu^*)^+ = 0$. Applying Corollary 10, we then get

$$\mu^* = (\mu^*)^+ - (\mu^*)^- = -\mu^-,$$

which is the desired result.

We may now present the

Proof of Theorem 5. Assume $\mu \in \mathcal{M}(\Omega)$ is a good measure for every g. Given a Borel set $\Sigma \subset \Omega$ with zero H^1-capacity, let $\lambda = \mu^+\lfloor_\Sigma$. In view of Theorem 14, there exists $\tilde{g}$ for which $\lambda^* = 0$. On the other hand, by Theorems 4 and 6′, λ is a good measure for $\tilde{g}$. Thus, $\lambda = \lambda^* = 0$. In other words, $\mu^+(\Sigma) = 0$. Since Σ was arbitrary, μ^+ is diffuse. This establishes the theorem.

We conclude this section with the following

Open problem 2. Let $g : \mathbb{R} \to \mathbb{R}$ be any given continuous, nondecreasing function satisfying (0.5). Can one always find some nonnegative $\mu \in \mathcal{M}(\Omega)$ such that μ is good for g, but μ is *not* diffuse?

After this paper was finished, A.C. Ponce [P] has given a positive answer to the above problem.

7. Signed measures and general nonlinearities g.

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function, such that $g(0) = 0$. But we will *not* impose in this section that $g(t) = 0$ if $t < 0$. We shall follow the same approximation scheme as in the Introduction. Namely, let (g_n) be a sequence of nondecreasing continuous functions, $g_n : \mathbb{R} \to \mathbb{R}$, $g_n(0) = 0$, satisfying (0.8), such that both (g_n^+) and (g_n^-) verify (0.7), and

$$g_n^+(t) \uparrow g^+(t), \quad g_n^-(t) \uparrow g^-(t) \quad \forall t \in \mathbb{R} \qquad \text{as } n \uparrow \infty.$$

Let $\mu \in \mathcal{M}(\Omega)$. For each $n \geq 1$, we denote by u_n the unique solution of

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

First a simple observation:

Lemma 4. *Assume $\mu \geq 0$ or $\mu \leq 0$. Then there exists $u^* \in L^1(\Omega)$ such that $u_n \to u^*$ in $L^1(\Omega)$. If $\mu \geq 0$, then $u^* \geq 0$ is the largest subsolution of* (0.1). *If $\mu \leq 0$, then $u^* \leq 0$ is the smallest supersolution of* (0.1). *In both cases, we have*

$$\Big| \int_\Omega u^* \Delta\zeta \Big| \leq 2\|\mu\|_{\mathcal{M}} \|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}) \tag{7.2}$$

and

$$\int_\Omega |g(u^*)| \leq \|\mu\|_{\mathcal{M}}. \tag{7.3}$$

Proof. If $\mu \geq 0$, then $u_n \geq 0$ a.e. In particular, $g_n(u_n) = g_n^+(u_n)$ for every $n \geq 1$. Since (g_n^+) satisfies the assumptions of Proposition 1, we conclude that $u_n \to u^*$ in $L^1(\Omega)$, where $u^* \geq 0$ is the largest subsolution of (0.1).
If $\mu \leq 0$, then $u_n \leq 0$, so that $w_n = -u_n$ satisfies

$$\begin{cases} -\Delta w_n + \tilde{g}_n(w_n) = -\mu & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{g}_n(t) = g_n^-(-t)$, $\forall t \in \mathbb{R}$. Clearly, the sequence $(\tilde{g}_n)$ satisfies the assumptions of Proposition 1. Therefore, $u_n = -w_n \to -w^* = u^*$ in $L^1(\Omega)$. It is easy to see that $u^* \leq 0$ is the smallest supersolution of (0.1).

Given $\mu \in \mathcal{M}(\Omega)$ such that $\mu \geq 0$ or $\mu \leq 0$, we *define* $\mu^* \in \mathcal{M}(\Omega)$ by

$$\mu^* = -\Delta u^* + g(u^*) \quad \text{in } (C_0^2)^*. \tag{7.4}$$

The reduced measure μ^* is well-defined because of (7.2) and (7.3). It is easy to see that

(a) if $\mu \geq 0$, then $0 \leq \mu^* \leq \mu$;

(b) if $\mu \leq 0$, then $\mu \leq \mu^* \leq 0$.

We now consider the general case of a signed measure $\mu \in \mathcal{M}(\Omega)$. In view of (7.4), both measures $(\mu^+)^*$ and $(-\mu^-)^*$ are well-defined. Moreover,

$$-\mu^- \leq (-\mu^-)^* \leq 0 \leq (\mu^+)^* \leq \mu^+.$$

The convergence of the approximating sequence (u_n) is governed by the following:

Theorem 15. *Let u_n be given by* (7.1). *Then, $u_n \to u^*$ in $L^1(\Omega)$, where u^* is the unique solution of*

$$\begin{cases} -\Delta u^* + g(u^*) = (\mu^+)^* + (-\mu^-)^* & \text{in } \Omega, \\ u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Proof. By standard estimates, $\|\Delta u_n\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}$. Thus, without loss of generality, we may assume that for a subsequence, still denoted (u_n), $u_n \to u$ in $L^1(\Omega)$ and a.e. We shall show that u satisfies (7.5); by uniqueness (see Corollary B.1), this will imply that u is independent of the subsequence.
For each $n \geq 1$, let $v_n, \tilde{v}_n$ be the solutions of

$$\begin{cases} -\Delta v_n + g_n(v_n) = \mu^+ & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.6}$$

and

$$
\begin{cases} -\Delta \tilde{v}_n + g_n^+(\tilde{v}_n) = \mu & \text{in } \Omega, \\ \tilde{v}_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.7}
$$

so that $v_n \geq 0$ a.e., $v_n \downarrow v^*$ and $\tilde{v}_n \downarrow \tilde{v}^*$ in $L^1(\Omega)$. By comparison (see Corollary B.2), we have

$$
\tilde{v}_n \leq u_n \leq v_n \quad \text{a.e.}
$$

Thus,

$$
g_n^+(\tilde{v}_n) \leq g_n^+(u_n) \leq g_n^+(v_n) = g_n(v_n) \quad \text{a.e.} \tag{7.8}
$$

By Lemma 2, we know that

$$
\begin{aligned} g_n^+(\tilde{v}_n) &\overset{*}{\rightharpoonup} g^+(\tilde{v}^*) + \mu - \mu^*, \\ g_n(v_n) &\overset{*}{\rightharpoonup} g(v^*) + \mu^+ - (\mu^+)^*. \end{aligned}
$$

Here, both reduced measures μ^* and $(\mu^+)^*$ are computed with respect to the nonlinearity g^+; in particular, (see Corollary 10)

$$
\mu - \mu^* = \mu^+ - (\mu^+)^*. \tag{7.9}
$$

We claim that

$$
g_n^+(u_n) \overset{*}{\rightharpoonup} g^+(u) + \mu^+ - (\mu^+)^*. \tag{7.10}
$$

This will be a consequence of the following

Lemma 5. *Let $a_n, b_n, c_n \in L^1(\Omega)$ be such that*

$$
a_n \leq b_n \leq c_n \quad \textit{a.e.}
$$

Assume that $a_n \to a$, $b_n \to b$ and $c_n \to c$ a.e. in Ω for some $a, b, c \in L^1(\Omega)$. If $(c_n - a_n) \overset{*}{\rightharpoonup} (c - a)$ weak in $\mathcal{M}(\Omega)$, then*

$$
(c_n - b_n) \overset{*}{\rightharpoonup} (c - b) \quad \textit{weak}^* \textit{ in } \mathcal{M}(\Omega). \tag{7.11}
$$

Proof. Since

$$
0 \leq (c_n - b_n) \leq (c_n - a_n) \quad \text{a.e.,} \tag{7.12}
$$

the sequence $(c_n - b_n)$ is bounded in $L^1(\Omega)$. Passing to a subsequence if necessary, we may assume that there exists $\lambda \in \mathcal{M}(\Omega)$ such that

$$(c_n - b_n) \overset{*}{\rightharpoonup} \lambda.$$

By (7.12), we have $0 \le \lambda \le (c-a)$. Thus, λ is absolutely continuous with respect to the Lebesgue measure. In other words, $\lambda \in L^1(\Omega)$. Given $M > 0$, we denote by S_M the truncation operator $S_M(t) = \min\{M, \max\{t, -M\}\}$, $\forall t \in \mathbb{R}$. By dominated convergence, we have

$$S_M(a_n) \to S_M(a) \quad \text{strongly in } L^1(\Omega),$$

and similarly for $S_M(b_n)$ and $S_M(c_n)$. Since

$$0 \le \big[(c_n - S_M(c_n)) - (b_n - S_M(b_n))\big] \le \big[(c_n - S_M(c_n)) - (a_n - S_M(a_n))\big] \quad \text{a.e.},$$

as $n \to \infty$ we get

$$0 \le \lambda - (S_M(c) - S_M(b)) \le \big[(c - S_M(c)) - (a - S_M(a))\big] \quad \text{a.e.}$$

Let $M \to \infty$ in the expression above. We then get $\lambda = (c - b)$. This concludes the proof of the lemma.

We now apply the previous lemma with $a_n = g_n^+(\tilde{v}_n)$, $b_n = g_n^+(u_n)$ and $c_n = g_n(v_n)$. In view of (7.8) and (7.9), the assumptions of Lemma 5 are satisfied. It follows from (7.11) that

$$g_n(v_n) - g_n^+(u_n) \overset{*}{\rightharpoonup} g(v^*) - g^+(u). \tag{7.13}$$

Thus,

$$g_n^+(u_n) = g_n(v_n) - \big[g_n(v_n) - g_n^+(u_n)\big] \overset{*}{\rightharpoonup} g^+(u) + \mu^+ - (\mu^+)^*,$$

which is precisely (7.10). A similar argument shows that

$$g_n^-(u_n) \overset{*}{\rightharpoonup} g^-(u) + \mu^- + (-\mu^-)^*. \tag{7.14}$$

We conclude from (7.10) and (7.14) that

$$g_n(u_n) \overset{*}{\rightharpoonup} g(u) + \mu - \big[(\mu^+)^* + (-\mu^-)^*\big]. \tag{7.15}$$

Therefore, u satisfies (7.5), so that (7.5) has a solution $u^* = u$. By uniqueness, the whole sequence (u_n) converges to u^* in $L^1(\Omega)$.

Motivated by Theorem 15, for any $\mu \in \mathcal{M}(\Omega)$, we *define* the reduced measure μ^* by

$$\mu^* = (\mu^+)^* + (-\mu^-)^*. \tag{7.16}$$

[This definition is coherent if μ is either a positive or a negative measure].

One can derive a number of properties satisfied by μ^*. For instance, the statements of Theorems 8–10 remain true. Moreover,

Theorem 2′. *There exists a Borel set $\Sigma \subset \Omega$ with* $\operatorname{cap}(\Sigma) = 0$ *such that*

$$|\mu - \mu^*|(\Omega \backslash \Sigma) = 0.$$

8. Examples.

We describe here some simple examples, where the measure μ^* can be explicitly identified. Throughout this section, we assume again that $g(t) = 0$ for $t \leq 0$.

Example 2. $N = 1$ and g is arbitrary.

This case is very easy since every measure is diffuse (recall that the only set of zero capacity is the empty set). Hence, by Corollary 2, every measure is good. Thus, $\mu^* = \mu$ for every μ.

Example 3. $N \geq 2$ and $g(t) = t^p$, $t \geq 0$, with $1 < p < \frac{N}{N-2}$.

In this case, we have again $\mu^* = \mu$ since, for *every* measure μ, problem (0.1) admits a solution. This result was originally established in 1975 by Ph. Bénilan and H. Brezis (see [BB, Appendix A], [B1], [B2], [B3], [B4] and also Theorem B.5 below). The crucial ingredient is the compactness of the imbedding of the space $\big\{u \in W_0^{1,1}\,;\, \Delta u \in \mathcal{M}\big\}$, equipped with the norm $\|u\|_{W^{1,1}} + \|\Delta u\|_{\mathcal{M}}$, into L^q for every $q < \frac{N}{N-2}$ (see Theorem B.1 below).

Example 4. $N \geq 3$ and $g(t) = t^p$, $t \geq 0$, with $p \geq \frac{N}{N-2}$.

In this case, we have

Theorem 16. *For every measure μ, we have*

$$\mu^* = \mu - (\mu_2)^+, \tag{8.1}$$

where $\mu = \mu_1 + \mu_2$ is the unique decomposition of μ (in the sense of Lemma A.1) relative to the $W^{2,p'}$-capacity.

Proof. By a result of Baras-Pierre [BP] (already mentioned in the Introduction) we know that a measure $\nu \geq 0$ is a good measure if and only if ν is diffuse with respect to the $W^{2,p'}$-capacity.

Set

$$\tilde{\mu} = \mu - (\mu_2)^+ = \mu_1 - (\mu_2)^- \quad \text{and} \quad \tilde{\nu} = (\mu_2)^+. \tag{8.2}$$

We claim that

$$(\tilde{\mu})^* = \tilde{\mu} \quad \text{and} \quad (\tilde{\nu})^* = 0. \tag{8.3}$$

Clearly, $(\tilde{\mu})^+ = (\mu_1)^+$. From the result of Baras-Pierre [BP], we infer that $(\mu_1)^+$ is a good measure. By Theorem 4, $\tilde{\mu}$ is also a good measure. Thus, $(\tilde{\mu})^* = \tilde{\mu}$. Since $\tilde{\nu}$ is a nonnegative measure concentrated on a set of zero $W^{2,p}$-capacity, it follows from [BP] that $(\tilde{\nu})^* \leq 0$. Since $(\tilde{\nu})^* \geq 0$, we conclude that (8.3) holds.
Applying Theorem 8, we get

$$\mu^* = (\tilde{\mu} + \tilde{\nu})^* = (\tilde{\mu})^* + (\tilde{\nu})^* = \tilde{\mu} = \mu - (\mu_2)^+,$$

which is precisely (8.1).

Remark 9. In this example we see that the measure $\mu - \mu^*$ is concentrated on a set Σ whose $W^{2,p'}$-capacity is zero. This is a better information than the general fact that $\mu - \mu^*$ is concentrated on a set Σ whose H^1-capacity is zero.

Example 5. $N = 2$ and $g(t) = \mathrm{e}^t - 1$, $t \geq 0$.

In this case, the identification of μ^* relies heavily on a result of Vázquez [Va].

Theorem 17. *Given any measure μ, let*

$$\mu = \mu_1 + \mu_2$$

where μ_2 is the purely atomic part of μ (this corresponds to the decomposition of μ in the sense of Lemma A.1, where $\mathcal{Z}$ consists of countable sets). Write

$$\mu_2 = \sum_i \alpha_i \delta_{a_i} \tag{8.4}$$

with $a_i \in \Omega$ distinct, and $\sum |\alpha_i| < \infty$. Then

$$\mu^* = \mu - \sum_i (\alpha_i - 4\pi)^+ \delta_{a_i}. \tag{8.5}$$

Proof. By a result of Vázquez [Va], we know that a measure ν is a good measure if and only if $\nu(\{x\}) \leq 4\pi$ for every $x \in \Omega$. (The paper of Vázquez deals with the equation (0.1) in all of $\mathbb{R}^2$ but the conclusion, and the proof, are the same for a bounded domain).

Clearly, $\mu_1(\{x\}) = 0$, $\forall x \in \Omega$. From the result of Vázquez [Va] we infer that μ_1 is a good measure. Thus,

$$(\mu_1)^* = \mu_1. \tag{8.6}$$

Let $a \in \Omega$ and $\alpha \in \mathbb{R}$. It is easy to see from [Va] that

$$(\alpha\delta_a)^* = \min\{\alpha, 4\pi\}\, \delta_a. \tag{8.7}$$

An induction argument applied to Theorem 8 and the continuity of the mapping $\mu \mapsto \mu^*$ show that

$$\mu^* = (\mu_1)^* + (\mu_2)^* = (\mu_1)^* + \sum_i (\alpha_i \delta_{a_i})^*. \tag{8.8}$$

By (8.6)–(8.8), we have

$$\mu^* = \mu_1 + \sum_i \min\{\alpha_i, 4\pi\}\, \delta_{a_i} = \mu - \sum_i (\alpha_i - 4\pi)^+ \delta_{a_i}. \tag{8.9}$$

This establishes (8.5).

We conclude this section with two interesting questions:

Open problem 3. Let $N = 2$ and $g(t) = (\mathrm{e}^{t^2} - 1)$, $t \geq 0$. Is there an explicit formula for μ^* ?

Open problem 4. Let $N \geq 3$ and $g(t) = (\mathrm{e}^t - 1)$, $t \geq 0$. Is there an explicit formula for μ^* ?

A partial answer to Open problem 4 has been obtained by Bartolucci-Leoni-Orsina-Ponce [BLOP]. More precisely, they have established the following:

Theorem 18. *Any measure μ such that $\mu \leq 4\pi\mathcal{H}^{N-2}$ is a good measure.*

Here, $\mathcal{H}^{N-2}$ denotes the $(N-2)$-Hausdorff measure. The converse of Theorem 18 is not true. This was suggested by L. Véron in a personal communication; explicit examples are given in [P]. The characterization of good measures is still open; see however [MV5].

9. Further directions and open problems.

9.1. Vertical asymptotes.

Let $g : (-\infty, +1) \to \mathbb{R}$ be a continuous, nondecreasing function such that $g(t) = 0$, $\forall t \leq 0$, and such $g(t) \to +\infty$ as $t \to +1$. Let (g_n) be a sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ which are continuous, nondecreasing and satisfy the following conditions:

$$0 \leq g_1(t) \leq g_2(t) \leq \ldots \leq g(t) \quad \forall t < 1, \tag{9.1}$$

$$g_n(t) \to g(t) \quad \forall t < 1 \quad \text{and} \quad g_n(t) \to +\infty \quad \forall t \geq 1. \tag{9.2}$$

If $N \geq 2$, then we also assume that

$$\text{each } g_n \text{ has subcritical growth, i.e.,} \quad g_n(t) \leq C(|t|^p + 1) \quad \forall t \in \mathbb{R}, \tag{9.3}$$

for some constant C and some $p < \frac{N}{N-2}$, possibly depending on n.

Given $\mu \in \mathcal{M}(\Omega)$, let u_n be a solution of (0.10). Then $u_n \downarrow u^*$ in Ω as $n \uparrow \infty$. Moreover (0.11) and (0.12) hold. We may therefore define $\mu^* \in \mathcal{M}(\Omega)$ by (0.13).

Open problem 5. Study the properties of u^* and the reduced measure μ^*.

Clearly, u^* is the largest subsolution. But there are some major differences in this case. When $N \geq 2$, Dupaigne-Ponce-Porretta [DPP] have shown that for any such g one can find a nonnegative measure μ for which the set $\{\nu \in \mathcal{G} \ ; \ \nu \leq \mu\}$ has *no* largest element. In particular, for such measure μ, the reduced measure μ^* cannot be the largest good measure $\leq \mu$. They have also proved that the set of good measures $\mathcal{G}$ is *not* convex for any g. We refer the reader to [DPP] for other results.

Similar questions arise when g is a multivalued graph. For example,

$$g(r) = \begin{cases} 0 & \text{if } r < 1, \\ [0, \infty) & \text{if } r = 1, \\ \emptyset & \text{if } r > 1. \end{cases}$$

This is a simple model of one-sided variational inequality. The objective is to solve in some natural "weak" sense the multivalued equation

$$\begin{cases} -\Delta u + g(u) \ni \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for any given $\mu \in \mathcal{M}(\Omega)$. This problem has been recently studied by Brezis-Ponce [BP4]. There were some partial results; see, e.g., Baxter [Ba], Dall'Aglio-Dal Maso [DD], Orsina-Prignet [OP], Brezis-Serfaty [BSe], and the references therein.

9.2. Nonlinearities involving ∇u.

Consider the model problem:

$$\begin{cases} -\Delta u + u|\nabla u|^2 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.4}$$

where $\mu \in \mathcal{M}(\Omega)$. Problems of this type have been extensively studied and it is known that they bear some similarities with the problems discussed in this paper. In particular, it has been proved in [BGO2] that (9.4) admits a solution if and only if the measure μ is diffuse, i.e., $|\mu|(A) = 0$ for every Borel set $A \subset \Omega$ such that $\operatorname{cap}(A) = 0$. Moreover, the solution is unique (see [BM]). When μ is a general measure, not necessarily diffuse, it would be interesting to apply to (9.4) the same strategy as in this paper. More precisely, to prove that approximate solutions converge to the solution of (9.4), where μ is replaced by its diffuse part μ_{d} (in the sense of Lemma A.1, relative to the Borel sets whose H^1-capacity are zero):

$$\begin{cases} -\Delta u + u|\nabla u|^2 = \mu_{\mathrm{d}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.5}$$

which possesses a unique solution. There are several "natural" approximations. For example, one may truncate the nonlinearity $g(u, \nabla u) = u|\nabla u|^2$ and replace it by $g_n(u, \nabla u) = \frac{n}{n+|g(u,\nabla u)|}\, g(u, \nabla u)$. It is easy to see (via a Schauder fixed point argument in $W_0^{1,1}$) that the corresponding equation

$$
\begin{cases} -\Delta u_n + g_n(u_n, \nabla u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.6}
$$

admits a solution u_n.

Open problem 6. Is it true that (u_n) converges to the solution of (9.5)?

Another possible approximation consists of smoothing μ: let u_n be a solution of

$$
\begin{cases} -\Delta u_n + u_n |\nabla u_n|^2 = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.7}
$$

where $\mu_n = \rho_n * \mu$, as in Section 4. It has been proved by Porretta [Po] that if $\mu \geq 0$, then $u_n \to u$ in $L^1(\Omega)$, where u is the solution of (9.5). We have been informed by A. Porretta that the same conclusion holds for any measure μ, by using a substantial modification of the argument in [Po].

9.3. Measures as boundary data.

Consider the problem

$$
\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{9.8}
$$

where μ is a measure on $\partial\Omega$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function satisfying (0.5). It has been proved by H. Brezis (1972, unpublished) that (9.8) admits a unique weak solution when μ is any L^1 function (for a general nonlinearity g). When g is a power, the study of (9.8) for measures was initiated by Gmira-Véron [GV], and has vastly expanded in recent years; see the papers of Marcus-Véron [MV1], [MV2], [MV3], [MV4]. Important motivations coming from the theory of probability — and the use of probabilistic methods — have reinvigorated the whole subject; see the pioneering papers of Le Gall [LG1], [LG2], the recent books of Dynkin [D1], [D2], and the numerous references therein. It is known that (9.8) has no solution if $g(t) = t^p$, $t \geq 0$, with $p \geq p_{\mathrm{c}} = \frac{N+1}{N-1}$ and $\mu = \delta_a$, $a \in \partial\Omega$ (see [GV]). Therefore, it is interesting to develop for (9.8) the same program as in this paper. More precisely, let (g_k) be a sequence of functions $g_k : \mathbb{R} \to \mathbb{R}$ which are continuous, nondecreasing, and satisfy (0.7) and (0.8). Assume in addition that each g_k is, e.g., bounded. Then, for every $\mu \in \mathcal{M}(\partial\Omega)$, there exists a unique solution u_k of

$$\begin{cases} -\Delta u_k + g_k(u_k) = 0 & \text{in } \Omega, \\ u_k = \mu & \text{on } \partial\Omega, \end{cases} \tag{9.9}$$

in the sense that $u_k \in L^1(\Omega)$ and

$$-\int_\Omega u_k \Delta\zeta + \int_\Omega g_k(u_k)\zeta = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall\zeta \in C_0^2(\overline{\Omega}), \tag{9.10}$$

where $\frac{\partial}{\partial n}$ denotes the derivative with respect to the outward normal of $\partial\Omega$.
We have the following:

Theorem 19. *As $k \uparrow \infty$, $u_k \downarrow u^*$ in $L^1(\Omega)$, where u^* satisfies*

$$\begin{cases} -\Delta u^* + g(u^*) = 0 & \textit{in } \Omega, \\ u^* = \mu^* & \textit{on } \partial\Omega, \end{cases} \tag{9.11}$$

for some $\mu^* \in \mathcal{M}(\partial\Omega)$ such that $\mu^* \leq \mu$. More precisely, $g(u^*)\rho_0 \in L^1(\Omega)$, where $\rho_0(x) = d(x, \partial\Omega)$, and

$$-\int_\Omega u^* \Delta\zeta + \int_\Omega g(u^*)\zeta = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu^* \quad \forall\zeta \in C_0^2(\overline{\Omega}). \tag{9.12}$$

In addition, u^* is the largest subsolution of (9.8), *i.e., if $v \in L^1(\Omega)$ is any function satisfying $g(v)\rho_0 \in L^1(\Omega)$ and*

$$-\int_\Omega v \Delta\zeta + \int_\Omega g(v)\zeta \leq -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall\zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \textit{ in } \Omega, \tag{9.13}$$

then $v \leq u^*$ a.e. in Ω.

Proof. By comparison (see Corollary B.2), we know that (u_k) is non-increasing. By standard estimates, we have

$$\int_\Omega |u_k| + \int_\Omega g_k(u_k)\rho_0 \leq C\|\mu\|_{\mathcal{M}(\partial\Omega)} \quad \forall k \geq 1.$$

In addition, (see [B5, Theorem 3])

$$\|u_k\|_{C^1(\overline{\omega})} \leq C_\omega \quad \forall k \geq 1,$$

for every $\omega \subset\subset \Omega$. Thus, u_k converges in $L^1(\Omega)$ to a limit, say u^*. Moreover,

$$g_k(u_k) \to g(u^*) \quad \text{in } L^\infty_{\text{loc}}(\Omega).$$

Let $\zeta_0 \in C_0^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta\zeta_0 = 1 & \text{in } \Omega, \\ \quad\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\big(g_k(u_k)\zeta_0\big)$ is uniformly bounded in $L^1(\Omega)$, then up to a subsequence

$$g_k(u_k)\zeta_0 \overset{*}{\rightharpoonup} g(u^*)\zeta_0 + \lambda \quad \text{in } \big[C(\overline{\Omega})\big]^*, \tag{9.14}$$

for some $\lambda \in \mathcal{M}(\partial\Omega)$, $\lambda \geq 0$. We claim that

$$\int_\Omega g_k(u_k)\zeta \to \int_\Omega g(u^*)\zeta + \int_{\partial\Omega} \frac{\partial\zeta}{\partial n} \frac{1}{\frac{\partial\zeta_0}{\partial n}}\, d\lambda \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{9.15}$$

In fact, given $\zeta \in C_0^2(\overline{\Omega})$, define $\gamma = \zeta/\zeta_0$. It is easy to see that $\gamma \in C(\overline{\Omega})$ and $\gamma = \frac{\partial\zeta}{\partial n}\frac{1}{\frac{\partial\zeta_0}{\partial n}}$ on $\partial\Omega$. Using γ as a test function in (9.14), we obtain (9.15).
Let $k \to \infty$ in (9.10). In view of (9.15), we conclude that u^* satisfies (9.12), where μ^* is given by

$$\mu^* = \mu + \frac{1}{\frac{\partial\zeta_0}{\partial n}}\,\lambda \leq \mu.$$

Finally, it follows from Corollary B.2 that if v is a subsolution of (9.8), then $v \leq u_k$ a.e., $\forall k \geq 1$, and thus $v \leq u^*$ a.e.

Some natural questions have been addressed and the following results will be presented in a forthcoming paper (see [BP3]):

(a) the reduced measure μ^* is the largest good measure $\leq \mu$; in other words, if $\nu \in \mathcal{M}(\partial\Omega)$ is a good measure (i.e., (9.8) has a solution with boundary data ν) and if $\nu \leq \mu$, then $\nu \leq \mu^*$;

(b) $\mu - \mu^*$ is concentrated on a subset of $\partial\Omega$ of zero $\mathcal{H}^{N-1}$-measure (i.e., $(N-1)$-dimensional Lebesgue measure on $\partial\Omega$) and this fact is "optimal", in the sense that any measure $\nu \geq 0$ which is singular with respect to $\mathcal{H}^{N-1}\lfloor_{\partial\Omega}$ can be written as $\nu = \mu - \mu^*$ for some $\mu \geq 0$ and some g;

(c) if μ is a measure on $\partial\Omega$ which is good for every g, then $\mu^+ \in L^1(\partial\Omega)$;

(d) given any g, there exists some measure $\mu \geq 0$ on $\partial\Omega$ which is good for g, but $\mu \notin L^1(\partial\Omega)$.

When $g(t) = t^p$, $t \geq 0$, with $p \geq \frac{N+1}{N-1}$, a known result (see, e.g., [MV3]) asserts that $\mu \in \mathcal{M}(\partial\Omega)$ is a good measure if and only if $\mu^+(A) = 0$ for every Borel set $A \subset \partial\Omega$ such that $C_{2/p,p'}(A) = 0$, where $C_{2/p,p'}$ refers to the Bessel capacity on $\partial\Omega$. In this case, we have

(e) the reduced measure μ^* is given by $\mu^* = \mu - (\mu_2)^+$, where $\mu = \mu_1 + \mu_2$ is the decomposition of μ, in the sense of Lemma A.1, relative to $C_{2/p,p'}$.

In contrast with Example 5, we do not know what the reduced measure μ^* is when $N = 2$ and $g(t) = \mathrm{e}^t - 1$, $t \geq 0$.

Similar issues can be investigated for the parabolic equations

$$\begin{cases} u_t - \Delta u + g(u) = \mu, \\ \qquad\qquad u(0) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} u_t - \Delta u + g(u) = 0, \\ \qquad\qquad u(0) = \mu. \end{cases}$$

Appendix A: Decomposition of measures into diffuse and concentrated parts.

The following result is taken from [FST]. We reproduce their proof for the convenience of the reader.

Lemma A.1. *Let μ be a bounded Borel measure in $\mathbb{R}^N$ and let $\mathcal{Z}$ be a collection of Borel sets such that:*

(a) *$\mathcal{Z}$ is closed with respect to finite or countable unions;*

(b) *$A \in \mathcal{Z}$ and $A' \subset A$ Borel $\Rightarrow A' \in \mathcal{Z}$.*

Then μ can be represented in the form

$$\mu = \mu_1 + \mu_2, \tag{A.1}$$

where μ_1 and μ_2 are bounded Borel measures such that

$$\mu_1(A) = 0 \quad \forall A \in \mathcal{Z} \quad \text{and} \quad \mu_2 \text{ vanishes outside a set } A_0 \in \mathcal{Z}.$$

This representation is unique.

Proof. First assume that μ is nonnegative. Denote

$$X_\mu = \sup \big\{\mu(A)\,;\, A \in \mathcal{Z}\big\}.$$

Let $\{A_n\}$ be an increasing sequence of sets in $\mathcal{Z}$ such that

$$\mu(A_n) \to X_\mu.$$

Let $A_0 = \bigcup_n A_n$ and put

$$\mu_1(B) = \mu(B \cap A_0^{\mathrm{c}}), \quad \mu_2(B) = \mu(B \cap A_0),$$

for every Borel set B. Since $A_0 \in \mathcal{Z}$, it remains to verify that μ_1 vanishes on sets of $\mathcal{Z}$. By contradiction, suppose that there exists $E \in \mathcal{Z}$ such that $\mu_1(E) > 0$. Let $E_1 = E \cap A_0^{\mathrm{c}}$. Then $\mu(E_1) > 0$ and $E_1 \in \mathcal{Z}$. It follows that $A_0 \cup E_1 \in \mathcal{Z}$ and $\mu(A_0 \cup E_1) > X_\mu$. Contradiction.
If μ is a signed measure, apply the above to μ^+ and μ^-. The uniqueness is obvious.

Appendix B: Standard existence, uniqueness and comparison results.

In this appendix, we collect some well-known results (and a few new ones) which are used throughout this paper. For the convenience of the reader, we shall sketch some of the proofs.

We start with the existence, uniqueness and regularity of solutions of the linear problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.1}$$

where $\mu \in \mathcal{M}(\Omega)$.

Theorem B.1. *Given $\mu \in \mathcal{M}(\Omega)$, there exists a unique $u \in L^1(\Omega)$ satisfying*

$$-\int_\Omega u\Delta\zeta = \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{B.2}$$

Moreover, $u \in W_0^{1,q}(\Omega)$ for every $1 \le q < \frac{N}{N-1}$, with the estimates

$$\|u\|_{L^{q^*}} \le C\|\nabla u\|_{L^q} \le C\|\mu\|_{\mathcal{M}}, \tag{B.3}$$

where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{N}$. In particular, $u \in L^p(\Omega)$ for every $1 \le p < \frac{N}{N-2}$, and u satisfies

$$\int_\Omega \nabla u \cdot \nabla \psi = \int_\Omega \psi \, d\mu \quad \forall \psi \in W_0^{1,r}(\Omega), \tag{B.4}$$

for any $r > N$.

The proof of Theorem B.1 relies on a standard duality argument and shall be omitted; see [S, Théorème 8.1].

We now establish a weak form of the maximum principle:

Proposition B.1. *Let* $v \in W_0^{1,1}(\Omega)$ *be such that*

$$-\int_\Omega v\Delta\varphi \leq 0 \quad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega. \tag{B.5}$$

Then

$$-\int_\Omega v\Delta\zeta \leq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega \tag{B.6}$$

and, consequently,

$$v \leq 0 \quad a.e. \tag{B.7}$$

Proof. From (B.5) we have

$$\int_\Omega \nabla v \cdot \nabla\varphi \leq 0 \quad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega$$

so that, by density of $C_c^\infty(\Omega)$ in $C_c^2(\Omega)$,

$$\int_\Omega \nabla v \cdot \nabla\varphi \leq 0 \quad \forall \varphi \in C_c^2(\Omega),\ \varphi \geq 0 \text{ in } \Omega.$$

Let (γ_n) be a sequence in $C_c^\infty(\Omega)$ such that $0 \leq \gamma_n \leq 1$, $\gamma_n(x) = 1$ if $d(x, \partial\Omega) > \frac{1}{n}$, and $|\nabla\zeta_n| \leq Cn$, $\forall n \geq 1$. For any $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \geq 0$, we have

$$\int_\Omega \nabla v \cdot (\gamma_n \nabla\zeta + \zeta\nabla\gamma_n) = \int_\Omega \nabla v \cdot \nabla(\gamma_n\zeta) \leq 0. \tag{B.8}$$

Note that

$$\int_\Omega |\nabla v||\nabla\gamma_n|\zeta \leq Cn \int_{d(x,\partial\Omega)\leq\frac{1}{n}} |\nabla v|\zeta \leq C \int_{d(x,\partial\Omega)\leq\frac{1}{n}} |\nabla v| \to 0 \quad \text{as } n \to \infty.$$

Thus, as $n \to \infty$ in (B.8), we obtain

$$\int_\Omega \nabla v \cdot \nabla\zeta \leq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega,$$

which yields (B.6) since $v \in W_0^{1,1}(\Omega)$. Inequality (B.7) is a trivial consequence of (B.6).

Lemma B.1. *Let $p : \mathbb{R} \to \mathbb{R}$, $p(0) = 0$, be a bounded nondecreasing continuous function. Given $f \in L^1(\Omega)$, let $u \in L^1(\Omega)$ be the unique solution of*

$$-\int_\Omega u\Delta\zeta = \int_\Omega f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{B.9}$$

Then

$$\int_\Omega fp(u) \geq 0. \tag{B.10}$$

Proof. Clearly, it suffices to establish the lemma for $p \in C^2(\mathbb{R})$. Assume for the moment $f \in C^\infty(\overline{\Omega})$. In this case, $u \in C_0^2(\overline{\Omega})$. Since $p(0) = 0$, we have $p(u) \in C_0^2(\overline{\Omega})$. Using $p(u)$ as a test function in (B.9), we get

$$\int_\Omega fp(u) = \int_\Omega p'(u)|\nabla u|^2 \geq 0.$$

This establishes the lemma for f smooth. The general case when f is just an L^1-function, not necessarily smooth, easily follows by density.

Proposition B.2. *Given $f \in L^1(\Omega)$, let u be the unique solution of* (B.9)*. Then, for every $M > 0$, we have*

$$\int\limits_{[u \geq M]} f \geq 0 \quad \text{and} \quad \int\limits_{[u \leq -M]} f \leq 0. \tag{B.11}$$

In particular,

$$\int\limits_{[|u| \geq M]} f \operatorname{sgn}(u) \geq 0. \tag{B.12}$$

Above, we denote by sgn the function $\operatorname{sgn}(t) = 1$ if $t > 0$, $\operatorname{sgn}(t) = -1$ if $t < 0$, and $\operatorname{sgn}(0) = 0$.

Proof. Clearly, it suffices to establish the first inequality in (B.11). Let (p_n) be a sequence of continuous functions in $\mathbb{R}$ such that each p_n is nondecreasing, $p_n(t) = 1$ if $t \geq M$ and $p_n(t) = 0$ if $t \leq M - \frac{1}{n}$. By the previous lemma,

$$\int_\Omega fp_n(u) \geq 0 \quad \forall n \geq 1.$$

As $n \to \infty$, the result follows.

Proposition B.3. *Let* $v \in L^1(\Omega)$, $f \in L^1(\Omega)$ *and* $\nu \in \mathcal{M}(\Omega)$ *satisfy*

$$-\int_\Omega v\Delta\zeta + \int_\Omega f\zeta = \int_\Omega \zeta \, d\nu \quad \forall\zeta \in C_0^2(\overline{\Omega}). \tag{B.13}$$

Then

$$\int\limits_{[v>0]} f \leq \|\nu^+\|_{\mathcal{M}} \tag{B.14}$$

and thus

$$\int_\Omega f \operatorname{sgn}(v) \leq \|\nu\|_{\mathcal{M}}. \tag{B.15}$$

Proof. Let $\nu_n = \rho_n * \nu$ (here we use the same notation as in Section 4). Let v_n denote the solution of (B.13) with ν replaced by ν_n. By Lemma B.1, we have

$$\int_\Omega (\nu_n - f)\, p(v_n) \geq 0,$$

where p is any function satisfying the assumptions of the lemma. Thus, if $0 \leq p(t) \leq 1$, $\forall t \in \mathbb{R}$, then we have

$$\int_\Omega fp(v_n) \leq \int_\Omega \nu_n p(v_n) \leq \int_\Omega (\nu_n)^+ \leq \|\nu^+\|_{\mathcal{M}}.$$

Let $n \to \infty$ to get

$$\int_\Omega fp(v) \leq \|\nu^+\|_{\mathcal{M}}. \tag{B.16}$$

Apply (B.16) to a sequence of nondecreasing continuous functions (p_n) such that $p_n(t) = 0$ if $t \leq 0$ and $p_n(t) = 1$ if $t \geq \frac{1}{n}$. As $n \to \infty$, we obtain (B.14).

An easy consequence of Proposition B.3 is the following

Corollary B.1. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous, nondecreasing function such that* $g(0) = 0$. *Given* $\mu \in \mathcal{M}(\Omega)$, *then the equation*

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.17}$$

has at most *one solution* $u \in L^1(\Omega)$ *with* $g(u) \in L^1(\Omega)$. *Moreover,*

$$\int_\Omega |g(u)| \le \|\mu\|_{\mathcal{M}} \quad \textit{and} \quad \int_\Omega |\Delta u| \le 2\|\mu\|_{\mathcal{M}}. \tag{B.18}$$

If (B.17) *has a solution for* $\mu_1, \mu_2 \in \mathcal{M}(\Omega)$, *say* u_1, u_2, *resp., then*

$$\int_\Omega \big[g(u_1) - g(u_2)\big]^+ \le \big\|(\mu_1 - \mu_2)^+\big\|_{\mathcal{M}}. \tag{B.19}$$

In particular,

$$\int_\Omega \big|g(u_1) - g(u_2)\big| \le \|\mu_1 - \mu_2\|_{\mathcal{M}}. \tag{B.20}$$

We now recall the following unpublished result of H. Brezis from 1972 (see, e.g., [GV]):

Proposition B.4. *Given* $f \in L^1(\Omega; \rho_0\, dx)$ *and* $h \in L^1(\partial\Omega)$, *there exists a unique* $u \in L^1(\Omega)$ *such that*

$$-\int_\Omega u\Delta\zeta = \int_\Omega f\zeta - \int_{\partial\Omega} h\frac{\partial\zeta}{\partial n} \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{B.21}$$

In addition, there exists $C > 0$ *such that*

$$\|u\|_{L^1} \le C\left(\|f\rho_0\|_{L^1(\Omega)} + \|h\|_{L^1(\partial\Omega)}\right). \tag{B.22}$$

We now establish the following

Lemma B.2. *Given* $f \in L^1(\Omega; \rho_0\, dx)$, *let* $u \in L^1(\Omega)$ *be the unique solution of* (B.21) *with* $h = 0$. *Then*

$$k \int_{d(x,\partial\Omega) < \frac{1}{k}} |u| \to 0 \quad \textit{as } k \to \infty. \tag{B.23}$$

Proof.
Step 1. Proof of the lemma when $f \ge 0$.

Since $f \ge 0$, we have $u \ge 0$. Let $H \in C^2(\mathbb{R})$ be a nondecreasing concave function such that $H(0) = 0$, $H''(t) = -1$ if $t \le 1$ and $H(t) = 1$ if $t \ge 2$. We denote by $\zeta_0 \in C_0^2(\overline{\Omega})$, $\zeta_0 \ge 0$, the solution of

$$\begin{cases} -\Delta\zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

For any $k \geq 1$, let $w_k = \frac{1}{k} H(k\zeta_0)$. By construction, $w_k \in C_0^2(\overline{\Omega})$ and

$$\Delta w_k = kH''(k\zeta_0)|\nabla\zeta_0|^2 + H'(k\zeta_0)\Delta\zeta_0 \leq -k\chi_{[\zeta_0 \leq \frac{1}{k}]}|\nabla\zeta_0|^2.$$

Thus,

$$-\int_\Omega u\Delta w_k \geq k \int\limits_{[\zeta_0 \leq \frac{1}{k}]} |\nabla\zeta_0|^2 u. \tag{B.24}$$

Use w_k as a test function in (B.21) (recall that $h = 0$). It follows from (B.24) that

$$k \int\limits_{[\zeta_0 \leq \frac{1}{k}]} |\nabla\zeta_0|^2 u \leq \int_\Omega w_k f. \tag{B.25}$$

By Hopf's lemma, we have $|\nabla\zeta_0|^2 \geq \alpha_0 > 0$ in some neighborhood of $\partial\Omega$ in $\overline{\Omega}$. In particular, there exists $c > 0$ such that $c\zeta_0(x) \leq d(x, \partial\Omega) \leq \frac{1}{c}\zeta_0(x)$ for all $x \in \overline{\Omega}$. Thus, for $k \geq 1$ sufficiently large, we have

$$\alpha_0 k \int\limits_{d(x,\partial\Omega) \leq \frac{c}{k}} |\nabla\zeta_0|^2 u \leq \int_\Omega w_k f. \tag{B.26}$$

Note that the right-hand side of (B.26) tends to 0 as $k \to \infty$. In fact, we have $w_k \leq C\zeta_0$, $\forall k \geq 1$, and $w_k \leq \frac{1}{k} H(k\zeta_0) \to 0$ a.e. Thus, by dominated convergence,

$$\int_\Omega w_k f \to 0 \quad \text{as } k \to \infty. \tag{B.27}$$

Combining (B.26) and (B.27), we obtain (B.23).

Step 2. Proof of the lemma completed.

Let $v \in L^1(\Omega)$ denote the unique solution of

$$-\int_\Omega v\Delta\zeta = \int_\Omega |f|\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{B.28}$$

By comparison, we have $|u| \leq v$. On the other hand, v satisfies the assumption of Step 1. Thus,

$$k \int\limits_{d(x,\partial\Omega) < \frac{1}{k}} |u| \leq k \int\limits_{d(x,\partial\Omega) < \frac{1}{k}} v \to 0 \quad \text{as } k \to \infty. \tag{B.29}$$

This establishes Lemma B.2.

The next result is a new variant of Kato's inequality, where the test function ζ need not have compact support in Ω:

Proposition B.5. *Let $u \in L^1(\Omega)$ and $f \in L^1(\Omega;\rho_0\,dx)$ be such that*

$$-\int_\Omega u\Delta\zeta \le \int_\Omega f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega. \tag{B.30}$$

Then

$$-\int_\Omega u^+\Delta\zeta \le \int\limits_{[u\ge 0]} f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega. \tag{B.31}$$

Proof. We first notice that

$$-\int_\Omega u^+\Delta\varphi \le \int\limits_{[u\ge 0]} f\varphi \quad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega. \tag{B.32}$$

In fact, by (B.30) we have $-\Delta u \le f$ in $\mathcal{D}'(\Omega)$. Then, Theorem 7 yields

$$(-\Delta u^+)_{\mathrm{d}} \le \chi_{[u\ge 0]}(-\Delta u)_{\mathrm{d}} \le \chi_{[u\ge 0]} f \quad \text{and} \quad (-\Delta u^+)_{\mathrm{c}} = (-\Delta u)_{\mathrm{c}}^+ \le (f)_{\mathrm{c}}^+ = 0.$$

Thus,

$$-\Delta u^+ = (-\Delta u^+)_{\mathrm{d}} + (-\Delta u^+)_{\mathrm{c}} \le \chi_{[u\ge 0]} f \quad \text{in } \mathcal{D}'(\Omega),$$

which is precisely (B.32).

Let $(\gamma_k) \subset C_c^\infty(\Omega)$ be a sequence such that $0 \le \gamma_k \le 1$ in Ω, $\gamma_k(x) = 1$ if $d(x,\partial\Omega) \ge \frac{1}{k}$, $\|\nabla\gamma_k\|_{L^\infty} \le k$, and $\|\Delta\gamma_k\|_{L^\infty} \le Ck^2$. Given $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \ge 0$, we apply (B.32) with $\varphi = \zeta\gamma_k$ to get

$$-\int_\Omega u^+\Delta(\zeta\gamma_k) \le \int\limits_{[u\ge 0]} f\zeta\gamma_k. \tag{B.33}$$

Consider again the unique solution $v \ge 0$ of (B.28). By comparison we have $u \le v$ a.e. and thus $u^+ \le v$ a.e. From Lemma B.2 we see that

$$\int_\Omega u^+|\nabla\zeta||\nabla\gamma_k| \le Ck \int\limits_{d(x,\partial\Omega)<\frac{1}{k}} u^+ \to 0 \quad \text{as } k \to \infty. \tag{B.34}$$

Similarly,

$$\int_\Omega u^+\zeta|\Delta\gamma_k| \le Ck \int\limits_{d(x,\partial\Omega)<\frac{1}{k}} u^+ \to 0 \quad \text{as } k \to \infty. \tag{B.35}$$

Let $k \to \infty$ in (B.33). Using (B.34) and (B.35), we obtain (B.31).

Remark B.1. There is an alternative proof of Proposition B.5. First, one shows that (B.30) implies that there exist two measures $\mu \leq 0$, $\lambda \leq 0$, where $\mu \in \mathcal{M}(\partial\Omega)$ and λ is locally bounded in Ω, with $\int_\Omega \rho_0 \, d|\lambda| < \infty$, satisfying

$$-\int_\Omega u\Delta\zeta = \int_\Omega f\zeta + \int_\Omega \zeta \, d\lambda - \int_{\partial\Omega} \frac{\partial \zeta}{\partial n} \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{B.36}$$

[The existence of λ is fairly straightforward, and the existence of μ is a consequence of Herglotz's theorem concerning positive superharmonic functions].
Then, inequality (B.31) follows from (B.36) using the same strategy as in the proof of Lemma 1.5 in [MV2].

As a consequence of Proposition B.5, we have the following

Corollary B.2. *Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be two continuous nondecreasing functions such that $g_1 \leq g_2$. Let $u_k \in L^1(\Omega)$, $k = 1, 2$, be such that $g_k(u_k) \in L^1(\Omega; \rho_0 \, dx)$. If*

$$-\int_\Omega (u_2 - u_1)\Delta\zeta + \int_\Omega \big[g_2(u_2) - g_1(u_1)\big]\zeta \leq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega, \tag{B.37}$$

then

$$u_2 \leq u_1 \quad a.e. \tag{B.38}$$

Proof. Applying Proposition B.5 to $u = u_2 - u_1$ and $f = g_1(u_1) - g_2(u_2)$, we have

$$-\int_\Omega (u_2 - u_1)^+ \Delta\zeta \leq -\int_\Omega \big[g_2(u_2) - g_1(u_1)\big]^+ \zeta \leq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega.$$

This immediately implies that $u_2 \leq u_1$ a.e.

We now present some general existence results for problem (B.17). Below, $g : \mathbb{R} \to \mathbb{R}$ denotes a continuous, nondecreasing function, such that $g(0) = 0$.

Theorem B.2 (Brezis-Strauss [BS]). *For every $f \in L^1(\Omega)$, the equation*

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.39}$$

has a unique solution $u \in L^1(\Omega)$ with $g(u) \in L^1(\Omega)$.

Proof. We first observe that if $f \in C^\infty(\overline{\Omega})$, then (B.39) always has a solution $u \in C^1(\overline{\Omega})$ (easily obtained via minimization).
For a general $f \in L^1(\Omega)$, let (f_n) be a sequence of smooth functions on $\overline{\Omega}$, converging to f in $L^1(\Omega)$. For each f_n, let u_n denote the corresponding solution of (B.39). By (B.20), the sequence $(g(u_n))$ is Cauchy in $L^1(\Omega)$. We then conclude from (B.3) that (u_n) is also Cauchy in $L^1(\Omega)$, so that

$$u_n \to u \quad \text{and} \quad g(u_n) \to g(u) \quad \text{in } L^1(\Omega).$$

Thus u is a solution of (B.39). The uniqueness follows from Corollary B.1.

Theorem B.3 (Brezis-Browder [BBr]). *For every $T \in H^{-1}(\Omega)$, the equation*

$$\text{(B.40)} \qquad \begin{cases} -\Delta u + g(u) = T & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$ with $g(u) \in L^1(\Omega)$.

Proof. Assume g is uniformly bounded. In this case, the existence of u presents no difficulty, e.g., via a minimization argument in $H_0^1(\Omega)$. In particular, we see that $u \in H_0^1(\Omega)$.
For a general nonlinearity g, let (g_n) be the sequence given by $g_n(t) = g(t)$ if $|t| \le n$, $g_n(t) = g(n)$ if $t > n$, and $g_n(t) = g(-n)$ if $t < -n$. Let $u_n \in H_0^1(\Omega)$ be the solution of (B.40) corresponding to g_n. Note that u_n satisfies

$$\int_\Omega \nabla u_n \cdot \nabla v + \int_\Omega g_n(u_n) v = \langle T, v\rangle \quad \forall v \in H_0^1(\Omega).$$

Using $v = u_n$ as a test function, we get

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega g_n(u_n) u_n = \langle T, u_n\rangle \le C \left(\int_\Omega |\nabla u_n|^2\right)^{1/2}.$$

Thus,

$$\text{(B.41)} \qquad \int_\Omega g_n(u_n) u_n \le C \quad \text{and} \quad \int_\Omega |\nabla u_n|^2 \le C,$$

for some constant $C > 0$ independent of $n \ge 1$. Since (u_n) is uniformly bounded in $H_0^1(\Omega)$, then up to a subsequence we can find $u \in H_0^1(\Omega)$ such that

$$u_n \to u \quad \text{in } L^1 \text{ and a.e.}$$

By (B.41), for any $M > 0$, we also have

$$\int_{[|u_n| \geq M]} |g_n(u_n)| \leq \frac{1}{M} \int_\Omega g_n(u_n) u_n \leq \frac{C}{M}.$$

We claim that

$$g_n(u_n) \quad \text{is equi-integrable.}$$

In fact, for any Borel set $E \subset \Omega$, we estimate

$$\int_E |g_n(u_n)| = \int_{\substack{E \\ [|u_n| < M]}} |g_n(u_n)| + \int_{\substack{E \\ [|u_n| \geq M]}} |g_n(u_n)| \leq A_M |E| + \frac{C}{M},$$

where $A_M = \max\{g(M), -g(-M)\}$. Given $\varepsilon > 0$, let $M > 0$ sufficiently large so that $\frac{C}{M} < \varepsilon$. With M fixed, we take $|E|$ small enough so that $A_M|E| < \varepsilon$. We conclude that

$$\int_E |g_n(u_n)| < 2\varepsilon \quad \forall n \geq 1.$$

Thus, $(g_n(u_n))$ is equi-integrable. Since $u_n \to u$ a.e., it follows from Egorov's lemma that $g_n(u_n) \to g(u)$ in $L^1(\Omega)$. Therefore, u satisfies (B.40). By Proposition B.3, this solution is unique.

Combining the techniques from both proof, we have the following:

Theorem B.4. *For every $f \in L^1(\Omega)$ and every $T \in H^{-1}(\Omega)$, the equation*

$$\begin{cases} -\Delta u + g(u) = f + T & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.42}$$

has a unique solution $u \in L^1(\Omega)$ with $g(u) \in L^1(\Omega)$.

Proof. Let f_n be a sequence in $C^\infty(\overline{\Omega})$ converging to f in $L^1(\Omega)$. Since $f_n + T \in H^{-1}$, we can apply Theorem B.3 to obtain a solution u_n of (B.42) for $f_n + T$. For every $n_1, n_2 \geq 1$, we have

$$-\Delta(u_{n_1} - u_{n_2}) + g(u_{n_1}) - g(u_{n_2}) = f_{n_1} - f_{n_2} \quad \text{in } (C_0^2)^*. \tag{B.43}$$

It follows from Proposition B.3 that

$$\int_\Omega \left| g(u_{n_1}) - g(u_{n_2}) \right| \leq \int_\Omega |f_{n_1} - f_{n_2}|.$$

Thus, $(g(u_n))$ is a Cauchy sequence. Returning to (B.43), we conclude from (B.3) that (u_n) is Cauchy in $L^1(\Omega)$. Passing to the limit as $n \to \infty$, we find a solution $u \in L^1(\Omega)$ of (B.42). By Proposition B.3, the solution is unique.

Corollary B.3. *Let $\mu \in \mathcal{M}(\Omega)$. If μ is diffuse, then* (B.17) *admits a unique solution $u \in L^1(\Omega)$ with $g(u) \in L^1(\Omega)$.*

Proof. It suffices to observe that, by a result of Boccardo-Gallouët-Orsina [BGO1], every diffuse measure μ belongs to $L^1 + H^{-1}$.

Concerning the existence of solutions for *every* measure $\mu \in \mathcal{M}(\Omega)$, we have

Theorem B.5 (Bénilan-Brezis [BB]). *Assume $N \geq 2$ and*

$$|g(t)| \leq C(|t|^p + 1) \quad \forall t \in \mathbb{R}, \tag{B.44}$$

for some $p < \frac{N}{N-2}$. Then, for every $\mu \in \mathcal{M}(\Omega)$, problem (B.17) *has a unique solution $u \in L^1(\Omega)$.*

Assumption (B.44) is optimal, in the sense that if $N \geq 3$, $g(t) = |t|^{p-1}t$ and $p \geq \frac{N}{N-2}$, then (B.17) has no weak solution for $\mu = \delta_a$, where $a \in \Omega$:

Theorem B.6 (Bénilan-Brezis [BB]; Brezis-Véron [BV]). *Assume $N \geq 3$. If $p \geq \frac{N}{N-2}$, then, for any $a \in \Omega$, the problem*

$$\begin{cases} -\Delta u + |u|^{p-1}u = \delta_a & \text{in } \Omega, \\ \hfill u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution $u \in L^p(\Omega)$.

Appendix C: Correspondence between $\big[C_0(\overline{\Omega})\big]^*$ and $\big[C(\overline{\Omega})\big]^*$.

In this section we establish the following

Proposition C.1. *Given $\mu \in \big[C_0(\overline{\Omega})\big]^*$, there exists a unique $\tilde{\mu} \in \big[C(\overline{\Omega})\big]^*$ such that*

$$\tilde{\mu} = \mu \quad \text{on } C_0(\overline{\Omega}) \quad \text{and} \quad |\tilde{\mu}|(\partial\Omega) = 0. \tag{C.1}$$

In addition, the map $\mu \mapsto \tilde{\mu}$ is a linear isometry.

In order to prove Proposition C.1, we shall need the following

Lemma C.1. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\zeta \in C_0(\overline{\Omega})$, $|\zeta| \leq 1$ in $\overline{\Omega}$, and* $\operatorname{supp} \zeta \subset \overline{\Omega} \backslash \Omega_\delta$, *then*

$$|\langle \mu, \zeta \rangle| \leq \varepsilon.$$

Here, we denote by Ω_δ the set $\big\{x \in \Omega\, ;\, d(x, \partial\Omega) > \delta\big\}$.

Proof. We argue by contradiction. Assume there exist $\varepsilon_0 > 0$ and a sequence $(\zeta_n) \subset C_0(\overline{\Omega})$ such that $|\zeta_n| \leq 1$ in $\overline{\Omega}$, $\operatorname{supp} \zeta_n \subset \overline{\Omega} \backslash \Omega_{1/n}$, and

$$\langle \mu, \zeta_n \rangle > \varepsilon_0 \quad \forall n \geq 1.$$

Without loss of generality, we may assume that each ζ_n has compact support in Ω (this is always possible, by density of $C_c^\infty(\Omega)$ in $C_0(\overline{\Omega})$). In particular, we can extract a subsequence (ζ_{n_j}) such that $\operatorname{supp} \zeta_{n_j}$ are all disjoint. For any $k \geq 1$, let $\tilde{\zeta}_k = \sum_{j=1}^k \zeta_{n_j}$. By construction,

$$\|\tilde{\zeta}_k\|_{L^\infty} \leq 1 \quad \text{and} \quad \operatorname{supp} \tilde{\zeta}_k \subset \Omega.$$

Moreover,

$$k\varepsilon_0 < \langle \mu, \tilde{\zeta}_k \rangle \leq \|\mu\|_{\mathcal{M}}.$$

Since $k \geq 1$ was arbitrary, this gives a contradiction.

Proof of Proposition C.1. Let $\mu \in \big[C_0(\overline{\Omega})\big]^*$. Given $\zeta \in C(\overline{\Omega})$, let (ζ_n) be any sequence in $C_0(\overline{\Omega})$ such that

$$\|\zeta_n\|_{L^\infty} \leq C \quad \text{and} \quad \zeta_n \to \zeta \quad \text{in } L^\infty_{\mathrm{loc}}(\Omega).$$

It easily follows from Lemma C.1 that $(\langle \mu, \zeta_n \rangle)$ is Cauchy in $\mathbb{R}$. In particular, the limit $\lim\limits_{n\to\infty} \langle \mu, \zeta_n \rangle$ exists and is independent of the sequence (ζ_n). Set

$$\langle \tilde{\mu}, \zeta \rangle = \lim_{n\to\infty} \langle \mu, \zeta_n \rangle.$$

Clearly, $\tilde{\mu}$ is a continuous linear functional on $C(\overline{\Omega})$ and

$$\langle \tilde{\mu}, \zeta \rangle = \langle \mu, \zeta \rangle \quad \forall \zeta \in C_0(\overline{\Omega}).$$

In addition, Lemma C.1 implies that $|\tilde{\mu}|(\partial\Omega) = 0$; in particular, $\|\tilde{\mu}\|_{C^*} = \|\mu\|_{(C_0)^*}$. The uniqueness of $\tilde{\mu}$ follows immediately from (C.1).

Appendix D: A new decomposition for diffuse measures.

The goal of this section is to establish Theorem 3. Let G denote the Green function of the Laplacian in Ω. Given $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, set

$$G(\mu)(x) = \int_\Omega G(x, y)\, d\mu(y).$$

Note that $G(\mu)$ is well-defined for every $x \in \Omega$, possibly taking values $+\infty$.

We first present some well-known results in Potential Theory:

Lemma D.1. *Let $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, be such that $G(\mu) < \infty$ everywhere in Ω. Given $\varepsilon > 0$, there exists $L \subset \Omega$ compact such that*

$$\mu(\Omega \backslash L) < \varepsilon \quad \textit{and} \quad G(\mu\lfloor_L) \in C_0(\overline{\Omega}). \tag{D.1}$$

Proof. If μ has compact support in Ω, then Lemma D.1 is precisely Theorem 6.21 in [H]. For an arbitrary $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, such that $G(\mu) < \infty$ in Ω, we proceed as follows. By inner regularity of μ, there exists $K \subset \Omega$ compact such that $\mu(\Omega \backslash K) < \frac{\varepsilon}{2}$. Since $G(\mu\lfloor_K) \leq G(\mu)$, the function $G(\mu\lfloor_K)$ is also finite everywhere in Ω. Then, by Theorem 6.21 in [H], there exists $L \subset \Omega$ compact such that

$$\mu\lfloor_K(\Omega \backslash L) < \frac{\varepsilon}{2} \quad \text{and} \quad G(\mu\lfloor_{K \cap L}) \in C_0(\overline{\Omega}).$$

We conclude that (D.1) holds with L replaced by $K \cap L$.

As a consequence of Lemma D.1, we have

Proposition D.1. *Let $u \in W_0^{1,1}(\Omega)$ be such that Δu is a diffuse measure in Ω. Then, there exists a sequence $(u_n) \subset C_0(\overline{\Omega})$ such that $\Delta u_n \in \mathcal{M}(\Omega)$, $\forall n \geq 1$,*

$$u = \sum_{n=1}^{\infty} u_n \quad \textit{a.e. in } \Omega \quad \textit{and} \quad \|\Delta u\|_{\mathcal{M}} = \sum_{n=1}^{\infty} \|\Delta u_n\|_{\mathcal{M}}. \tag{D.2}$$

Proof. We shall split the proof of Proposition D.1 into three steps.

Step 1. Let $\mu \geq 0$ be a measure such that $G(\mu) < \infty$ everywhere in Ω. Then, there exist disjoint Borel sets $A_n \subset \Omega$ such that

$$\mu\Big(\Omega \backslash \bigcup_{n=1}^{\infty} A_n\Big) = 0 \quad \text{and} \quad G(\mu\lfloor_{A_n}) \in C_0(\overline{\Omega}) \quad \forall n \geq 1. \tag{D.3}$$

This result easily follows from Lemma D.1 by an induction argument.

Step 2. Let $\mu \geq 0$ be a diffuse measure in Ω. Then, there exist disjoint Borel sets $A_n \subset \Omega$ such that

$$\mu\Big(\Omega \backslash \bigcup_{n=1}^{\infty} A_n\Big) = 0 \quad \text{and} \quad G(\mu\lfloor_{A_n}) \in C_0(\overline{\Omega}) \quad \forall n \geq 1. \tag{D.4}$$

For each $k \geq 1$, let

$$E_k = \big\{x \in \Omega \; ; \; G(\mu)(x) \leq k\big\}.$$

Since $G(\mu)$ is lower semicontinuous (by Fatou), E_k is closed in Ω. Clearly, we have $G(\mu\lfloor_{E_k}) \leq k$ in E_k and $G(\mu\lfloor_{E_k})$ is harmonic in $\Omega\backslash E_k$. Therefore, by the maximum principle, $G(\mu\lfloor_{E_k}) \leq k$ everywhere in Ω.
Applying the previous step to the measures $\mu\lfloor_{E_k\backslash E_{k-1}}$, one can find disjoint Borel sets $A_n \subset \Omega$ such that

$$\mu\Big(F\backslash \bigcup_{n=1}^{\infty} A_n\Big) = 0 \quad \text{and} \quad G(\mu\lfloor_{A_n}) \in C_0(\overline{\Omega}) \quad \forall n \geq 1,$$

where

$$F = \big\{x \in \Omega\ ;\ G(\mu)(x) < \infty\big\}.$$

Since μ is diffuse and $\Omega\backslash F$ has zero capacity (see e.g. [H, Theorem 7.33]), we have $\mu(\Omega\backslash F) = 0$. Thus,

$$\mu\Big(\Omega\backslash \bigcup_{n=1}^{\infty} A_n\Big) = 0,$$

from which the result follows.

Step 3. Proof of Proposition D.1 completed.

Set $\mu = -\Delta u$. Applying Step 2 to μ^+, one can find disjoint Borel sets (A_n) such that

$$\mu^+\Big(\Omega\backslash \bigcup_{n=1}^{\infty} A_n\Big) = 0 \quad \text{and} \quad G(\mu^+\lfloor_{A_n}) \in C_0(\overline{\Omega}) \quad \forall n \geq 1.$$

Similarly, there exist disjoint Borel sets (B_n) such that

$$\mu^-\Big(\Omega\backslash \bigcup_{n=1}^{\infty} B_n\Big) = 0 \quad \text{and} \quad G(\mu^-\lfloor_{B_n}) \in C_0(\overline{\Omega}) \quad \forall n \geq 1.$$

Since

$$\mu = \mu^+ - \mu^- = \sum_{n=1}^{\infty} \mu^+\lfloor_{A_n} - \sum_{n=1}^{\infty} \mu^-\lfloor_{B_n},$$

we have

$$u = \sum_{n=1}^{\infty} G(\mu^+\lfloor_{A_n}) - \sum_{n=1}^{\infty} G(\mu^-\lfloor_{B_n}) \quad \text{a.e.}$$

and

$$\|\Delta u\|_{\mathcal{M}} = \sum_{n=1}^{\infty} \big\|\mu^+\lfloor_{A_n}\big\|_{\mathcal{M}} + \sum_{n=1}^{\infty} \big\|\mu^-\lfloor_{B_n}\big\|_{\mathcal{M}}.$$

This concludes the proof of the proposition.

We can now present the

Proof of Theorem 3. Let $u \in W_0^{1,1}(\Omega)$ be the unique solution of

$$-\Delta u = \mu \quad \text{in } (C_0^2)^*.$$

Let $(u_n) \subset C_0(\overline{\Omega})$ be the sequence given by Proposition D.1. For $\delta > 0$ fixed, take $w_n \in C_0^2(\overline{\Omega})$ such that

$$\|u_n - w_n\|_{L^\infty} \leq \frac{\delta}{2^n} \quad \text{and} \quad \|\Delta w_n\|_{L^1} \leq \|\Delta u_n\|_{\mathcal{M}}.$$

Let

$$v = \sum_{n=1}^{\infty} (u_n - w_n) \quad \text{and} \quad f = -\sum_{n=1}^{\infty} \Delta w_n.$$

Since

$$\|v\|_{L^\infty} \leq \sum_{n=1}^{\infty} \|u_n - w_n\|_{L^\infty} \leq \delta, \tag{D.5}$$

we have $v \in C_0(\overline{\Omega})$ and $\|v\|_{L^\infty} \leq \delta$. Moreover,

$$\|f\|_{L^1} \leq \sum_{n=1}^{\infty} \|\Delta w_n\|_{L^1} \leq \sum_{n=1}^{\infty} \|\Delta u_n\|_{\mathcal{M}} = \|\mu\|_{\mathcal{M}} \tag{D.6}$$

implies $f \in L^1(\Omega)$. Finally, by construction, we have

$$\mu = f - \Delta v \quad \text{in } (C_0^2)^*. \tag{D.7}$$

In particular, $\Delta v = f - \mu$ is a measure and $\|\Delta v\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}$. Thus,

$$\|\nabla v\|_{L^2}^2 \leq \|v\|_{L^\infty} \|\Delta v\|_{\mathcal{M}} \leq 2\delta \|\mu\|_{\mathcal{M}}. \tag{D.8}$$

Since $v \in C_0(\overline{\Omega}) \cap H_0^1$, (0.21) immediately follows from (D.7). Moreover, replacing δ by $\frac{\delta}{2}\|\mu\|_{\mathcal{M}}$ in (D.5) and (D.8), we conclude that (0.22) holds. The proof of Theorem 3 is complete.

Note that our construction of $f \in L^1$ and $v \in L^\infty$ satisfying (0.21) is not linear with respect to μ. Here is a natural question:

Open problem 7. Can one find a bounded linear operator

$$T \,:\, \mu \in \mathcal{M}_{\mathrm{d}}(\Omega) \longmapsto (f, v) \in L^1 \times L^\infty$$

such that (0.21) and (0.22) hold?

After receiving a preprint of our work, A. Ancona [A2] has provided a negative answer to the question above.

Appendix E: Equivalence between cap_{H^1} and $\mathrm{cap}_{\Delta,1}$.

Given a compact set $K \subset \Omega$, let $\mathrm{cap}_{\Delta,1}(K)$ denote the capacity associated to the Laplacian. More precisely,

$$\mathrm{cap}_{\Delta,1}(K) = \inf\left\{ \int_\Omega |\Delta\varphi| \;;\; \varphi \in C_{\mathrm{c}}^\infty(\Omega),\ \varphi \geq 1 \text{ in some neighborhood of } K \right\}.$$

In order to avoid confusion, throughout this section we shall denote by cap_{H^1} the Newtonian capacity with respect to Ω (which we simply denote cap everywhere else in this paper).

The main result in this appendix is the following

Theorem E.1. *For every compact set $K \subset \Omega$, we have*

$$\mathrm{cap}_{\Delta,1}(K) = 2\,\mathrm{cap}_{H^1}(K). \tag{E.1}$$

Remark E.1. In an earlier version of this work, we had only established the equivalence between cap_{H^1} and $\mathrm{cap}_{\Delta,1}$. The exact formula (E.1) has been suggested to us by A. Ancona.

We first prove the following

Lemma E.1. *Let $K \subset \Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C_{\mathrm{c}}^\infty(\Omega)$ such that $0 \leq \psi \leq 1$ in Ω, $\psi = 1$ in some neighborhood of K, and*

$$\int_\Omega |\Delta\psi| \leq 2\,\mathrm{cap}_{H^1}(K) + \varepsilon. \tag{E.2}$$

Proof. Let $\omega \subset\subset \Omega$ be an open set such that $K \subset \omega$ and

$$\mathrm{cap}_{H^1}(\overline{\omega}) \leq \mathrm{cap}_{H^1}(K) + \frac{\varepsilon}{4}.$$

Let u denote the capacitary potential of $\overline{\omega}$. More precisely, let $u \in H_0^1(\Omega)$ be such that $u = 1$ in $\overline{\omega}$ and

$$\int_\Omega |\nabla u|^2 = \operatorname{cap}_{H^1}(\overline{\omega}).$$

Note that u is superharmonic in Ω and harmonic in $\Omega \backslash \overline{\omega}$. In particular, $0 \leq u \leq 1$. Since $\operatorname{supp} \Delta u \subset [u = 1]$, u is continuous (see [H, Theorem 6.20]) and

$$\|\Delta u\|_{\mathcal{M}} = -\int_\Omega \Delta u = -\int_\Omega u \Delta u = \int_\Omega |\nabla u|^2 = \operatorname{cap}_{H^1}(\overline{\omega}).$$

Given $\delta > 0$ small, set

$$v = \frac{(u - \delta)^+}{1 - \delta}.$$

Since v has compact support in Ω, we have

$$\int_\Omega \Delta v = 0. \tag{E.3}$$

Moreover, Δv is a diffuse measure (note that $v \in H_0^1(\Omega)$) and

$$\operatorname{supp} \Delta v \subset [v = 0] \cup [v = 1]. \tag{E.4}$$

Thus, by Corollary 1.3 in [BP2], we have

$$\Delta v \geq 0 \quad \text{in } [v = 0] \quad \text{and} \quad \Delta v \leq 0 \quad \text{in } [v = 1]. \tag{E.5}$$

It then follows from (E.3)–(E.5) that

$$\|\Delta v\|_{\mathcal{M}} = 2 \int\limits_{[v=1]} |\Delta v|.$$

Since $\Delta v = \frac{1}{1-\delta} \Delta u$ in $[v = 1]$, we conclude that

$$\|\Delta v\|_{\mathcal{M}} \leq \frac{2}{1 - \delta} \|\Delta u\|_{\mathcal{M}}.$$

Using the same notation as in Section 4, we now take $n \geq 1$ sufficiently large so that the function $\psi = \rho_n * v$ has compact support in Ω and $\psi = 1$ in some neighborhood of K. We claim that ψ satisfies all the required properties. In fact, since $0 \leq \psi \leq 1$ in Ω, we only have to show that (E.2) holds. Note that

$$\int_\Omega |\Delta \psi| \leq \|\Delta v\|_{\mathcal{M}} \leq \frac{2}{1 - \delta} \|\Delta u\|_{\mathcal{M}} = \frac{2}{1 - \delta} \operatorname{cap}_{H^1}(\overline{\omega}).$$

Choosing $\delta > 0$ so that

$$\frac{\delta}{1-\delta} \operatorname{cap}_{H^1}(\overline{\omega}) < \frac{\varepsilon}{4},$$

we have

$$\int_\Omega |\Delta \psi| \le 2 \left(1 + \frac{\delta}{1-\delta}\right) \operatorname{cap}_{H^1}(\overline{\omega}) \le 2 \operatorname{cap}_{H^1}(K) + \varepsilon,$$

which is precisely (E.2).

We now present the

Proof of Theorem E.1. In view of Lemma E.1, it suffices to show that

$$\operatorname{cap}_{H^1}(K) \le \frac{1}{2} \operatorname{cap}_{\Delta,1}(K). \tag{E.6}$$

Let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi \ge 1$ in some neighborhood of K. Set $\tilde{\varphi} = \min\{1, \varphi^+\}$. For $n \ge 1$ sufficiently large, the function $\tilde{\varphi}_n = \rho_n * \tilde{\varphi}$ belongs to $C_c^\infty(\Omega)$ and $\tilde{\varphi}_n = 1$ in some neighborhood of K. We then have

$$\operatorname{cap}_{H^1}(K) \le \int_\Omega |\nabla \tilde{\varphi}_n|^2 \le \int_\Omega |\nabla \tilde{\varphi}|^2 = \int_\Omega \nabla \tilde{\varphi} \cdot \nabla \varphi = -\int_\Omega \tilde{\varphi} \Delta \varphi.$$

Recall that φ has compact support in Ω and $0 \le \tilde{\varphi} \le 1$. Thus, $\int_\Omega \Delta \varphi = 0$ and we have

$$\operatorname{cap}_{H^1}(K) \le -\int_\Omega \Big(\tilde{\varphi} - \frac{1}{2}\Big) \Delta \varphi \le \frac{1}{2} \int_\Omega |\Delta \varphi|.$$

Since φ was arbitrary, we conclude that (E.6) holds. This establishes Theorem E.1.

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References

[A1] A. Ancona, *Une propriété d'invariance des ensembles absorbants par perturbation d'un opérateur elliptique*, Comm. Partial Differential Equations **4** (1979), 321–337.

[A2] ______, *Sur une question de H. Brezis, M. Marcus et A.C. Ponce*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

[BP] P. Baras and M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), 185–206.

[BLOP] D. Bartolucci, F. Leoni, L. Orsina and A.C. Ponce, *Semilinear equations with exponential nonlinearity and measure data*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

[Ba] J.R. Baxter, *Inequalities for potentials of particle systems*, Illinois J. Math. **24** (1980), 645–652.

[BM] G. Barles and F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions*, Arch. Rational Mech. Anal. **133** (1995), 77–101.

[BB] Ph. Bénilan and H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation*, J. Evol. Equ. **3** (2004), 673–770. Dedicated to Ph. Bénilan.

[BGO1] L. Boccardo, T. Gallouët and L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 539–551.

[BGO2] ______, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. **73** (1997), 203–223.

[B1] H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation.* In: Contemporary developments in continuum mechanics and partial differential equations (G.M. de la Penha and L.A. Medeiros, eds.) Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, North Holland, Amsterdam, 1978, pp. 74–80.

[B2] ______, *Some variational problems of the Thomas-Fermi type.* In: Variational inequalities and complementarity problems (R.W. Cottle, F. Giannessi and J.-L. Lions, eds.) Proc. Internat. School, Erice, 1978, Wiley, Chichester, 1980, pp. 53–73.

[B3] ______, *Problèmes elliptiques et paraboliques non linéaires avec données mesures.* Goulaouic-Meyer-Schwartz Seminar, 1981/1982, École Polytech., Palaiseau, 1982, pp. X.1–X.12.

[B4] ______, *Nonlinear elliptic equations involving measures.* In: Contributions to nonlinear partial differential equations (C. Bardos, A. Damlamian, J.I. Diaz and J. Hernandez, eds.) Madrid, 1981, Pitman, Boston, MA, 1983, pp. 82–89.

[B5] ______, *Semilinear equations in $\mathbb{R}^N$ without condition at infinity*, Appl. Math. Optim. **12** (1984), 271–282.

[BBr] H. Brezis and F.E. Browder, *Strongly nonlinear elliptic boundary value problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **5** (1978), 587–603.

[BCMR] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), 73–90.

[BMP] H. Brezis, M. Marcus and A.C. Ponce, *A new concept of reduced measure for nonlinear elliptic equations*, C. R. Acad. Sci. Paris, Ser. I **339** (2004), 169–174.

[BP1] H. Brezis and A.C. Ponce, *Remarks on the strong maximum principle*, Differential Integral Equations **16** (2003), 1–12.

[BP2] ______, *Kato's inequality when Δu is a measure*, C. R. Acad. Sci. Paris, Ser. I **338** (2004), 599–604.

[BP3] ______, *Reduced measures on the boundary*, to appear in J. Funct. Anal.

[BP4] ———, *Obstacle problems involving measures*, to appear.

[BSe] H. Brezis and S. Serfaty, *A variational formulation for the two-sided obstacle problem with measure data*, Commun. Contemp. Math. **4** (2002), 357–374.

[BS] H. Brezis and W.A. Strauss, *Semilinear second-order elliptic equations in L^1*, J. Math. Soc. Japan **25** (1973), 565–590.

[BV] H. Brezis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), 1–6.

[DD] P. Dall'Aglio and G. Dal Maso, *Some properties of the solutions of obstacle problems with measure data*, Ricerche Mat. **48** (1999), suppl., 99–116. Papers in memory of Ennio De Giorgi.

[DM] C. Dellacherie and P.-A. Meyer, *Probabilités et potentiel,* Chapitres I à IV, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. XV, Actualités Scientifiques et Industrielles, No. 1372, Hermann, Paris, 1975.

[DS] N. Dunford and J.T. Schwartz, *Linear operators. Part I*, Wiley, New York, 1958.

[DP] L. Dupaigne and A.C. Ponce, *Singularities of positive supersolutions in elliptic PDEs*, Selecta Math. (N.S.) **10** (2004), 341–358.

[DVP] C. De La Vallée-Poussin, *Sur l'intégrale de Lebesgue*, Trans. Amer. Math. Soc. **16** (1915), 435–501.

[DPP] L. Dupaigne, A.C. Ponce and A. Porretta, *Elliptic equations with vertical asymptotes in the nonlinear term*, to appear.

[D1] E.B. Dynkin, *Diffusions, superdiffusions and partial differential equations*, American Mathematical Society, Providence, RI, 2002.

[D2] ———, *Superdiffusions and positive solutions of nonlinear partial differential equations*, American Mathematical Society, Providence, RI, 2004.

[FTS] M. Fukushima, K. Sato and S. Taniguchi, *On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures*, Osaka J. Math. **28** (1991), 517–535.

[GM] T. Gallouët and J.-M. Morel, *Resolution of a semilinear equation in L^1*, Proc. Roy. Soc. Edinburgh Sect. A **96** (1984), 275–288; Corrigenda: Proc. Roy. Soc. Edinburgh Sect. A **99** (1985), 399.

[GV] A. Gmira and L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64** (1991), 271–324.

[GV] M. Grillot and L. Véron, *Boundary trace of the solutions of the prescribed Gaussian curvature equation.*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 527–560.

[GRe] M. Grun-Rehomme, *Caractérisation du sous-différential d'intégrandes convexes dans les espaces de Sobolev*, J. Math. Pures Appl. **56** (1977), 149–156.

[H] L.L. Helms, *Introduction to potential theory*, Wiley-Interscience, New York, 1969.

[K] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148 (1973).

[LG1] J.-F. Le Gall, *The Brownian snake and solutions of $\Delta u = u^2$ in a domain*, Probab. Theory Related Fields **102** (1995), 393–432.

[LG2] ———, *A probabilistic Poisson representation for positive solutions of $\Delta u = u^2$ in a planar domain*, Comm. Pure Appl. Math. **50** (1997), 69–103.

[MV1] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rational Mech. Anal. **144** (1998), 201–231.

[MV2] ———, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. **77** (1998), 481–524.

[MV3] ———, *Removable singularities and boundary traces*, J. Math. Pures Appl. **80** (2001), 879–900.

[MV4] ———, *Capacitary estimates of solutions of a class of nonlinear elliptic equations*, C. R. Acad. Sci. Paris, Ser. I **336** (2003), 913–918.

[MV5] ———, *Nonlinear capacities associated to semilinear elliptic equations*, in preparation.

[P] A.C. Ponce, *How to construct good measures.* To appear in Elliptic and parabolic problems (C. Bandle, H. Berestycki, B. Brighi, A. Brillard, M. Chipot, J.-M. Coron, C. Sbordone, I. Shafrir, V. Valente and G. Vergara-Caffarelli, eds.) Gaeta, 2004, Birkhäuser. A special tribute to the work of Haïm Brezis.

[Po] A. Porretta, *Absorption effects for some elliptic equations with singularities*, to appear in Bull. Un. Mat. Ital.

[S] G. Stampacchia, *Équations elliptiques du second ordre à coefficients discontinus*, Les Presses de l'Université de Montréal, Montréal, 1966.

[Va] J.L. Vázquez, *On a semilinear equation in $\mathbb{R}^2$ involving bounded measures*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), 181–202.

(1) Rutgers University
Dept. of Mathematics, Hill Center, Busch Campus
110 Frelinghuysen Rd., Piscataway, NJ 08854, USA
E-mail address: `brezis@math.rutgers.edu`

(2) Laboratoire J.-L. Lions
Université P. et M. Curie, B.C. 187
4 Pl. Jussieu
75252 Paris Cedex 05, France
E-mail address: `brezis@ccr.jussieu.fr`

(3) Technion
Dept. of Mathematics
Haifa 32000, Israel
E-mail address: `marcusm@tx.technion.ac.il`

(4) Institute for Advanced Study
Princeton, NJ 08540, USA
E-mail address: `augponce@math.ias.edu`