How to construct good measures[1]

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Dedicated to H. Brezis in the occasion of his 60th birthday

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Abstract. Given any continuous nondecreasing function $g : \mathbb{R} \to \mathbb{R}$, with $g(t) = 0$, $\forall t \leq 0$, we show that there always exists some positive measure μ, concentrated on a set of zero Newtonian capacity, for which the problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

admits a solution. This provides an affirmative answer to Open problem 2 raised by Brezis-Marcus-Ponce [3]. When $N \geq 3$ and $g(t) = \mathrm{e}^t - 1$, $\forall t \geq 0$, Bartolucci-Leoni-Orsina-Ponce [1] proved that any measure $\mu \leq 4\pi\mathcal{H}^{N-2}$ is good for problem (0.1). We present examples of other good measures which are not $\leq 4\pi\mathcal{H}^{N-2}$.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $g(t) = 0$, $\forall t \leq 0$. Given a bounded measure $\mu \in \mathcal{M}(\Omega)$, then u is a solution of

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega)$, and

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ on } \partial\Omega.$$

We say that μ is a *good measure* (relative to g) if (1.1) has a solution u. We observe that u, whenever it exists, is unique. The study of problem (1.1), when $\mu \in L^1(\Omega)$, was initiated by Brezis-Strauss [5]. They established that every measure in $L^1(\Omega)$ is good. Later, Bénilan-Brezis [2] (see also Brezis-Véron [6])

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proved that (1.1) need not have a solution for a given measure μ. In fact, if $N \geq 3$ and $g(t) = t^p$, $\forall t \geq 0$, for some $p \geq \frac{N}{N-2}$, then there exists no u satisfying (1.1) for $\mu = \delta_a$, $a \in \Omega$.

Let $\mathcal{G}(g)$ denote the set of good measures associated to g. One can show (see [3]) that $\mathcal{G}(g)$ is convex and closed with respect to the strong topology in $\mathcal{M}(\Omega)$.

A measure μ is *diffuse* if $\mu(A) = 0$ for every Borel set $A \subset \Omega$ such that $\operatorname{cap}(A) = 0$, where "cap" denotes the Newtonian (H^1) capacity. If $\mu \in \mathcal{M}(\Omega)$ and μ^+ is diffuse, then μ is good for every nonlinearity g (see [3, Corollary 3]). The converse is also true. Namely, if μ is good for every g, then μ^+ is diffuse (see [3, Theorem 5]). We can summarize this as

$$\Big\{\mu \in \mathcal{M}(\Omega) : \mu^+ \text{ is a diffuse measure}\Big\} = \bigcap_g \mathcal{G}(g),$$

where the intersection is taken over all continuous nondecreasing functions $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = 0$, $\forall t \leq 0$.

One of our main results is the following

Theorem 1 *Given any g, we have*

$$\Big\{\mu \in \mathcal{M}(\Omega) : \mu^+ \textit{ is a diffuse measure}\Big\} \subsetneqq \mathcal{G}(g).$$

In other words, for any fixed g, there exists a measure $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, such that $\mu \in \mathcal{G}(g)$, but μ is not *diffuse.*

Theorem 1 gives a positive answer to Open problem 2 in [3]. As we shall see below, the proof of Theorem 1 is constructive. In fact, it gives a recipe for explicitly obtaining the measure μ. Of course, such μ will heavily depend on the function g.

In dimension $N \geq 3$, Theorem 1 can be improved. Recall that any Borel set $A \subset \Omega$ such that $\mathcal{H}^{N-2}(A) < \infty$ has zero capacity (but the converse is false; see [7]). When $N \geq 3$, it is always possible to find good measures μ of the form $\mu = \alpha \mathcal{H}^{N-2}\lfloor_K$ for some compact set $K \subset \Omega$ and $\alpha > 0$. More precisely, we have

Theorem 2 *Assume $N \geq 3$. Given any g, there exists a compact set $K \subset \Omega$, $\mathcal{H}^{N-2}(K) \in (0, \infty)$, such that $\mu = \alpha \mathcal{H}^{N-2}\lfloor_K$ is good (relative to g) for every $\alpha > 0$.*

Theorem 2 is no longer true in dimension $N = 2$. In fact, problem (1.1) has no solution when $g(t) = \mathrm{e}^t - 1$, $\forall t \geq 0$, and $\mu = \alpha\delta_a$, $a \in \Omega$, for any $\alpha > 4\pi$ (see Vázquez [12]).

One can also construct good measures $\mu \geq 0$ concentrated on a set of zero $\mathcal{H}^{N-2}$-measure. In fact,

Theorem 3 *Assume $N \geq 3$. For any g, there exists a good measure $\mu \geq 0$ such that $\mathcal{H}^{N-2}(\operatorname{supp} \mu) = 0$.*

When $N \geq 3$ and $g(t) = \mathrm{e}^t - 1$, $\forall t \geq 0$, it has been established in [1] that if $\mu \leq 4\pi\mathcal{H}^{N-2}$, then μ is good. According to Theorems 2 and 3 above, there are other good measures which are *not* $\leq 4\pi\mathcal{H}^{N-2}$. The existence of such measures was suggested by L. Véron in a personal communication.

The construction presented here has been applied in the study of other related problems; see [4] and [8]. An alternative approach for obtaining good measures which are not diffuse might be found in some recent work of Marcus-Véron [10].

This paper is organized as follows. In Section 2, we define a Cantor-type set F associated to a subsequence (ℓ_{k_j}); as we shall see later on, the proofs of Theorems 1–3 rely on suitable choices of (ℓ_k) and (k_j). We then introduce a positive measure μ_F supported on F. In Section 3, we estimate the potential generated by μ_F in terms of (ℓ_{k_j}). In Section 4, we present the proofs of Theorems 2 and 3; as a corollary, we obtain Theorem 1 when $N \geq 3$. Finally, in Section 5, we prove Theorem 1 in the case $N = 2$.

2 Construction of the Cantor set F associated to the subsequence (ℓ_{k_j})

We shall assume for simplicity that $\Omega = Q_1$, the unit cube centered at 0. One of the main ingredients in the proofs of Theorems 1–3 will be the construction of a (generalized) Cantor set $F \subset \Omega$; see e.g. [11]. We begin by describing the building blocks used in the definition of F.

Let $n \geq 1$ be an integer and let $0 < s \ll t$. We shall associate to the triple (s, t, n) a compact set $E(s, t, n) \subset [-\frac{t}{2}, \frac{t}{2}]^N$ in the following way. Let

$$\alpha = \frac{t - ns}{n - 1}. \tag{2.1}$$

For $j = 1, \ldots, n$, set

$$a_j = (j-1)(s+\alpha) - \frac{t}{2} \quad \text{and} \quad b_j = a_j + s.$$

In particular, $a_1 = -\frac{t}{2}$ and $b_n = \frac{t}{2}$. We then define

$$E(s, t, n) = \bigcup_{1 \leq i_1, \ldots, i_N \leq n} [a_{i_1}, b_{i_1}] \times \cdots \times [a_{i_N}, b_{i_N}].$$

Thus, the set $E(s, t, n)$ is the union of n^N cubes of side s, uniformly distributed in $[-\frac{t}{2}, \frac{t}{2}]^N$. The distance between two components of $E(s, t, n)$ is $\geq \alpha$.

We now turn to the construction of F.

Let (ℓ_k) be a decreasing sequence of positive numbers such that

$$\ell_1 \leq \frac{1}{4} \quad \text{and} \quad \ell_{k+1} \leq \theta\, \ell_k \quad \forall k \geq 1, \tag{2.2}$$

for some $\theta \in (0, \frac{1}{2})$. The Cantor set F associated to the subsequence (ℓ_{k_j}) is defined by induction as follows.

Let $F_0 = Q_1$, $k_0 = 0$ and $\ell_0 = 1$. Let F_j be the set obtained after the j-th step; F_j is the disjoint union of 2^{Nk_j} cubes Q_i of side ℓ_{k_j}. Let $x_1, \ldots, x_{2^{Nk_j}}$ denote the centers of each component of F_j (although it is not indicated, such points do depend on j). We then set

$$F_{j+1} = \bigcup_{i=1}^{2^{Nk_j}} E\Big(\ell_{k_{j+1}}, \gamma\ell_{k_j}, 2^{(k_{j+1}-k_j)}\Big) + x_i, \tag{2.3}$$

where $\gamma = \frac{1}{2} + \theta \in (\frac{1}{2}, 1)$. In particular, F_{j+1} is the union of $2^{Nk_{j+1}}$ disjoint cubes of side $\ell_{k_{j+1}}$. Moreover, since we are taking $t = \gamma\ell_{k_j}$, we have

$$d(F_{j+1}, \partial F_j) = \frac{1-\gamma}{2}\,\ell_{k_j} = \frac{1-2\theta}{4}\,\ell_{k_j}. \tag{2.4}$$

We also point out that the distance between any two components of F_{j+1} inside the cube $[-\gamma\ell_{k_j}, \gamma\ell_{k_j}]^N + x_i$ is $\geq \alpha$, where α is given by (2.1). Since (2.2) holds with $\theta < \frac{1}{2}$, we have

$$\alpha \sim \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}}.$$

We finally set

$$F = \bigcap_{j=0}^{\infty} F_j.$$

We would like to emphasize the main feature in the construction of F. In order to obtain a standard Cantor set, inside each component Q_i of F_j one would take 2^N small cubes. In our case, we select $2^{N(k_{j+1}-k_j)}$ small cubes inside Q_i. This possibility of choosing many more cubes turns out to be crucial in the proofs of some of our main results.

3 Potential generated by the uniform measure μ_F concentrated on F

In this section, we present some basic estimates which will be used throughout this paper.

For each $j \geq 1$, let $\mu_j = \frac{1}{|F_{j+1}|}\chi_{F_{j+1}}$, where F_{j+1} is given by (2.3). The *uniform measure concentrated on* F, μ_F, is the weak* limit of (μ_j) in $\mathcal{M}(\Omega)$ as $j \to \infty$. In particular, $\mu_F \geq 0$ and $\mu_F(\Omega) = 1$. A key property satisfied by μ_F is given by the next

Lemma 1 *For every $x \in F_{j+1}$, $j \geq 0$, we have*

$$\mu_F\big(B_r(x)\big) \sim \begin{cases} \frac{1}{2^{Nk_{j+1}}} & \text{if } \ell_{k_{j+1}} \lesssim r \lesssim \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}}, \\ \frac{1}{2^{Nk_j}} \left(\frac{r}{\ell_{k_j}}\right)^N & \text{if } \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}} \lesssim r \lesssim \ell_{k_j}. \end{cases} \tag{3.1}$$

Here, we implicitly assume that $k_0 = 0$. We say that $a \lesssim b$ if there exists $C > 0$, depending on N and θ, such that $a \leq C\, b$. By $a \sim b$, we mean that $a \lesssim b$ and $b \lesssim a$.

Proof. We shall use the same notation as in the construction of F. Note that if

$$\ell_{k_{j+1}} \lesssim r \lesssim \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}},$$

then $B_r(x)$ contains a single component $Q_{i,n}$ of F_{j+1}. Since

$$\mu_F(Q_{i,n}) = \frac{1}{2^{Nk_{j+1}}},$$

the first estimate in (3.1) follows.
We now assume

$$\frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}} \lesssim r \lesssim \ell_{k_j}.$$

Let Q_i be the component of F_j containing x. Recall that there are $2^{N(k_{j+1}-k_j)}$ components $Q_{i,n}$ of F_{j+1} contained in Q_i. Thus, the number of cubes $Q_{i,n}$ contained in $B_r(x)$ is of the order of $2^{N(k_{j+1}-k_j)} \left(\frac{r}{\ell_{k_j}}\right)^N$. Since, for each $Q_{i,n}$, $\mu_F(Q_{i,n}) = \frac{1}{2^{Nk_{j+1}}}$, we then have

$$\mu_F\big(B_r(x)\big) \sim 2^{N(k_{j+1}-k_j)} \left(\frac{r}{\ell_{k_j}}\right)^N \mu_F(Q_{i,n}) = \frac{1}{2^{Nk_j}} \left(\frac{r}{\ell_{k_j}}\right)^N.$$

The proof of the lemma is complete.

Let $v \in L^1(Q_1)$ be the unique solution of

$$\begin{cases} -\Delta v = \mu_F & \text{in } Q_1, \\ \quad v = 0 & \text{on } \partial Q_1. \end{cases} \tag{3.2}$$

A basic estimate satisfied by v is given by the following

Proposition 1 *Assume $N \geq 3$. Let $F \subset Q_1$ be the Cantor set associated to the subsequence (ℓ_{k_j}) and let v be the solution of (3.2). Then, there exist constants $C_1, C_2 > 0$ (depending on N and θ) such that*

$$C_1 \left(\frac{1}{\ell_{k_1}^{N-2}} + \sum_{i=1}^{j} \frac{1}{2^{Nk_i} \ell_{k_i}^{N-2}} \right) \leq v(x) \leq C_2 \left(\frac{1}{\ell_{k_1}^{N-2}} + \sum_{i=1}^{j} \frac{1}{2^{Nk_i} \ell_{k_i}^{N-2}} \right), \tag{3.3}$$

for every $x \in \partial F_j$, $j \geq 1$.

Proof. Let

$$w(x) = \frac{1}{N\omega_N} \int_0^\infty \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr \quad \forall x \in Q_1,$$

where $\omega_N = |B_1|$. By (2.4), for every $x \in \partial F_j$ we have

$$\mu_F\big(B_r(x)\big) = 0 \quad \text{if } r \lesssim \ell_{k_j},$$

so that

$$w(x) \sim \int_{\ell_{k_j}}^\infty \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr \quad \forall x \in \partial F_j.$$

Thus,

$$w(x) \sim \sum_{i=1}^{j-1} \int_{\ell_{k_{i+1}}}^{\ell_{k_i}} \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr + \int_{\ell_{k_1}}^\infty \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr \sim \sum_{i=1}^{j-1} \big(A_i + B_i\big) + \frac{1}{\ell_{k_1}^{N-2}},$$

where, by Lemma 1 and (2.2),

$$A_i = \int_{\ell_{k_{i+1}}}^{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}} \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr \sim \frac{1}{2^{Nk_{i+1}}} \int_{\ell_{k_{i+1}}}^{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}} \frac{dr}{r^{N-1}} \sim \frac{1}{2^{Nk_{i+1}}\ell_{k_{i+1}}^{N-2}}$$

and

$$B_i = \int_{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}}^{\ell_{k_i}} \frac{\mu_F\big(B_r(x)\big)}{r^{N-1}}\, dr \sim \frac{1}{2^{Nk_i}\ell_{k_i}^N} \int_{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}}^{\ell_{k_i}} r\, dr \sim \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}}.$$

Therefore,

$$w(x) \sim \sum_{i=1}^{j-1} \left(\frac{1}{2^{Nk_{i+1}}\ell_{k_{i+1}}^{N-2}} + \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}} \right) + \frac{1}{\ell_{k_1}^{N-2}} \sim \sum_{i=1}^{j} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}} + \frac{1}{\ell_{k_1}^{N-2}}. \quad (3.4)$$

In other words, w satisfies (3.3). On the other hand, we have

$$d(F_1, \partial Q_1) = \frac{1-\gamma}{2} = \frac{1-2\theta}{4} > 0.$$

Since $w \geq 0$ and $-\Delta w = \mu_F$ in Q_1 (see Lemma 2 below), there exist constants $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\tilde{C}_1 w \leq v \leq \tilde{C}_2 w \quad \text{on } F_1. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain (3.3). This concludes the proof of the proposition.

We now establish a well-known fact used in the proof of Proposition 1:

Lemma 2 *Given* $\mu \in \mathcal{M}(\mathbb{R}^N)$*, let*

$$w(x) = \frac{1}{N\omega_N} \int_0^\infty \frac{\mu\big(B_r(x)\big)}{r^{N-1}}\, dr \quad \forall x \in \mathbb{R}^N. \tag{3.6}$$

Then,

$$-\Delta w = \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Proof. We shall prove the lemma for $N \geq 3$; the case $N = 2$ is similar.
We make the change of variables $r = s^{-\frac{1}{N-2}}$ in (3.6). Since $\frac{dr}{r^{N-1}} = -\frac{ds}{N-2}$, we get

$$\begin{aligned}
N(N-2)\omega_N\, w(x) &= (N-2) \int_0^\infty \mu\Big(\big\{y \in \mathbb{R}^N : |x-y| < r\big\}\Big) \frac{dr}{r^{N-1}} \\
&= \int_0^\infty \mu\Big(\Big\{y \in \mathbb{R}^N : |x-y| < s^{-\frac{1}{N-2}}\Big\}\Big)\, ds \\
&= \int_0^\infty \mu\Big(\Big\{y \in \mathbb{R}^N : \frac{1}{|x-y|^{N-2}} > s\Big\}\Big)\, ds = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-2}},
\end{aligned}$$

from which the result follows.

The counterpart of Proposition 1 in dimension $N = 2$ is given by

Proposition 2 *Assume* $N = 2$*. Let* $F \subset Q_1$ *be the Cantor set associated to the subsequence* (ℓ_{k_j}) *and let* v *be the solution of* (3.2)*. Then, for every* $j \geq 1$*, we have*

$$v \sim \left(\log \frac{1}{\ell_{k_1}} + \sum_{i=1}^{j} \frac{1}{4^{k_i}} \log \frac{1}{\ell_{k_i}} \right) \quad \text{on } \partial F_j. \tag{3.7}$$

The proof of Proposition 2 follows along the same lines and shall be omitted.

4 Proofs of Theorems 2 and 3

We start by recalling the definition of the (spherical) Hausdorff measure $\mathcal{H}^s$ in $\mathbb{R}^N$, where $0 \leq s \leq N$. Let $A \subset \mathbb{R}^N$ be a Borel set. Given $\delta > 0$, let

$$\mathcal{H}^s_\delta(A) = \inf \Big\{ \sum_i \omega_s r_i^s : K \subset \bigcup_i B_{r_i} \text{ with } r_i < \delta,\, \forall i \Big\},$$

where the infimum is taken over all coverings of A with open balls B_{r_i} of radii $r_i < \delta$, and $\omega_s = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$. When s is a positive integer, then ω_s is the measure of the unit ball in $\mathbb{R}^s$. We then set

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A).$$

We have the following

Lemma 3 *Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). Then,*

$$\mathcal{H}^s(F) \sim \liminf_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s. \tag{4.1}$$

Moreover, if $\mathcal{H}^s(F) \in (0,\infty)$, then

$$\mu_F = \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s\lfloor_F. \tag{4.2}$$

Proof.
Proof of (4.1). For $j \geq 1$ fixed, let (B_i) be a covering of F with 2^{Nk_j} balls of radii ℓ_{k_j}, where each ball B_i is concentric to some component of F_j. Then,

$$\mathcal{H}^s_\delta(F) \leq \omega_s\, 2^{Nk_j}\ell_{k_j}^s,$$

for every $\delta > \ell_{k_j}$. Thus,

$$\mathcal{H}^s(F) \leq \omega_s \liminf_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s, \tag{4.3}$$

which gives $\lesssim$ in (4.1).
Conversely, if $\liminf\limits_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s = 0$, then it follows from (4.3) that $\mathcal{H}^s(F) = 0$ and we are done. We now assume that

$$\liminf_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s > 0$$

(the limit above possibly being infinite). Given $0 < a < \liminf\limits_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s$, let $j_0 \geq 1$ be sufficiently large so that

$$2^{Nk_j}\ell_{k_j}^s \geq a \quad \forall j \geq j_0. \tag{4.4}$$

It then follows from Lemma 1 and (4.4) that there exists $C > 0$ such that

$$\mu_F\big(B_r(x)\big) \leq \frac{Cr^s}{a} \quad \forall x \in F, \quad \forall r \in (0, \ell_{j_0}). \tag{4.5}$$

Let $\delta \in (0, \ell_{j_0})$ and let (B_{r_i}) be a covering of F with balls of radii $r_i < \delta$. Without loss of generality, we may assume that each B_{r_i} is centered at some point of F. Thus, in view of (4.5), we have

$$\sum_i r_i^s \geq \frac{a}{C}\sum_i \mu_F(B_{r_i}) \geq \frac{a}{C}\mu_F\Big(\bigcup_i B_{r_i}\Big) = \frac{a}{C}\mu_F(F) = \frac{a}{C}.$$

This lower bound holds for any covering (B_{r_i}) such that $r_i < \delta$, $\forall i$. Therefore,

$$\mathcal{H}^s(F) \geq \mathcal{H}^s_\delta(F) \geq \frac{\omega_s}{C}a.$$

Since $a < \liminf_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s$ was arbitrary, we conclude that

$$\mathcal{H}^s(F) \geq \frac{\omega_s}{C} \liminf_{j\to\infty} 2^{Nk_j}\ell_{k_j}^s.$$

This establishes (4.1).

Proof of (4.2). Assume $\mathcal{H}^s(F) \in (0,\infty)$. Let Q_i be a component of F_j, $j \geq 1$. By symmetry, we have

$$\mathcal{H}^s(F) = 2^{Nk_j}\mathcal{H}^s(Q_i \cap F).$$

Since $\mu_F(Q_i) = 2^{-Nk_j}$, we get

$$\mu_F(Q_i) = \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s\lfloor_F(Q_i). \tag{4.6}$$

Given $A \subset \mathbb{R}^N$ open, we may write $A \cap F = \bigcup_i (Q_i \cap F)$, where (Q_i) is a family of disjoint connected components among all F_j, $j \geq 1$. It then follows from (4.6) that

$$\mu_F(A) = \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s\lfloor_F(A) \quad \text{for every open set } A \subset \mathbb{R}^N.$$

Since μ_F and $\mathcal{H}^s\lfloor_F$ are Radon measures, (4.2) follows. This concludes the proof of the lemma.

We recall the following result (see [3, Theorem 4]):

Proposition 3 *Suppose $\mu_1 \in \mathcal{M}(\Omega)$ is a good measure for problem* (1.1). *Then, any measure $\mu_2 \leq \mu_1$ is also good.*

We now establish the

Proposition 4 *Assume $N \geq 3$. Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). There exists $C > 0$ (depending on N and θ) such that if*

$$\sum_{j=1}^{\infty} g\left(C\alpha_0 \sum_{i=1}^{j+1} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}}\right) 2^{Nk_j}\ell_{k_j}^N < \infty \quad \textit{for some } \alpha_0 > 0, \tag{4.7}$$

then $\alpha_0\mu_F \in \mathcal{G}(g)$.

Proof. Let

$$a = \frac{1}{\ell_{k_1}^{N-2}} \quad \text{and} \quad b_j = \sum_{i=1}^{j} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}} \quad \forall j \geq 1.$$

Let v be the solution of (3.2). By Proposition 1, there exists $C_2 > 0$ such that

$$v(x) \leq C_2(a + b_j) \quad \forall x \in \partial F_j.$$

Note that v is harmonic in $(\operatorname{int} F_j)\backslash F_{j+1}$. Thus, by the maximum principle,

$$v(x) \leq C_2(a + b_{j+1}) \quad \forall x \in F_j\backslash F_{j+1}.$$

Assume that $\lim_{j\to\infty} b_j < \infty$. In this case, we have $v \in L^\infty(\Omega)$; hence, $g(\alpha_0 v) \in L^1(\Omega)$. We then conclude that $\alpha_0\mu_F + g(\alpha_0 v)$ is good. By Proposition 3, $\alpha_0\mu_F$ is also a good measure.

We now assume that

$$\lim_{j\to\infty} b_j = \infty. \tag{4.8}$$

Since

$$|F_j\backslash F_{j+1}| \leq |F_j| = 2^{Nk_j}\ell_{k_j}^N,$$

then, for every $\alpha > 0$, we have

$$\begin{aligned}\int_\Omega g(\alpha v) &= \sum_{j=1}^\infty \int_{F_j\backslash F_{j+1}} g(\alpha v) + \int_{\Omega\backslash F_1} g(\alpha v)\\ &\leq \sum_{j=1}^\infty g\big(C_2\alpha(a + b_{j+1})\big)\, 2^{Nk_j}\ell_{k_j}^N + O(1).\end{aligned}$$

Using (4.8), we have $C_2\alpha(a + b_{j+1}) \leq 2C_2\alpha b_{j+1}$ for every $j \geq 1$ sufficiently large. Therefore, if (4.7) holds with $C = 2C_2$, then $g(\alpha_0 v) \in L^1(\Omega)$, so that $\alpha_0\mu_F + g(\alpha_0 v)$ is a good measure. Applying Proposition 3 above, we conclude that $\alpha_0\mu_F \in \mathcal{G}(g)$.

We now present the

Proof of Theorem 2. Set $\ell_k = 2^{-\frac{N}{N-2}k}$, $\forall k \geq 1$. We now fix an increasing sequence of positive integers (k_j) such that

$$\frac{g(j^2)}{2^{\frac{2N}{N-2}k_j}} \leq \frac{1}{2^j} \quad \forall j \geq 1. \tag{4.9}$$

Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). We claim that $\alpha\mu_F$ is good for every $\alpha > 0$.

In fact, since $2^{Nk_i}\ell_{k_i}^{N-2} = 1$ for every $i \geq 1$, we have

$$\sum_{i=1}^{j+1} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}} = j + 1 \leq 2j \quad \forall x \in F_j\backslash F_{j+1}.$$

Moreover,

$$2^{Nk_j}\ell_{k_j}^N = \frac{1}{2^{\frac{2N}{N-2}k_j}}.$$

Thus, for every $\beta > 0$, we have

$$\sum_{j=1}^\infty g\left(\beta \sum_{i=1}^{j+1} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}}\right) 2^{Nk_j}\ell_{k_j}^N \leq \sum_{j=1}^\infty \frac{g\big(2\beta j\big)}{2^{\frac{2N}{N-2}k_j}}. \tag{4.10}$$

Since $2\beta j \leq j^2$ for $j \geq 1$ sufficiently large, it then follows from (4.9) that the right-hand side of (4.10) is finite for every $\beta > 0$. Applying Proposition 4, we conclude that $\alpha\mu_F$ is a good measure for every $\alpha > 0$.
On the other hand, since $2^{Nk_j}\ell_{k_j}^{N-2} = 1$, $\forall j \geq 1$, we deduce from Lemma 3 that $\mathcal{H}^{N-2}(F) \in (0, \infty)$. Thus, by (4.2), we have

$$\mu_F = \frac{1}{\mathcal{H}^{N-2}(F)}\mathcal{H}^{N-2}\lfloor_F.$$

Therefore, $\alpha\mathcal{H}^{N-2}\lfloor_F$ is good for every $\alpha > 0$.

Proof of Theorem 3. Let (k_j) be an increasing sequence of positive integers such that

$$\frac{g(j^3)}{2^{\frac{2N}{N-2}k_j}} \leq \frac{1}{2^j} \quad \forall j \geq 1. \tag{4.11}$$

Let

$$\ell_k = \frac{1}{j^{\frac{1}{N-2}}\, 2^{\frac{Nk}{N-2}}} \quad \text{if } k_{j-1} < k \leq k_j,$$

with the convention that $k_0 = 0$. Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). By Lemma 3, we know that $\mathcal{H}^{N-2}(F) = 0$. We now show that μ_F is a good measure relative to g.
Since $2^{Nk_i}\ell_{k_i}^{N-2} = \frac{1}{i}$, we have

$$\sum_{i=1}^{j+1} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}} = \frac{(j+1)(j+2)}{2} \leq 3j^2.$$

Moreover,

$$2^{Nk_j}\ell_{k_j}^N = \frac{1}{j^{\frac{N}{N-2}}\, 2^{\frac{2N}{N-2}k_j}} \leq \frac{1}{2^{\frac{2N}{N-2}k_j}}.$$

Thus, for every $\beta > 0$, we have

$$\sum_{j=1}^{\infty} g\left(\beta \sum_{i=1}^{j+1} \frac{1}{2^{Nk_i}\ell_{k_i}^{N-2}}\right) 2^{Nk_j}\ell_{k_j}^N \leq \sum_{j=1}^{\infty} \frac{g(3\beta j^2)}{2^{\frac{2N}{N-2}k_j}}. \tag{4.12}$$

Since $3\beta j^2 \leq j^3$ for $j \geq 1$ sufficiently large, it then follows from (4.11) that the right-hand side of (4.12) is finite for every $\beta > 0$. Applying Proposition 4, we conclude that μ_F is a good measure. The proof of Theorem 3 is complete.

5 Proof of Theorem 1

When $N \geq 3$, Theorem 1 follows from Theorem 2 (or Theorem 3) and the following well-known

Proposition 5 *Let $K \subset \Omega$ be a compact set. If $\mathcal{H}^{N-2}(K) < \infty$, then* $\operatorname{cap}(K) = 0$.

We refer the reader to e.g. [7] for a proof of Proposition 5.

We now deal with the case $N = 2$. We shall need the following

Lemma 4 *Assume $N = 2$. Let $F \subset \Omega$ be the Cantor set associated to the subsequence (ℓ_{k_j}). Then,*

$$\operatorname{cap}(F) = 0 \quad \textit{if and only if} \quad \sum_{j=1}^{\infty} \frac{1}{4^{k_j}} \log \frac{1}{\ell_{k_j}} = \infty. \tag{5.1}$$

When F is a standard Cantor set, (5.1) is Carleson's test (see [7, p.31]) for determining whether F has zero capacity. The same proof as in [7] can be used to establish Lemma 4. We present a different argument based on Proposition 2 above.

Proof of Lemma 4. $(\Leftarrow)$ Suppose

$$\sum_{j=1}^{\infty} \frac{1}{4^{k_j}} \log \frac{1}{\ell_{k_j}} = \infty.$$

It then follows from Proposition 2 that $v = +\infty$ on F, where v is the solution of (3.2). Since v is superharmonic, we can apply Theorem 7.33 in [9] to conclude that $\operatorname{cap}(F) = 0$.

$(\Rightarrow)$ Assume that

$$\sum_{j=1}^{\infty} \frac{1}{4^{k_j}} \log \frac{1}{\ell_{k_j}} < \infty. \tag{5.2}$$

Let v be the solution of (3.2). It follows from (5.2) and Proposition 2 that v is uniformly bounded in Ω. Thus, the measure μ_F is diffuse. Since μ_F is concentrated in F, we must have $\operatorname{cap}(F) > 0$. The proof of Lemma 4 is complete.

Remark 1 Here is the counterpart of (5.1) in dimension $N \geq 3$:

$$\operatorname{cap}(F) = 0 \quad \text{if and only if} \quad \sum_{j=1}^{\infty} \frac{1}{2^{Nk_j} \ell_{k_j}^{N-2}} = \infty. \tag{5.3}$$

The proof of (5.3) follows along the same lines.

The analog of Proposition 4 in dimension $N = 2$ is given by the next

Proposition 6 *Assume $N = 2$. Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). There exists $C > 0$ (depending on θ) such that if*

$$\sum_{j=1}^{\infty} g\left(C\alpha_0 \sum_{i=1}^{j+1} \frac{1}{4^{k_i}} \log \frac{1}{\ell_{k_i}}\right) 4^{k_j} \ell_{k_j}^2 < \infty \quad \textit{for some } \alpha_0 > 0, \tag{5.4}$$

then $\alpha_0 \mu_F \in \mathcal{G}(g)$.

The proof of Proposition 6 is based on Proposition 2 and shall be omitted.

We may now present the

Proof of Theorem 1 completed. Let $\ell_k = 4^{-4^k}$, $\forall k \geq 1$. We now fix an increasing sequence of positive integers (k_j) such that

$$\frac{g(j^2)}{4^{4^{k_j}}} \leq \frac{1}{2^j} \quad \forall j \geq 1. \tag{5.5}$$

Let F be the Cantor set associated to the subsequence (ℓ_{k_j}). Note that

$$\frac{1}{4^{k_i}} \log \frac{1}{\ell_{k_i}} = \log 4 \quad \forall i \geq 1.$$

In particular,

$$\sum_{i=1}^{j+1} \frac{1}{4^{k_i}} \log \frac{1}{\ell_{k_i}} = (j+1) \log 4 \leq 4j.$$

It then follows from Lemma 4 that $\operatorname{cap}(F) = 0$. We now show that μ_F is a good measure.
Since

$$|F_j \backslash F_{j+1}| \leq |F_j| = 4^{k_j} \ell_{k_j}^2 = \frac{1}{4^{2\, 4^{k_j} - k_j}} \leq \frac{1}{4^{4^{k_j}}},$$

then, for every $\beta > 0$, we have

$$\sum_{j=1}^{\infty} g\left(\beta \sum_{i=1}^{j+1} \frac{1}{4^{k_i}} \log \frac{1}{\ell_{k_i}}\right) 4^{k_j} \ell_{k_j}^2 \leq \sum_{j=1}^{\infty} \frac{g(4\beta j)}{4^{4^{k_j}}}. \tag{5.6}$$

In view of (5.5), we conclude that the right-hand side of (5.6) is finite for every $\beta > 0$. Thus, by Proposition 6 above, μ_F is good. The proof of Theorem 1 is complete.

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