Reduced measures on the boundary

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Abstract. We study the existence of solutions of the nonlinear problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where μ is a bounded measure and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $g(t) = 0$, $\forall t \leq 0$. Problem (0.1) admits a solution for every $\mu \in L^1(\partial\Omega)$, but this need not be the case when μ is a general bounded measure. We introduce a concept of reduced measure μ^* (in the spirit of [4]); this is the "closest" measure to μ for which (0.1) admits a solution.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function such that $g(0) = 0$. In this paper, we are interested in the problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a bounded measure on $\partial\Omega$. The right concept of weak solution of (1.1) is the following:

$$\begin{cases} u \in L^1(\Omega),\ g(u)\rho_0 \in L^1(\Omega) \text{ and} \\ -\displaystyle\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \end{cases} \tag{1.2}$$

where $\rho_0(x) = d(x, \partial\Omega)$, $\forall x \in \Omega$, $\frac{\partial}{\partial n}$ denotes the derivative with respect to the outward normal of $\partial\Omega$, and

$$C_0^2(\overline{\Omega}) = \big\{\zeta \in C^2(\overline{\Omega})\ ;\ \zeta = 0 \text{ on } \partial\Omega\big\}.$$

If u is a solution of (1.1), then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $\forall p < \infty$ (see [3, Theorem 5]).

It has been proved by H. Brezis (1972, unpublished; see [15]) that (1.1) admits a unique weak solution when μ is any L^1-function (for a general nonlinearity

g). When g is a power, the study of (1.1) for measures was initiated by Gmira-Véron [15] (in the same spirit as [1]). They proved that if $g(t) = |t|^{p-1}t$ and $1 < p < \frac{N+1}{N-1}$, then (1.1) has a solution for any measure μ. They also showed that if $p \geq \frac{N+1}{N-1}$ and $\mu = \delta_a$, $a \in \partial\Omega$, then (1.1) has no solution. The set of measures μ for which (1.1) has a solution has been completely characterized when $p \geq \frac{N+1}{N-1}$. In this case, (1.1) has a solution if and only if $\mu(A) = 0$ for every Borel set $A \subset \partial\Omega$ such that $C_{2/p,p'}(A) = 0$, where $C_{2/p,p'}$ denotes the Bessel capacity on $\partial\Omega$ associated to $W^{2/p,p'}$. This result was established by Le Gall [17] (for $p = 2$) and by Dynkin-Kuznetsov [12] (for $p < 2$) using probabilistic tools and by Marcus-Véron [20] (for $p > 2$) using purely analytical methods; see also Marcus-Véron [21] for a unified approach for any $p \geq \frac{N+1}{N-1}$.

Our goal in this paper is to develop for (1.1) the same program as in [4] for the problem

$$\begin{cases} -\Delta u + g(u) = \lambda & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where λ, in this case, is a measure in Ω. We shall analyze the nonexistence mechanism behind (1.1) for a general nonlinearity g. In [4] we have shown that the Newtonian (H^1) capacity in Ω, cap_{H^1}, plays a major role in the study of (1.3); one of the main results there asserts that (1.3) has a solution for every g if and only if $\lambda(E) = 0$ for every Borel set $E \subset \Omega$ such that $\text{cap}_{H^1}(E) = 0$. For problem (1.1), the analogous quantity is the Hausdorff measure $\mathcal{H}^{N-1}$ on $\partial\Omega$ (i.e., $(N-1)$-dimensional Lebesgue measure on $\partial\Omega$). In fact, many of the results in [4] remain valid provided one replaces in the statements the H^1-capacity by the $(N-1)$-Hausdorff measure. Some of the proofs, however, have to be substantially modified.

Concerning the function g we will assume *throughout the rest of the paper* that $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and that

$$g(t) = 0 \quad \forall t \leq 0. \tag{1.4}$$

The space of bounded measures on $\partial\Omega$ is denoted by $\mathcal{M}(\partial\Omega)$ and is equipped with the standard norm

$$\|\mu\|_{\mathcal{M}} = \sup\left\{ \int_{\partial\Omega} \varphi \, d\mu \, ; \, \varphi \in C(\partial\Omega) \text{ and } \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

By a (weak) *solution* u of (1.1) we mean that (1.2) holds. A (weak) *subsolution* of (1.1) is a function v satisfying

$$\begin{cases} v \in L^1(\Omega), \ g(v)\rho_0 \in L^1(\Omega) \text{ and} \\ -\displaystyle\int_\Omega v\Delta\zeta + \int_\Omega g(v)\zeta \leq -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n} \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}), \ \zeta \geq 0 \text{ in } \Omega. \end{cases} \tag{1.5}$$

We will say that $\mu \in \mathcal{M}(\partial\Omega)$ is a *good measure* if (1.1) admits a solution. If μ is a good measure, then equation (1.1) has exactly one solution u (see [20]; although this result is stated there when g is a power, the proof remains unchanged for a general nonlinearity g). We denote by $\mathcal{G}$ the set of good measures (relative to g); when we need to make explicit the dependence on g we shall write $\mathcal{G}(g)$. Recall that L^1-functions on $\partial\Omega$ belong to $\mathcal{G}(g)$ for every g.

In the sequel we denote by (g_k) a sequence of functions $g_k : \mathbb{R} \to \mathbb{R}$ which are continuous, nondecreasing and satisfy the following conditions:

$$0 \le g_1(t) \le g_2(t) \le \ldots \le g(t) \quad \forall t \in \mathbb{R}, \tag{1.6}$$

$$g_k(t) \to g(t) \quad \forall t \in \mathbb{R}. \tag{1.7}$$

We assume in addition that each g_k has subcritical growth, i.e., that there exist $C > 0$ and $p < \frac{N+1}{N-1}$ (possibly depending on k) such that

$$g_k(t) \le C\big(|t|^p + 1\big) \quad \forall t \in \mathbb{R}. \tag{1.8}$$

A good example to keep in mind is $g_k(t) = \min\{g(t), k\}$, $\forall t \in \mathbb{R}$.

Since (1.8) holds, then for every $\mu \in \mathcal{M}(\partial\Omega)$ there exists a unique solution u_k of

$$\begin{cases} -\Delta u_k + g_k(u_k) = 0 & \text{in } \Omega, \\ \qquad\qquad\quad u_k = \mu & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

The convergence of the sequence (u_k) follows from the next result, established in [4, Section 9.3]:

Theorem 1 *As $k \uparrow \infty$, $u_k \downarrow u^*$ in $L^1(\Omega)$, with $g(u^*)\rho_0 \in L^1(\Omega)$, and u^* satisfies*

$$\begin{cases} -\Delta u^* + g(u^*) = 0 & \textit{in } \Omega, \\ \qquad\qquad\quad u^* = \mu^* & \textit{on } \partial\Omega, \end{cases} \tag{1.10}$$

for some $\mu^* \in \mathcal{M}(\partial\Omega)$ such that $\mu^* \le \mu$. In addition, u^* is the largest subsolution of (1.1).

Remark 1 An alternative approximation mechanism consists of keeping g *fixed* and considering a sequence of functions $\mu_k \in L^1(\partial\Omega)$ weakly converging to μ. Let v_k be the solution of (1.1) associated to μ_k. It would be interesting to prove that $v_k \to u^*$ in $L^1(\Omega)$ for some appropriate choices of sequences (μ_k) (for measures in Ω, see [4, Theorem 11]).

An important consequence of Theorem 1 is that u^* — and thus μ^* — *does not depend on the choice of the truncating sequence* (g_k). We call μ^* the *reduced measure* associated to μ. If g has subcritical growth, then $\mu^* = \mu$ for every $\mu \in \mathcal{M}(\partial\Omega)$ (see Example 1 below). However, if g has critical or supercritical growth, then μ^* might be different from μ. In this case, μ^* depends both on the measure μ and on the nonlinearity g.

By definition, μ^* is a good measure $\le \mu$ (since (1.10) has a solution u^*). One of the main properties satisfied by μ^* is the following

Theorem 2 *The reduced measure μ^* is the largest good measure $\leq \mu$.*

A consequence of Theorem 2 is

Corollary 1 *There exists a Borel set $\Sigma \subset \partial\Omega$ with $\mathcal{H}^{N-1}(\Sigma) = 0$ such that*

$$(\mu - \mu^*)(\partial\Omega \setminus \Sigma) = 0. \tag{1.11}$$

To see this, let μ_{a} and μ_{s} denote, respectively, the absolutely continuous and the singular parts of μ with respect to $\mathcal{H}^{N-1}$. Since $\mu_{\mathrm{a}} \in L^1(\partial\Omega)$, then μ_{a} is good. Thus, $\mu_{\mathrm{a}} - \mu_{\mathrm{s}}^-$ is also a good measure (see Proposition 1 below). We then conclude from Theorem 2 that $\mu_{\mathrm{a}} - \mu_{\mathrm{s}}^- \leq \mu^* \leq \mu$. Hence,

$$0 \leq \mu - \mu^* \leq \mu - \mu_{\mathrm{a}} + \mu_{\mathrm{s}}^- = \mu_{\mathrm{s}}^+$$

and so $\mu - \mu^*$ is concentrated on a set of zero $\mathcal{H}^{N-1}$-measure.

Remark 2 Corollary 1 is the "best one can say" about $\mu - \mu^*$ for a *general* nonlinearity g. In fact, given any measure $\mu \geq 0$ concentrated on a set of zero $\mathcal{H}^{N-1}$-measure, there exists some g such that $\mu^* = 0$ (see Theorem 7 below). In particular, $\mu - \mu^*$ can be *any* nonnegative measure concentrated on a set of zero $\mathcal{H}^{N-1}$-measure in $\partial\Omega$.

It is not difficult to see that if $\mu \in \mathcal{M}(\partial\Omega)$ and $\mu^+ \in L^1(\partial\Omega)$, then $\mu \in \mathcal{G}(g)$ for every g (see Proposition 5 below). The converse is also true:

Theorem 3 *Let $\mu \in \mathcal{M}(\partial\Omega)$. If $\mu \in \mathcal{G}(g)$ for every g, then $\mu^+ \in L^1(\partial\Omega)$.*

A key ingredient in the proof of Theorem 3 is the following

Theorem 4 *For every compact set $K \subset \partial\Omega$, we have*

$$\mathcal{H}^{N-1}(K) = \inf\left\{\int_\Omega |\Delta\zeta| \,;\ \zeta \in C_0^2(\overline{\Omega}),\ -\frac{\partial\zeta}{\partial n} \geq 1 \text{ in some neighborhood of } K\right\}.$$

Remark 3 As we have already pointed out, the measure $\mathcal{H}^{N-1}$ plays here the same role as cap_{H^1} in [4]. There, for every compact set $K \subset \Omega$ we showed that

$$\mathrm{cap}_{H^1}(K) = \frac{1}{2}\inf\left\{\int_\Omega |\Delta\varphi| \,;\ \varphi \in C_c^\infty(\Omega),\ \varphi \geq 1 \text{ in some neighborhood of } K\right\},$$

which is the counterpart of Theorem 4.

We now address a *different* question. Could it happen that, for some fixed g_0, the only good measures μ are those satisfying $\mu^+ \in L^1(\partial\Omega)$? The answer is negative. In fact,

Theorem 5 *For any g, there exists a good measure $\mu \geq 0$ such that $\mu \notin L^1(\partial\Omega)$.*

A natural question is to combine the results of [4] with those in the present paper, i.e., consider the problem

$$\begin{cases} -\Delta u + g(u) = \lambda & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\partial\Omega)$. We say that the pair (λ, μ) is good if (1.12) has a solution in the usual weak sense (with $g(u)\rho_0 \in L^1(\Omega)$). Surprisingly, the problem "uncouples". More precisely,

Theorem 6 *Let $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\partial\Omega)$. The pair (λ, μ) is good if and only if λ is a good measure for* (1.3) *and μ is a good measure for* (1.1)*. Furthermore, $(\lambda, \mu)^* = (\lambda^*, \mu^*)$.*

This paper is organized as follows. In the next section we prove Theorem 2. In Section 3, we present several properties satisfied by the mapping $\mu \mapsto \mu^*$ and by the set of good measures $\mathcal{G}$. Theorem 4 will be established in Section 4. We show in Section 5 that for every singular measure $\mu \geq 0$ there exists some g such that $\mu^* = 0$; we then deduce Theorem 3 as a corollary. Theorem 5 will be proved in Section 6. In Section 7, we give the explicit value of μ^* in the case where $g(t) = t^p$, $t \geq 0$, for any $p > 1$. In the last section we present the proof of Theorem 6.

Some of the results in this paper were announced in [4].

2 Proof of Theorem 2

The main ingredient in the proof of Theorem 2 is the following:

Lemma 1 *Given $f \in L^1(\Omega; \rho_0\, dx)$, $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\partial\Omega)$, let $w \in L^1(\Omega)$ be the unique solution of*

$$-\int_\Omega w\Delta\zeta = \int_\Omega f\zeta + \int_\Omega \zeta\, d\lambda - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

If $w \geq 0$ a.e. in Ω, then $\mu \geq 0$ on $\partial\Omega$.

This result is fairly well-known. We present a proof for the convenience of the reader. For measures in Ω, the counterpart of Lemma 1 is the "Inverse" maximum principle of [8] (see [4]).

Proof. Given $\phi \in C^\infty(\partial\Omega)$, $\phi \geq 0$ on $\partial\Omega$, let $\zeta \in C_0^2(\overline{\Omega})$, $\zeta > 0$ in Ω, be such that $-\frac{\partial\zeta}{\partial n} = \phi$ on $\partial\Omega$. Let $\delta_j \downarrow 0$ be a sequence of regular values of ζ. For each $j \geq 1$, set $\zeta_j = \zeta - \delta_j$ and $\omega_j = [\zeta > \delta_j]$. In particular, $\zeta_j \in C_0^2(\overline{\omega}_j)$, $\zeta_j \geq 0$ in

ω_j, and $-\frac{\partial \zeta_j}{\partial n} \geq 0$ on $\partial\omega_j$. By standard elliptic estimates (see [25]), we know that $w \in W^{1,p}_{\text{loc}}(\Omega)$, $\forall p < \frac{N}{N-1}$; thus, w has a nonnegative L^1-trace on $\partial\omega_j$. Therefore,

$$-\int_{\omega_j} w\Delta\zeta_j = \int_{\omega_j} f\zeta_j + \int_{\omega_j} \zeta_j \, d\lambda - \int_{\partial\omega_j} \frac{\partial \zeta_j}{\partial n} w \geq \int_{\omega_j} f\zeta_j + \int_{\omega_j} \zeta_j \, d\lambda.$$

As $j \to \infty$, we conclude that

$$\int_\Omega w\Delta\zeta + \int_\Omega f\zeta + \int_\Omega \zeta \, d\lambda \leq 0.$$

Thus,

$$\int_{\partial\Omega} \phi \, d\mu = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial n} \, d\mu = -\Big(\int_\Omega w\Delta\zeta + \int_\Omega f\zeta + \int_\Omega \zeta \, d\lambda \Big) \geq 0.$$

Since $\phi \geq 0$ was arbitrary, we conclude that $\mu \geq 0$.

We can now establish Theorem 2:

Proof of Theorem 2. Assume ν is a good measure $\leq \mu$. Let v denote the solution of

$$\begin{cases} -\Delta v + g(v) = 0 & \text{in } \Omega, \\ v = \nu & \text{on } \partial\Omega. \end{cases}$$

Since $\nu \leq \mu$, it follows that v is a subsolution of (1.1). Thus, by Theorem 1, $v \leq u^*$ a.e. Applying Lemma 1 to the function $w = u^* - v$, we then conclude that $\mu^* - \nu \geq 0$.

3 Some properties of $\mathcal{G}$ and μ^*

Here is a list of properties which can be established exactly as in [4]. For this reason, we shall omit their proofs.

Proposition 1 *Suppose μ_1 is a good measure. Then, any measure $\mu_2 \leq \mu_1$ is also a good measure.*

Proposition 2 *If μ_1, μ_2 are good measures, then so is* $\sup\{\mu_1, \mu_2\}$.

Proposition 3 *The set $\mathcal{G}$ of good measures is convex.*

Proposition 4 *We have*

$$\mathcal{G} + L^1(\partial\Omega) \subset \mathcal{G}.$$

Proposition 5 *Let $\mu \in \mathcal{M}(\partial\Omega)$. Then, $\mu \in \mathcal{G}$ if and only if $\mu^+ \in \mathcal{G}$.*

Proposition 6 *Let $\mu \in \mathcal{M}(\partial\Omega)$. Then, $\mu \in \mathcal{G}$ if and only if $\mu_{\rm s} \in \mathcal{G}$, where $\mu_{\rm s}$ denotes the singular part of μ with respect to $\mathcal{H}^{N-1}$.*

Proposition 7 *Let $\mu \in \mathcal{M}(\partial\Omega)$. Then, $\mu \in \mathcal{G}$ if and only if there exist $f_0 \in L^1(\Omega;\rho_0\,dx)$ and $v_0 \in L^1(\Omega)$ such that $g(v_0) \in L^1(\Omega;\rho_0\,dx)$ and*

$$\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\,d\mu = \int_\Omega f_0\zeta + \int_\Omega v_0\Delta\zeta \quad \forall\zeta \in C_0^2(\overline{\Omega}). \tag{3.1}$$

Proposition 7 is the analog of a result of Gallouët-Morel [14]; see also [4, Theorem 6].

Proposition 8 *For every measure μ, we have*

$$0 \le \mu - \mu^* \le \mu^+. \tag{3.2}$$

Proposition 9 *For every measure μ, we have*

$$(\mu^*)^+ = (\mu^+)^* \quad \text{and} \quad (\mu^*)^- = \mu^-. \tag{3.3}$$

Proposition 10 *Let $\mu \in \mathcal{M}(\partial\Omega)$. Then,*

$$\|\mu - \mu^*\|_{\mathcal{M}} = \min_{\nu\in\mathcal{G}} \|\mu - \nu\|_{\mathcal{M}}. \tag{3.4}$$

Moreover, μ^* is the unique good measure which achieves the minimum in (3.4)*.*

Proposition 11 *Let $\mu \in \mathcal{M}(\partial\Omega)$ and $h \in L^1(\Omega;\rho_0\,dx)$. The problem*

$$\begin{cases} -\Delta v + g(v) = h & \text{in } \Omega,\\ \qquad\qquad v = \mu & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

has a solution if and only if $\mu \in \mathcal{G}(g)$.

By a solution v of (3.5) we mean that $v \in L^1(\Omega)$ satisfies $g(v) \in L^1(\Omega;\rho_0\,dx)$ and

$$-\int_\Omega v\Delta\zeta + \int_\Omega g(v)\zeta = \int_\Omega h\zeta - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\,d\nu \quad \forall\zeta \in C_0^2(\overline{\Omega}). \tag{3.6}$$

In view of Lemma 2 below such a solution, whenever it exists, is unique.

The proofs of Propositions 7 and 11 require an extra argument. We shall present a proof based on Lemmas 2–6 below.

Given $h \in L^1(\Omega;\rho_0\,dx)$, let $\mathcal{A}_g(h)$ denote the set of measures μ for which (3.5) has a solution. By Lemma 2 below, $\mathcal{A}_g(h)$ is closed with respect to the strong topology in $\mathcal{M}(\partial\Omega)$. Our goal is to show that $\mathcal{A}_g(h)$ is independent of h and $\mathcal{A}_g(h) = \mathcal{G}(g)$, $\forall h$. In the sequel, we shall denote by ζ_0 the solution of

$$\begin{cases} -\Delta\zeta_0 = 1 & \text{in } \Omega,\\ \quad\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

We start with the following

Lemma 2 *Let $h_i \in L^1(\Omega; \rho_0\, dx)$, $i = 1, 2$. Given $\mu_i \in \mathcal{A}_g(h_i)$, let v_i denote the solution of* (3.5) *corresponding to h_i, μ_i. Then,*

$$\int_\Omega |v_1 - v_2| + \int_\Omega \big|g(v_1) - g(v_2)\big|\, \zeta_0 \le \int_\Omega |h_1 - h_2|\, \zeta_0 + C \int_{\partial\Omega} |\mu_1 - \mu_2|. \tag{3.7}$$

Proof. Apply Lemma 1.5 in [20].

Lemma 3 *Assume g satisfies*

$$g(t) \le C\big(|t|^p + 1\big) \quad \forall t \in \mathbb{R}, \tag{3.8}$$

for some $p < \frac{N+1}{N-1}$. Then, for every $h \in L^1(\Omega; \rho_0\, dx)$, we have $\mathcal{A}_g(h) = \mathcal{M}(\partial\Omega)$.

Proof. This result is established in [15] for $h = 0$. The same proof there also applies for $h \in L^\infty(\Omega)$. The general case when $h \in L^1(\Omega; \rho_0\, dx)$ then follows by density using Lemma 2 above.

Given $\mu \in \mathcal{M}(\partial\Omega)$, let v_k be the solution of

$$\begin{cases} -\Delta v_k + g_k(v_k) = h & \text{in } \Omega, \\ \qquad\qquad\quad v_k = \mu & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where (g_k) is a sequence of functions satisfying (1.6)–(1.8).

Lemma 4 *Given $\mu \in \mathcal{A}_g(h)$, let v denote the solution of* (3.5). *Assume v_k satisfies* (3.9). *Then,*

$$v_k \to v \quad \text{in } L^1(\Omega) \quad \text{and} \quad g_k(v_k) \to g(v) \quad \text{in } L^1(\Omega; \rho_0\, dx). \tag{3.10}$$

Proof The lemma follows by mimicking the proof of Proposition 3 in [4] and using Lemma 2 above.

Lemma 5 *Let $h_1, h_2 \in L^1(\Omega; \rho_0\, dx)$. If $h_1 \le h_2$ a.e., then $\mathcal{A}_g(h_1) \supset \mathcal{A}_g(h_2)$.*

Proof. Let $\mu \in \mathcal{A}_g(h_2)$ and let (g_k) be a sequence satisfying (1.6)–(1.8). Denote by $v_{i,k}$, $i = 1, 2$, the solution of

$$\begin{cases} -\Delta v_{i,k} + g_k(v_{i,k}) = h_i & \text{in } \Omega, \\ \qquad\qquad\qquad v_{i,k} = \mu & \text{on } \partial\Omega. \end{cases}$$

Let v_i be such that $v_{i,k} \downarrow v_i$ in $L^1(\Omega)$ as $k \uparrow \infty$. By Lemma 4 above, we have

$$g_k(v_{2,k}) \to g(v_2) \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

By [4, Corollary B.2], $h_1 \leq h_2$ a.e. implies $v_{1,k} \leq v_{2,k}$ a.e.; thus, $g_k(v_{1,k}) \leq g_k(v_{2,k})$ a.e. It then follows by dominated convergence that

$$g_k(v_{1,k}) \to g(v_1) \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

Therefore, $\mu \in \mathcal{A}_g(h_1)$. This concludes the proof of the lemma.

Lemma 6 *Assume μ satisfies* (3.1) *for some $f_0 \in L^1(\Omega; \rho_0\, dx)$ and $v_0 \in L^1(\Omega)$, with $g(v_0) \in L^1(\Omega; \rho_0\, dx)$. Then, problem* (3.5) *has a solution for every $h \in L^1(\Omega; \rho_0\, dx)$.*

Proof. Fix $\alpha < 1$. Given $m \geq 1$, let $M_m = \dfrac{m\|\zeta_0\|_{L^\infty}}{1-\alpha}$. Since

$$\alpha v_0 + m\zeta_0 \leq v_0 \quad \text{a.e. on the set } [v_0 \geq M_m],$$

we have $g(\alpha v_0 + m\zeta_0) \in L^1(\Omega; \rho_0\, dx)$; moreover,

$$-\int_\Omega (\alpha v_0 + m\zeta_0)\Delta\zeta = \int_\Omega (\alpha f_0 + m)\zeta - \alpha \int_{\partial\Omega} \frac{\partial \zeta}{\partial n}\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Thus, $\alpha\mu \in \mathcal{A}_g(\tilde{h}_m)$, where

$$\tilde{h}_m = \alpha f_0 + m + g(\alpha v_0 + m\zeta_0).$$

Given $h \in L^1(\Omega; \rho_0\, dx)$, let

$$h_m = \min\{h, \tilde{h}_m\}.$$

Since $h_m \leq \tilde{h}_m$ a.e., it follows from Lemma 5 that $\alpha\mu \in \mathcal{A}_g(h_m)$, $\forall m \geq 1$. Note that $h_m \to h$ in $L^1(\Omega; \rho_0\, dx)$ as $m \to \infty$; thus, by Lemma 2 we get $\alpha\mu \in \mathcal{A}_g(h)$. Since this holds true for every $\alpha < 1$, we must have $\mu \in \mathcal{A}_g(h)$.

Proof of Proposition 7. Clearly, if μ is a good measure, then (3.1) holds. Conversely, assume μ satisfies (3.1) for some v_0, f_0. It then follows from the previous lemma that (3.5) has a solution for $h = 0$. In other words, μ is good.

Proof of Proposition 11. If μ is good, then (3.1) holds. Thus, by Lemma 6 above we conclude that problem (3.5) has a solution for every $h \in L^1(\Omega; \rho_0\, dx)$. Conversely, if (3.5) has a solution for some $h \in L^1(\Omega; \rho_0\, dx)$, then (3.1) holds. Applying Proposition 7, we deduce that μ is good.

4 Proof of Theorem 4

Given a compact set $K \subset \partial\Omega$, we define the capacity

$$c_{\partial\Omega}(K) = \inf\left\{\int_\Omega |\Delta\zeta|\ ;\ \zeta \in C_0^2(\overline{\Omega}),\ -\frac{\partial\zeta}{\partial n} \geq 1 \text{ in some neighborhood of } K\right\}.$$

In order to establish Theorem 4 we will need a few technical results. We start with

Lemma 7 *Let $K \subset \partial\Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C_0^2(\overline{\Omega})$ such that $\psi \geq 0$ in Ω, $-\frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of K and*

$$\int_\Omega |\Delta \psi| \leq c_{\partial\Omega}(K) + \varepsilon. \tag{4.1}$$

Proof. Given $\varepsilon > 0$, let $\zeta \in C_0^2(\overline{\Omega})$ be such that $-\frac{\partial \zeta}{\partial n} \geq 1$ in some neighborhood of K and

$$\int_\Omega |\Delta \zeta| \leq c_{\partial\Omega}(K) + \frac{\varepsilon}{2}. \tag{4.2}$$

We now extend ζ as a C^2-function in the whole space $\mathbb{R}^N$. We then let

$$f_k(x) = \int_{\mathbb{R}^N} \rho_k(x-y)\, |\Delta \zeta(y)|\, dy \quad \forall x \in \overline{\Omega},$$

where (ρ_k) is any sequence of nonnegative mollifiers such that $\operatorname{supp} \rho_k \subset B_{1/k}$, $\forall k \geq 1$. As $k \to \infty$, we have

$$f_k \to |\Delta \zeta| \quad \text{uniformly in } \overline{\Omega}. \tag{4.3}$$

Let $v_k \in C_0^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta v_k = f_k & \text{in } \Omega, \\ \quad v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $f_k \geq 0$, we have $v_k \geq 0$ in Ω. Moreover, (4.3) implies

$$\frac{\partial v_k}{\partial n} \to \frac{\partial v}{\partial n} \quad \text{uniformly on } \partial\Omega, \tag{4.4}$$

where v is the solution of

$$\begin{cases} -\Delta v = |\Delta \zeta| & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, $\zeta \leq v$ in Ω. Since $\zeta = v = 0$ on $\partial\Omega$, we have

$$-\frac{\partial \zeta}{\partial n} \leq -\frac{\partial v}{\partial n} \quad \text{on } \partial\Omega,$$

which implies that $-\frac{\partial v}{\partial n} \geq 1$ in some neighborhood of K. In view of (4.4), we can fix $k_0 \geq 1$ sufficiently large so that $\frac{\partial v_{k_0}}{\partial n} \geq \alpha$ in some neighborhood of K, where $\alpha < 1$. We may also assume that

$$\int_{A_{k_0}} |\Delta \zeta| < \frac{\varepsilon}{4},$$

where $A_{k_0} = N_{\frac{1}{k_0}}(\Omega) \setminus \overline{\Omega}$.
Set

$$\psi = \frac{1}{\alpha} v_{k_0},$$

so that $\psi \geq 0$ in Ω and $-\frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of K. Moreover,

$$\int_\Omega |\Delta \psi| = \frac{1}{\alpha} \int_\Omega |\Delta v_{k_0}| \leq \frac{1}{\alpha} \left(\int_\Omega |\Delta \zeta| + \frac{\varepsilon}{4} \right) \leq \frac{1}{\alpha} \left(c_{\partial\Omega}(K) + \frac{3\varepsilon}{4} \right).$$

Therefore, by taking

$$\alpha = \frac{c_{\partial\Omega}(K) + \frac{3\varepsilon}{4}}{c_{\partial\Omega}(K) + \varepsilon} < 1,$$

we conclude that ψ satisfies (4.1).

We next prove the

Lemma 8 *Let $K \subset \partial\Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C_0^2(\overline{\Omega})$ such that $0 \leq \psi \leq \varepsilon$ in Ω, $-\frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of K,*

$$\int_\Omega |\Delta \psi| \leq \mathcal{H}^{N-1}(K) + \varepsilon \quad \textit{and} \quad \left\| \frac{\psi}{\rho_0} \right\|_{L^\infty} \leq 1 + \varepsilon. \tag{4.5}$$

Proof. Let $\delta > 0$ be such that

$$\mathcal{H}^{N-1}\big(N_\delta(K) \cap \partial\Omega\big) \leq \mathcal{H}^{N-1}(K) + \varepsilon.$$

We now *fix* $\zeta \in C_0^2(\overline{\Omega})$ such that $\zeta > 0$ in Ω, $-\frac{\partial \zeta}{\partial n} = 1$ in $N_{\frac{\delta}{2}}(K) \cap \partial\Omega$, $\frac{\partial \zeta}{\partial n} = 0$ in $\partial\Omega \setminus N_\delta(K)$, $0 \leq -\frac{\partial \zeta}{\partial n} \leq 1$ on $\partial\Omega$, and $\left\| \frac{\zeta}{\rho_0} \right\|_{L^\infty} \leq 1 + \varepsilon$. Let $a \in (0, \varepsilon)$ be sufficiently small so that

$$\int_{[\zeta < a]} |\Delta \zeta| < \varepsilon.$$

Let

$$u = a - (a - \zeta)^+ \quad \text{in } \overline{\Omega}.$$

In particular, $0 \leq u < \varepsilon$ in Ω. It is easy to see that $\Delta u \in \mathcal{M}(\Omega)$ and $\Delta u = \Delta \zeta$ in $[\zeta < a]$. Since u is bounded and achieves its maximum everywhere on the set $[\zeta \geq a]$, we can apply Corollary 1.3 in [5] to deduce that

$$-\Delta u \geq 0 \quad \text{in } [\zeta \geq a].$$

Thus,

$$\begin{aligned} \|\Delta u\|_{\mathcal{M}} &= -\int_{[\zeta \geq a]} \Delta u + \int_{[\zeta < a]} |\Delta \zeta| \\ &\leq -\int_\Omega \Delta u + 2 \int_{[\zeta < a]} |\Delta \zeta| \leq -\int_\Omega \Delta u + 2\varepsilon. \end{aligned} \tag{4.6}$$

On the other hand, proceeding as in the proof of Lemma 7, one can find $\psi \in C_0^2(\overline{\Omega})$ such that $0 \le \psi \le \varepsilon$ in Ω, $-\frac{\partial \psi}{\partial n} \ge 1$ on $\partial\Omega$,

$$\left\| \frac{\psi}{\rho_0} \right\|_{L^\infty} \le \left\| \frac{u}{\rho_0} \right\|_{L^\infty} + \varepsilon \le 1 + 2\varepsilon, \tag{4.7}$$

and

$$\int_\Omega |\Delta\psi| \le \|\Delta u\|_{\mathcal{M}} + \varepsilon. \tag{4.8}$$

By (4.6) and (4.8), we have

$$\int_\Omega |\Delta\psi| \le -\int_\Omega \Delta u + 3\varepsilon.$$

Since $u = \zeta$ in a neighborhood of $\partial\Omega$,

$$\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_{\partial\Omega} \frac{\partial \zeta}{\partial n}.$$

Thus,

$$\int_\Omega |\Delta\psi| \le -\int_{\partial\Omega} \frac{\partial \zeta}{\partial n} + 3\varepsilon \le \mathcal{H}^{N-1}\big(N_\delta(K) \cap \partial\Omega\big) + 3\varepsilon \le \mathcal{H}^{N-1}(K) + 4\varepsilon.$$

This concludes the proof of the lemma.

Proof of Theorem 4. Given $\varepsilon > 0$, let $\psi \in C_0^2(\overline{\Omega})$ be the function given by Lemma 7. Since $\psi \ge 0$ in Ω, we have $-\frac{\partial \psi}{\partial n} \ge 0$ on $\partial\Omega$. Thus, integrating by parts and using (4.1) we get

$$\mathcal{H}^{N-1}(K) \le -\int_{\partial\Omega} \frac{\partial \psi}{\partial n} = -\int_{\partial\Omega} \Delta\psi \le \int_{\partial\Omega} |\Delta\psi| \le c_{\partial\Omega}(K) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\mathcal{H}^{N-1}(K) \le c_{\partial\Omega}(K).$$

The reverse inequality immediately follows from Lemma 8.

5 Nonnegative measures which are good for every g must belong to $L^1(\partial\Omega)$

We start with

Theorem 7 *Given a Borel set $\Sigma \subset \partial\Omega$ of zero $\mathcal{H}^{N-1}$-measure, there exists g such that*

$$\mu^* = -\mu^- \quad \textit{for every measure } \mu \textit{ concentrated on } \Sigma.$$

In particular, for every nonnegative $\mu \in \mathcal{M}(\partial\Omega)$ concentrated on a set of zero $\mathcal{H}^{N-1}$-measure, there exists some g such that $\mu^* = 0$.

Proof. Let $\Sigma \subset \partial\Omega$ be a Borel set such that $\mathcal{H}^{N-1}(\Sigma) = 0$. Let (K_k) be an increasing sequence of compact subsets of Σ such that

$$\mu^+\Big(\Sigma \setminus \bigcup_k K_k\Big) = 0. \tag{5.1}$$

For each $k \geq 1$, K_k has zero $\mathcal{H}^{N-1}$-measure. By Lemma 8, one can find $\psi_k \in C_0^2(\overline{\Omega})$ such that $0 \leq \psi_k \leq \min\{\frac{1}{k}, 2\rho_0\}$ in Ω, $-\frac{\partial\psi_k}{\partial n} \geq 1$ in some neighborhood of K_k, and

$$\int_\Omega |\Delta\psi_k| \leq \frac{1}{k} \quad \forall k \geq 1.$$

In particular,

$$\frac{\Delta\psi_k}{\rho_0} \to 0 \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

Passing to a subsequence if necessary, we may assume that

$$\frac{\Delta\psi_k}{\rho_0} \to 0 \quad \text{a.e.} \quad \text{and} \quad \frac{|\Delta\psi_k|}{\rho_0} \leq G \in L^1(\Omega; \rho_0\, dx) \quad \forall k \geq 1.$$

According to a theorem of De La Vallée-Poussin (see [6, Remarque 23] or [7, Théorème II.22]), there exists a convex function $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, $h(s) > 0$ for $s > 0$,

$$\lim_{t\to\infty} \frac{h(t)}{t} = +\infty, \quad \text{and} \quad h(G) \in L^1(\Omega; \rho_0\, dx).$$

Set $h(s) = +\infty$ for $s < 0$. Let $g = h^*$ be the convex conjugate of h. Note that h^* is finite in view of the coercivity of h, and we have $h^*(t) = 0$ if $t \leq 0$.
We claim that g satisfies all the required properties. In fact, let μ be any measure concentrated on Σ and set $\nu = (\mu^*)^+$, where the reduced measure μ^* is computed with respect to g. By Proposition 5, ν is a good measure. Let $u \in L^1(\Omega)$, $u \geq 0$ a.e., be such that $g(u)\rho_0 \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\nu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{5.2}$$

Recall that $\psi_k \geq 0$ in Ω and $\psi_k = 0$ on $\partial\Omega$; thus, $-\frac{\partial\psi_k}{\partial n} \geq 0$ on $\partial\Omega$. Using ψ_k as a test function in (5.2), we get

$$\nu(K_k) \leq -\int_{\partial\Omega} \frac{\partial\psi_k}{\partial n}\, d\nu \leq -\int_\Omega \big|u\Delta\psi_k + g(u)\psi_k\big|. \tag{5.3}$$

Note that

$$\big|u\Delta\psi_k + g(u)\psi_k\big| \to 0 \quad \text{a.e.}$$

and

$$\begin{aligned}
\big|u\Delta\psi_k + g(u)\psi_k\big| &\le u\frac{|\Delta\psi_k|}{\rho_0}\rho_0 + g(u)\frac{\psi_k}{\rho_0}\rho_0\\
&\le g(u)\rho_0 + h\left(\frac{|\Delta\psi_k|}{\rho_0}\right)\rho_0 + 2g(u)\rho_0\\
&\le 3g(u)\rho_0 + G\rho_0 \in L^1(\Omega).
\end{aligned}$$

By dominated convergence, we conclude that the right-hand side of (5.3) converges to 0 as $k \to \infty$. Thus,

$$(\mu^*)^+(K_k) = \nu(K_k) = 0 \quad \forall k \ge 1,$$

so that, by (5.1) and Proposition 8, $(\mu^*)^+(\Sigma) = 0$. Since μ is concentrated on Σ, we have $(\mu^*)^+ = 0$; thus, by Proposition 9,

$$\mu^* = (\mu^*)^+ - (\mu^*)^- = -\mu^-,$$

which is the desired result.

We now present the

Proof of Theorem 3. Assume $\mu \in \mathcal{M}(\partial\Omega)$ is good for every g. Given a Borel set $\Sigma \subset \partial\Omega$ of zero $\mathcal{H}^{N-1}$-measure, let $\nu = \mu^+\lfloor_\Sigma$. By Theorem 7, there exists some g_0 such that $\nu^* = 0$. On the other hand, by Propositions 1 and 5, ν is good for g_0. Thus, $\nu = \nu^* = 0$. In other words,

$$\mu^+(\Sigma) = 0 \quad \text{for every Borel set } \Sigma \subset \partial\Omega \text{ such that } \mathcal{H}^{N-1}(\Sigma) = 0.$$

We conclude that $\mu^+ \in L^1(\partial\Omega)$.

6 How to construct good measures which are not in $L^1(\partial\Omega)$

In this section, we establish Theorem 5. We shall closely follow the strategy used in [24] to construct good measures for problem (1.3) which are not diffuse.

Let (ℓ_k) be a decreasing sequence of positive numbers such that

$$\ell_1 < \frac{1}{2} \quad \text{and} \quad \ell_{k+1} < \frac{1}{2}\ell_k \quad \forall k \ge 1. \tag{6.1}$$

We start by briefly recalling the construction of the Cantor set $F \subset [-\frac{1}{2}, \frac{1}{2}]^{N-1}$ associated to the subsequence (ℓ_{k_j}). We refer the reader to [24, Section 2] for details.

We proceed by induction as follows. Let $F_0 = [-\frac{1}{2}, \frac{1}{2}]^{N-1}$, $\ell_0 = 1$ and $k_0 = 0$. Let F_j be the set obtained after the j-th step; F_j is the union of $2^{(N-1)k_j}$ cubes Q_i of side ℓ_{k_j}. Inside each Q_i, select $2^{(N-1)(k_{j+1}-k_j)}$ cubes $Q_{i,n}$ of side $\ell_{k_{j+1}}$ uniformly distributed in Q_i; the distance between the centers of any two cubes $Q_{i,n}$ is $\gtrsim \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}}$. Let

$$F_{j+1} = \bigcup_{i,n} Q_{i,n}.$$

The set F is given by

$$F = \bigcap_{j=0}^{\infty} F_j.$$

We now fix a diffeomorphism

$$\Phi : (-1,1)^{N-1} \to \Phi\big((-1,1)^{N-1}\big) \subset \partial\Omega$$

and define $\hat{F} = \Phi(F)$. From now on, we shall identify $\hat{F}$ with F, and simply denote $\hat{F}$ by F.

For each $j \geq 1$, let

$$\mu_j = \frac{1}{\mathcal{H}^{N-1}(F_{j+1})} \chi_{F_{j+1}};$$

in particular, $\mu_j \in L^1(\partial\Omega)$. The *uniform measure concentrated on* F, μ_F, is the weak* limit of (μ_j) in $\mathcal{M}(\partial\Omega)$ as $j \to \infty$. In particular, $\mu_F \geq 0$ and $\mu_F(\partial\Omega) = 1$. An important property satisfied by μ_F is given by the next

Lemma 9 *For every $x \in \partial\Omega$, we have*

$$\mu_F\big(B_r(x) \cap \partial\Omega\big) \lesssim \begin{cases} \frac{1}{2^{(N-1)k_{j+1}}} & \text{if } \ell_{k_{j+1}} \lesssim r \lesssim \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}}, \\ \frac{1}{2^{(N-1)k_j}} \left(\frac{r}{\ell_{k_j}}\right)^{N-1} & \text{if } \frac{\ell_{k_j}}{2^{(k_{j+1}-k_j)}} \lesssim r \lesssim \ell_{k_j}. \end{cases} \tag{6.2}$$

We say that $a \lesssim b$ if there exists $C > 0$, depending only on N, such that $a \leq C\, b$. By $a \sim b$, we mean that $a \lesssim b$ and $b \lesssim a$. We refer the reader to [24] for a proof of Lemma 9; although a slightly stronger assumption than (6.1) is made there, the proof of (6.2) remains unchanged.

Let $v \in L^1(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ \quad v = \mu_F & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Our next step is to establish the following

Proposition 12 *Let $F \subset \partial\Omega$ be the Cantor set associated to the subsequence (ℓ_{k_j}) and let v be the solution of* (6.3). *Assume that*

$$\frac{2^{k_{j+1}}\ell_{k_{j+1}}}{2^{k_j}\ell_{k_j}} \sim 1 \quad \forall j \geq 1. \tag{6.4}$$

Then, there exists $C > 0$ such that

$$v(x) \leq C\Bigg\{ \frac{1}{\ell_{k_1}^{N-1}} + \sum_{i=1}^{j} \frac{1}{2^{(N-1)k_i}\ell_{k_i}^{N-1}} \left(\frac{\ell_{k_j}}{\ell_{k_i}}\right) + \\ + \sum_{i=j+1}^{\infty} \frac{1}{2^{(N-1)k_i}\ell_{k_i}^{N-1}} \left(\frac{\ell_{k_i}}{\ell_{k_{j+1}}}\right)^{N+1} \Bigg\} \tag{6.5}$$

for every $x \in \Omega$ such that $\ell_{k_{j+1}} < d(x, \partial\Omega) \leq \ell_{k_j}$, $j \geq 1$.

Proof. We shall suppose for simplicity that $\Omega = \mathbb{R}^N_+$ is the upper-half space. In this case, the solution v of (6.3) can be explicitly written as (see Lemma 10 below)

$$v(z,t) = Nc_N \int_0^\infty \frac{st}{(s^2+t^2)^{\frac{N}{2}+1}}\, \mu_F\big(B_s(z) \cap \partial\mathbb{R}^N_+\big)\, ds \quad \forall z \in \mathbb{R}^{N-1} \quad \forall t > 0,$$

where $c_N = \frac{\Gamma(N/2)}{\pi^{N/2}}$. Applying Lemma 9, we have

$$v(z,t) \lesssim \sum_{i=1}^{\infty} (A_i + B_i) + C_0, \tag{6.6}$$

where

$$A_i = \frac{1}{2^{(N-1)k_{i+1}}} \int_{\ell_{k_{i+1}}}^{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}} \frac{st}{(s^2+t^2)^{\frac{N}{2}+1}}\, ds,$$

$$B_i = \frac{t}{2^{(N-1)k_i}\ell_{k_i}^{N-1}} \int_{\frac{\ell_{k_i}}{2^{(k_{i+1}-k_i)}}}^{\ell_{k_i}} \frac{s^N}{(s^2+t^2)^{\frac{N}{2}+1}}\, ds,$$

$$C_0 = \int_{\ell_{k_1}}^{\infty} \frac{st}{(s^2+t^2)^{\frac{N}{2}+1}}\, ds.$$

An elementary (but tedious) computation using (6.4) shows that

$$A_i \lesssim \begin{cases} \dfrac{1}{2^{(N-1)k_{i+1}}\ell_{k_{i+1}}^{N-1}}\left(\dfrac{\ell_{k_{i+1}}}{t}\right)^{N+1} & \text{if } t > \ell_{k_{i+1}}, \\ \dfrac{1}{2^{(N-1)k_{i+1}}\ell_{k_{i+1}}^{N-1}}\left(\dfrac{t}{\ell_{k_{i+1}}}\right) & \text{if } t \leq \ell_{k_{i+1}}, \end{cases} \tag{6.7}$$

$$B_i \lesssim \begin{cases} \dfrac{1}{2^{(N-1)k_i}\ell_{k_i}^{N-1}}\left(\dfrac{\ell_{k_i}}{t}\right)^{N+1} & \text{if } t > \ell_{k_i}, \\ \dfrac{1}{2^{(N-1)k_i}\ell_{k_i}^{N-1}} & \text{if } \ell_{k_{i+1}} < t \leq \ell_{k_i}, \\ \dfrac{1}{2^{(N-1)k_{i+1}}\ell_{k_{i+1}}^{N-1}}\left(\dfrac{t}{\ell_{k_{i+1}}}\right) & \text{if } t \leq \ell_{k_{i+1}}, \end{cases} \tag{6.8}$$

$$C_0 \lesssim \begin{cases} \dfrac{1}{t^{N-1}} & \text{if } t > \ell_{k_1}, \\ \dfrac{t}{\ell_{k_1}^N} & \text{if } t \leq \ell_{k_1}. \end{cases} \tag{6.9}$$

We now assume that $\ell_{k_{j+1}} < t \leq \ell_{k_j}$. Inserting (6.7)–(6.9) into (6.6), we obtain (6.5). In order to conclude the proof of Proposition 12, we establish the following

Lemma 10 *Given $\nu \in \mathcal{M}(\mathbb{R}^{N-1})$, let w be the solution of*

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}_+^N, \\ \quad w = \nu & \text{on } \partial\mathbb{R}_+^N. \end{cases} \tag{6.10}$$

Then,

$$w(z,t) = Nc_N \int_0^\infty \frac{st}{(s^2+t^2)^{\frac{N}{2}+1}} \nu\big(\tilde{B}_s(z)\big)\, ds \quad \forall z \in \mathbb{R}^{N-1} \quad \forall t > 0, \tag{6.11}$$

where $\tilde{B}_s(z)$ denotes the ball in $\partial\mathbb{R}_+^N$ of radius s centered at z.

Proof. Assume $\mu = f \in C_c^\infty(\mathbb{R}^{N-1})$. Then, w is given as the Poisson integral of f:

$$w(z,t) = c_N \int_{\mathbb{R}^{N-1}} \frac{t}{\big(|x-z|^2+t^2\big)^{\frac{N}{2}}} f(x)\, dx \quad \forall z \in \mathbb{R}^{N-1} \quad \forall t > 0.$$

Thus,

$$\begin{aligned} w(z,t) &= c_N \int_0^\infty \frac{t}{(s^2+t^2)^{\frac{N}{2}}} \left(\int_{\partial\tilde{B}_s(z)} f\right) ds \\ &= c_N \int_0^\infty \frac{t}{(s^2+t^2)^{\frac{N}{2}}} \frac{d}{ds}\left(\int_{\tilde{B}_s(z)} f\right) ds. \end{aligned}$$

Integrating by parts with respect to s, we obtain (6.11) for $\mu = f$. This establishes (6.11) when μ is a smooth function. The general case easily follows using a density argument (see, e.g., [20, Lemma 1.4]).

We may now turn to the

Proof of Theorem 5. Let (k_j) be an increasing sequence of positive integers such that

$$g(2^{Nj}) \le 2^{2k_j} \quad \forall j \ge 1. \tag{6.12}$$

Let (ℓ_k) be any sequence satisfying (6.1) and such that

$$\ell_{k_j} = \frac{1}{2^{j+k_j}} \quad \forall j \ge 1.$$

Let F be the Cantor set associated to (ℓ_{k_j}). Since

$$2^{(N-1)k_j} \ell_{k_j}^{N-1} = \frac{1}{2^{(N-1)j}} \to 0 \quad \text{as } j \to \infty,$$

we have $|F| = 0$; thus, $\mu_F \notin L^1(\partial\Omega)$. We claim that μ_F is a good measure. In fact, let v be the solution of (6.3). A simple computation shows that

$$\sum_{i=1}^{j} \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left(\frac{\ell_{k_j}}{\ell_{k_i}} \right) + \sum_{i=j+1}^{\infty} \frac{1}{2^{(N-1)k_i} \ell_{k_i}^{N-1}} \left(\frac{\ell_{k_i}}{\ell_{k_{j+1}}} \right)^{N+1} \le C\, 2^{(N-1)j}$$

for some constant $C > 0$ sufficiently large. It follows from Proposition 12 that

$$v(x) \le \tilde{C}\, 2^{(N-1)j} \quad \text{if } \ell_{k_{j+1}} < d(x, \partial\Omega) \le \ell_{k_j} \quad \forall j \ge 1.$$

Denoting $\Omega_j = \big\{ x \in \Omega \,;\, d(x, \partial\Omega) > \ell_{k_j} \big\}$, we then have

$$\begin{aligned}
\int_\Omega g(v)\rho_0 &= \sum_{j=1}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} g(v)\rho_0 + \int_{\Omega \setminus \Omega_1} g(v)\rho_0 \\
&\le C \sum_{j=1}^{\infty} g\big(\tilde{C}\, 2^{(N-1)j}\big)\, \ell_{k_j} |\Omega_{j+1} \setminus \Omega_j| + O(1).
\end{aligned}$$

Since $|\Omega_{j+1} \setminus \Omega_j| \le C \ell_{k_j}$, we get

$$\int_\Omega g(v)\rho_0 \le C \sum_{j=1}^{\infty} \frac{g\big(\tilde{C}\, 2^{(N-1)j}\big)}{2^{2(j+k_j)}} + O(1). \tag{6.13}$$

Note that, for $j \ge 1$ sufficiently large, we have $\tilde{C}\, 2^{(N-1)j} \le 2^{Nj}$. We deduce from (6.12) and (6.13) that $g(v) \in L^1(\Omega; \rho_0\, dx)$. By Proposition 7, we conclude that μ_F is a good measure.

7 The case where $g(t) = t^p$

We describe here some examples where the measure μ^* can be explicitly identified.

Example 1 $g(t) = t^p$, $t \geq 0$, with $1 < p < \frac{N+1}{N-1}$.

In this case, every measure is good (see [15]); thus, $\mu^* = \mu$, $\forall \mu \in \mathcal{M}(\partial\Omega)$.

Example 2 $g(t) = t^p$, $t \geq 0$, with $p \geq \frac{N+1}{N-1}$.

By [21], a nonnegative measure ν is good if and only if $\nu(A) = 0$ for every Borel set $A \subset \partial\Omega$ such that $C_{2/p,p'}(A) = 0$. Recall (see [13]) that any measure μ can be uniquely decomposed as

$$\mu = \mu_1 + \mu_2,$$

where $\mu_1(A) = 0$ for every Borel set $A \subset \partial\Omega$ such that $C_{2/p,p'}(A) = 0$, and μ_2 is concentrated on a set of zero $C_{2/p,p'}$-capacity. Using the same argument as in [4, Section 8], one then shows that for every $\mu \in \mathcal{M}(\partial\Omega)$ we have

$$\mu^* = \mu - \mu_2^+.$$

Here is an interesting

Open Problem 1 Let $N = 2$ and $g(t) = \mathrm{e}^t - 1$, $t \geq 0$. Is there a simple characterization of the set of good measures relative to g? Is there an explicit formula of μ^* in terms of μ?

There are some partial results in this direction; see [16] and also [23].

8 Proof of Theorem 6

We start with the following

Lemma 11 *Let $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\partial\Omega)$. Assume that there exists $w \in L^1(\Omega)$ such that $g(w) \in L^1(\Omega; \rho_0\, dx)$ and*

$$-\int_\Omega w\Delta\zeta + \int_\Omega g(w)\zeta \geq \int_\Omega \zeta\, d\lambda - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0. \tag{8.1}$$

Then, the pair (λ, μ) is good.

Proof. Since (8.1) holds, there exist $\mu_0 \in \mathcal{M}(\partial\Omega)$ and a locally bounded measure λ_0 in Ω, with $\int_\Omega \rho_0\, d|\lambda_0| < \infty$, such that $\mu_0 \geq \mu$ on $\partial\Omega$, $\lambda_0 \geq \lambda$ in Ω, and

$$-\int_\Omega w\Delta\zeta + \int_\Omega g(w)\zeta = \int_\Omega \zeta\, d\lambda_0 - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu_0 \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

(The existence of λ_0 and μ_0 is sketched in [4, Remark B.1]).
Let (g_k) be a sequence of bounded functions satisfying (1.6)–(1.7). Let u_k, w_k be the solutions associated to $(\lambda, \mu), (\lambda_0, \mu_0)$, resp. Then, as in the proof of Lemma 5 above, we have

$$g_k(u_k) \le g_k(w_k) \to g(w) \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

On the other hand, $u_k \downarrow u$ in $L^1(\Omega)$. Thus, by dominated convergence,

$$g_k(u_k) \to g(u) \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

We conclude that u satisfies (1.12). Therefore, (λ, μ) is good.

Proof of Theorem 6.
Step 1. Proof of

$$(\lambda, \mu)^* = (\lambda^*, \mu^*). \tag{8.2}$$

Let u_k be such that

$$\begin{cases} -\Delta u_k + g_k(u_k) = \lambda & \text{in } \Omega, \\ \hfill u_k = \mu & \text{on } \partial\Omega. \end{cases}$$

Then, $u_k \downarrow \hat{u}$ in $L^1(\Omega)$. By Fatou, we deduce that $g(\hat{u}) \in L^1(\Omega; \rho_0\, dx)$ and

$$-\int_\Omega \hat{u}\Delta\zeta + \int_\Omega g(\hat{u})\zeta \le \int_\Omega \zeta\, d\lambda - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0.$$

By [4, Remark B.1], there exist $\hat{\mu} \in \mathcal{M}(\partial\Omega)$ and a locally bounded measure $\hat{\lambda}$ in Ω, with $\int_\Omega \rho_0\, d|\hat{\lambda}| < \infty$, such that

$$-\int_\Omega \hat{u}\Delta\zeta + \int_\Omega g(\hat{u})\zeta = \int_\Omega \zeta\, d\hat{\lambda} - \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}\, d\hat{\mu} \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Note that $\hat{\lambda} \le \lambda$ in Ω and $\hat{\mu} \le \mu$ on $\partial\Omega$. We claim that

(*a*) $(\hat{\lambda})_{\mathrm{d}} = \lambda_{\mathrm{d}} = (\lambda^*)_{\mathrm{d}}$;

(*b*) $(\hat{\lambda})_{\mathrm{c}} = (\lambda^*)_{\mathrm{c}}$;

(*c*) $\hat{\mu} = \mu^*$.

The subscripts "d" and "c" denote the diffuse and the concentrated parts of the measure with respect to cap_{H^1} (see [13]). We then deduce from (*a*) and (*b*) that $\hat{\lambda} = \lambda^*$; in particular, $\hat{\lambda} \in \mathcal{M}(\Omega)$.

Proof of (*a*). The second equality in (*a*) is established in [4]. Proceeding exactly as in the proof of Lemma 1 there, one shows that

$$\hat{\lambda} \ge \lambda_{\mathrm{d}} - \lambda_{\mathrm{c}}^-.$$

Thus, $(\hat{\lambda})_{\mathrm{d}} \geq \lambda_{\mathrm{d}}$. Since $\hat{\lambda} \leq \lambda$, we conclude that $(\hat{\lambda})_{\mathrm{d}} = \lambda_{\mathrm{d}}$.

Proof of (b). Since the pair $(\lambda^*, 0)$ is good, it follows from Lemma 11 above that $(\lambda^*, -\mu^-)$ is also good. Let v_1 be the solution of (1.12) corresponding to $(\lambda^*, -\mu^-)$. By [4, Corollary B.2], we have $v_1 \leq u_k$ a.e., $\forall k \geq 1$. Thus,

$$v_1 \leq \hat{u} \quad \text{a.e.}$$

By the "Inverse" maximum principle (see [8]), we obtain

$$(\lambda^*)_{\mathrm{c}} = (-\Delta v_1)_{\mathrm{c}} \leq (-\Delta \hat{u})_{\mathrm{c}} = (\hat{\lambda})_{\mathrm{c}}. \tag{8.3}$$

We conclude from (a) and (8.3) that

$$\lambda^* \leq \hat{\lambda} \leq \lambda.$$

In particular, $\hat{\lambda} \in \mathcal{M}(\Omega)$. Since $(\hat{\lambda}, \hat{\mu})$ is good, we can apply Lemma 11 to deduce that $(\hat{\lambda}, -(\hat{\mu})^-)$ is also good. Let v_2 denote the corresponding solution. Clearly, v_2 is a subsolution of (1.3). Thus,

$$v_2 \leq v^* \quad \text{a.e.},$$

where v^* is the largest subsolution of (1.3), i.e., v^* is the solution of (1.3) with data λ^*. Applying the "Inverse" maximum principle, we conclude that

$$(\hat{\lambda})_{\mathrm{c}} = (-\Delta v_2)_{\mathrm{c}} \leq (-\Delta v^*)_{\mathrm{c}} = (\lambda^*)_{\mathrm{c}}. \tag{8.4}$$

We deduce from (8.3) and (8.4) that $(\hat{\lambda})_{\mathrm{c}} = (\lambda^*)_{\mathrm{c}}$.

Proof of (c). The argument in this case is the same as in the proof of (b) and is omitted (one should use Lemma 1 in Section 2 above, instead of the "Inverse" maximum principle).

It now follows from (a)–(c) that $\hat{\lambda} = \lambda^*$ and $\hat{\mu} = \mu^*$. This concludes the proof of Step 1.

Step 2. Proof of the theorem completed.

Assume (λ, μ) is good. Thus, $(\lambda, \mu)^* = (\lambda, \mu)$. We deduce from the previous step that $\lambda^* = \lambda$ and $\mu^* = \mu$. In other words, λ is a good measure for (1.3) and μ is good for (1.1). Similarly, the converse follows. The proof of Theorem 6 is complete.

Open Direction 1 In all the problems above, the equation in Ω is nonlinear but the boundary condition is the usual Dirichlet condition. It might be interesting to investigate problems involving nonlinear boundary conditions. Here is a typical example:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + g(u) = \mu & \text{on } \partial\Omega, \end{cases} \tag{8.5}$$

where g and μ are as in the Introduction. This type of problems arises in Mechanics for various choices of g, possibly graphs; see, e.g., [9]. They have been studied in [2] when $\mu \in L^2(\partial\Omega)$.

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