The continuity of functions with N-th derivative measure

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Abstract. We study the continuity of functions u whose mixed derivative $\partial_1 \ldots \partial_N u$ is a measure. If $u \in W^{1,1}(\mathbb{R}^N)$, then we prove that u is continuous. The same conclusion holds for $u \in W^{k,p}(Q)$, with $kp > N - 1$, where Q denotes a cube in $\mathbb{R}^N$. The key step in the proof consists in showing that the measure $\partial_1 \cdots \partial_N u$ does not charge hyperplanes orthogonal to the coordinate axes.

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1 Introduction

The classical Sobolev embedding theorem states that

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{N},$$

for every $k \geq 1$ integer and $1 \leq p < \infty$ such that $kp < N$. In the borderline case, namely $kp = N$, then functions in $W^{k,p}(\mathbb{R}^N)$ need not be bounded (or even locally bounded), except when $k = N$ and $p = 1$. In fact,

Proposition 1.1 *If $u \in W^{N,1}(\mathbb{R}^N)$, then $u \in L^\infty(\mathbb{R}^N)$ and*

$$\|u\|_{L^\infty} \leq \int_{\mathbb{R}^N} |\partial_1 \cdots \partial_N u|. \tag{1.1}$$

Moreover, u is continuous and

$$\big|u(x) - u(y)\big| \leq \int_{\overline{Q}_\varepsilon} |\partial_1 \cdots \partial_N u| + C\|\nabla u\|_{L^N(Q_\varepsilon)} \quad \forall x, y \in Q_\varepsilon, \tag{1.2}$$

where Q_ε is any parallel cube of side $2\varepsilon > 0$ and $C > 0$ does not depend on ε.

We denote by ∂_i the derivative with respect to the x_i-variable; a cube Q_ε is *parallel* if its sides are parallel to the hyperplanes $[x_i = 0]$ for every $i = 1, \ldots, N$. In Remark 3.1 we explain how one proves (1.2); in (1.2) we make use of the Sobolev embedding $W^{N,1} \hookrightarrow W^{1,N}$. The continuity of u can be deduced either from (1.1) (via approximation of u by convolution) or directly from (1.2).

If $u \in W^{N-1,1}(\mathbb{R}^N)$ and $D^N u$ is merely a finite measure, then (1.1) still holds (easily checked via approximation); in particular, one deduces that $u \in L^\infty(\mathbb{R}^N)$.

A natural question is whether u is continuous. Simple examples show that the answer is *no* if $N = 1$ (here, $W^{0,1} = L^1$). In dimension $N \geq 2$, it turns out that the answer is *yes*, but it is more delicate to prove. Using Lorentz spaces, L. Tartar established the following

Theorem 1.1 *Let $N \geq 2$. If $u \in W^{N-1,1}(\mathbb{R}^N)$ is such that $D^N u$ is a measure, then u is continuous.*

We refer the reader to [2] for the proof of L. Tartar (with a contribution by A. Cohen).

An alternative approach to prove Theorem 1.1 goes as follows. By approximation, (1.2) still holds for a.e. $x, y \in Q_\varepsilon$. Since $D^N u$ is the derivative of $D^{N-1}u$ in the sense of distributions, it follows from the theory of BV-functions (see, e.g., [3]) that

$$|D^N u|\big(\{z\}\big) = 0 \quad \forall z \in \mathbb{R}^N, \tag{1.3}$$

i.e., the measure $|D^N u|$ does not contain Dirac masses. Thus, by dominated convergence, one deduces that

$$\lim_{\varepsilon \to 0} \int_{\overline{Q}_\varepsilon} |D^N u| = 0 \tag{1.4}$$

for every family of cubes $(Q_\varepsilon)_{\varepsilon > 0}$. The continuity of u then easily follows from (1.2) and (1.4).

This argument simplifies Tartar's proof, but it still relies on the theory of BV-functions. On the other hand, in both inequalities (1.1) and (1.2), only $\partial_1 \cdots \partial_N u$ comes into play. The goal of this paper is to clarify the role of $\partial_1 \cdots \partial_N u$ and improve Theorem 1.1.

We assume from now on that $N \geq 2$. One of our main results is

Theorem 1.2 *If $u \in W^{1,1}(\mathbb{R}^N)$ and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous and bounded.*

We recall that $\partial_1 \cdots \partial_N u$ is a measure if there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^N} u \, \partial_1 \cdots \partial_N \varphi \right| \leq C \|\varphi\|_{L^\infty} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

As one sees by considering u equal to the characteristic function of a cube, the condition that $\partial_1 \cdots \partial_N u$ is a measure is not sufficient to ensure the continuity of u. In this example, u belongs to $BV(\mathbb{R}^N)$, but not to $W^{1,1}(\mathbb{R}^N)$.

In what follows, we denote by $Q = Q_1$ the cube $(-1,1)^N$. We point out that the conclusion of Theorem 1.2 still holds for functions $u \in W_0^{1,1}(Q)$ such that $\partial_1 \cdots \partial_N u$ is a measure; see Corollary 2.1. Without any restrictions on the boundary, one has the

Theorem 1.3 *Let $u \in W^{N-1,1}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. Then u is continuous in $\overline{Q}$.*

Note however that in dimension $N \geq 3$ there are functions $u \in W^{N-2,1}(Q)$, with $\partial_1 \cdots \partial_N u$ measure, which are not continuous. In fact, take any

$$v \in W^{N-2,1}\big((-1,1)^{N-1}\big)$$

which is discontinuous and define

$$u(x', x_N) = v(x') \quad \forall (x', x_N) \in Q.$$

Then, $\partial_1 \cdots \partial_N u = 0$ but u is not continuous.

In the same spirit, we have the

Theorem 1.4 *Let $u \in W^{k,p}(Q)$, $k \geq 1$ integer and $1 < p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. If $kp > N-1$, then u is continuous in $\overline{Q}$.*

According to Theorem 1.3, the conclusion of Theorem 1.4 holds if $kp = N-1$ and $p = 1$. Simple examples show that this is no longer true if $kp = N-1$ and $p > 1$.

As a corollary of Theorems 1.3 and 1.4, one immediately deduces counterparts of these results for functions $u \in W^{k,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set. In this case, one shows that u is continuous in Ω. Note, however, that u need not be continuous on $\overline{\Omega}$, even if Ω is a ball. In fact, for any $a \in (0, \frac{1}{2})$, let

$$u(x_1, x_2) = \frac{1}{(1-x_1)^a} \quad \forall (x_1, x_2) \in B_1 \subset \mathbb{R}^2.$$

Then, $u \in W^{1,1}(B_1)$ and $\partial_1 \partial_2 u = 0$ in B_1, but u is not uniformly bounded in B_1.

In Theorems 1.2–1.4, the measure $\partial_1 \cdots \partial_N u$ need not be the derivative of a BV-function. It is then unlikely that one can rely on the BV-theory in order to deduce (1.3) as above. One of the main ingredients to establish Theorems 1.2–1.4 is the next

Proposition 1.2 *Let $u \in W^{1,1}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. Then, $|\partial_1 \cdots \partial_N u|$ does not charge hyperplanes of the form $[x_i = t]$, $\forall i \in \{1, \ldots, N\}$, $\forall t \in (-1,1)$.*

Here,

$$[x_i = t] := \Big\{ x \in \mathbb{R}^N \, ; \, x_i = t \Big\}$$

is the hyperplane passing through te_i and orthogonal to the x_i-axis. We say that a positive measure μ does not charge a measurable set A if $\mu(A) = 0$. Note that the measure $|\partial_1 \cdots \partial_N u|$ *can* charge other hyperplanes. For instance, if

$$u(x) = \max \big\{ 1 - |x_1|, \ldots, 1 - |x_N|, 0 \big\} \quad \forall x \in \mathbb{R}^N,$$

then $u \in W^{1,1}(\mathbb{R}^N)$ and $\partial_1 \cdots \partial_N u$ is a measure supported on the diagonals of the cube Q.

The paper is organized as follows. In Sections 2–3 we establish Theorems 1.2–1.4. In Sections 4–5, we prove some results—including Proposition 1.2—which we used in Sections 2–3. Finally, in Section 6 we explain how to deduce counterparts of Theorems 1.2 and 1.4 for fractionary Sobolev spaces $W^{s,p}$, with $s > 0$.

2 Proof of Theorem 1.2

We assume Proposition 1.2, which will be established in Section 4 below. Let (ρ_n) be a sequence of nonnegative radial mollifiers such that $\rho_n \in C_0^\infty(B_{1/n})$, $\forall n \geq 1$. Since $u \in L^1(\mathbb{R}^N)$, for every fixed $n \geq 1$ the function $\rho_n * u$ converges uniformly to 0 as $|x| \to +\infty$. It is then easy to see that for each $x \in \mathbb{R}^N$

$$\rho_n * u(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \rho_n * (\partial_1 \cdots \partial_N u) = \int_{\mathbb{R}^N} (\rho_n * F)\, \partial_1 \cdots \partial_N u, \tag{2.1}$$

where

$$F(y) = \begin{cases} 1 & \text{if } y_i < x_i \text{ for every } i \in \{1, \ldots, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that

$$\rho_n * F(y) \to F(y) \quad \text{for every } y \in \mathbb{R}^N \text{ such that } y_i \neq x_i,\ \forall i \in \{1, \ldots, N\}.$$

Applying Proposition 1.2 to an arbitrary cube in $\mathbb{R}^N$, we have

$$|\partial_1 \cdots \partial_N u|\big([y_i = t]\big) = 0 \quad \forall t \in \mathbb{R}, \quad \forall i = 1, \ldots, N. \tag{2.2}$$

Hence,

$$\rho_n * F \to F \quad \partial_1 \cdots \partial_N u\text{-a.e.}$$

By dominated convergence, as $n \to \infty$ in (2.1), we then get

$$u(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \partial_1 \cdots \partial_N u \quad \text{a.e.} \tag{2.3}$$

In view of (2.2), the right-hand side of (2.3) is well-defined for *every* $x \in \mathbb{R}^N$ and, by dominated convergence, is continuous. Hence, u coincides a.e. with a continuous function. This concludes the proof of Theorem 1.2.

Remark 2.1 An easy variant of (2.3) shows that $u(x) \to 0$ uniformly as $|x| \to +\infty$. We leave the details to the reader.

As a consequence of Theorem 1.2, we have

Corollary 2.1 *If $u \in W_0^{1,1}(Q)$ and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous on $\overline{Q}$.*

Proof. Let $\bar{u} \in W^{1,1}(\mathbb{R}^N)$ be the extension of u as zero outside Q. By Lemma 2.1 below, $\partial_1 \cdots \partial_N \bar{u}$ is a measure in $\mathbb{R}^N$. Thus, $u = \bar{u}$ is continuous on $\overline{Q}$ in view of Theorem 1.2.

We are left to prove the

Lemma 2.1 *Let $u \in W_0^{1,1}(Q)$. If $\partial_1 \cdots \partial_N u$ is a measure, then*

$$\int_Q u\,\partial_1 \cdots \partial_N \varphi = (-1)^N \int_Q \varphi\,\partial_1 \cdots \partial_N u \quad \forall \varphi \in C^\infty(\overline{Q}). \tag{2.4}$$

Proof. By definition, equality in (2.4) holds for every $\varphi \in C_0^\infty(Q)$. We now show that (2.4) still holds for functions $\varphi \in C^\infty(\overline{Q})$ with support in $[-1,1] \times (-1,1)^{N-1}$. Consider a sequence $(\theta_n) \subset C_0^\infty\big((-1,1)\big)$ such that $0 \le \theta_n \le 1$ and $\theta_n(t) \to 1$ as $n \to \infty$ for every $t \in (-1,1)$. Let $\varphi \in C^\infty(\overline{Q})$ have its support in $[-1,1] \times (-1,1)^{N-1}$. Clearly,

$$I_n := \int_Q u(x)\,\partial_1 \cdots \partial_N \big(\theta_n(x_1)\varphi(x)\big)\,dx = (-1)^N \int_Q \theta_n(x_1)\varphi(x)\,\partial_1 \cdots \partial_N u.$$

Since $u \in W_0^{1,1}(Q)$, we also have

$$I_n = -\int_Q \big(\partial_1 u(x)\big)\,\theta_n(x_1)\,\partial_2 \cdots \partial_N \varphi(x)\,dx.$$

Thus, applying dominated convergence and using that $u \in W_0^{1,1}(Q)$,

$$I_n \to -\int_Q (\partial_1 u)\,\partial_2 \cdots \partial_N \varphi = \int_Q u\,\partial_1 \partial_2 \cdots \partial_N \varphi \quad \text{as } n \to \infty.$$

On the other hand, since $\partial_1 \cdots \partial_N u$ is a finite measure, it also follows by dominated convergence that

$$\int_Q \theta_n(x_1)\,\varphi(x)\,\partial_1 \cdots \partial_N u \to \int_Q \varphi\,\partial_1 \cdots \partial_N u.$$

We conclude that (2.4) holds with functions supported in $[-1,1] \times (-1,1)^{N-1}$. Repeating the process with respect to the remaining $N-1$ variables, one obtains the conclusion.

3 Proofs of Theorems 1.3 and 1.4

Theorems 1.3 and 1.4 are based on the following

Theorem 3.1 *Let $u \in W^{k,p}(Q)$, $k \geq 1$ integer and $1 \leq p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. Then, there exist $v, w_1, \ldots, w_N \in W^{k,p}(Q)$ and a constant $C > 0$ (depending on N) such that*

(i) $u = v + \sum_j w_j$;

(ii) $\|v\|_{W^{k,p}} + \sum_j \|w_j\|_{W^{k,p}} \leq C\|u\|_{W^{k,p}}$;

(iii) *v is continuous and*

$$\bigl|v(x) - v(y)\bigr| \leq \int_Q |\partial_1 \cdots \partial_N u| \quad \forall x, y \in Q; \tag{3.1}$$

(iv) *$\partial_j w_j = 0$ in $\mathcal{D}'(Q)$, for every $j = 1, \ldots, N$.*

We postpone the proof of Theorem 3.1 to Section 5.

Proofs of Theorems 1.3 and 1.4. Applying Theorem 3.1, we can write u as

$$u = v + \sum_{j=1}^{N} w_j, \tag{3.2}$$

where v is continuous and each w_j is a function of $(N-1)$-variables in $W^{N-1,1}$ ($W^{k,p}$ with $kp > N-1$, resp.). Therefore, each w_j is also continuous; hence, u is continuous as well.

Remark 3.1 (Modulus of continuity of $\boldsymbol{u}$)
Assume $p > N - 1$. From Theorem 3.1, one deduces the following estimate on the modulus of continuity of a function $u \in W^{1,p}(Q)$ such that $\partial_1 \cdots \partial_N u$ is a measure:

$$\bigl|u(x) - u(y)\bigr| \leq \int_{Q_\varepsilon} |\partial_1 \cdots \partial_N u| + \frac{C}{\varepsilon^{\frac{1}{p}}} \|\nabla u\|_{L^p(Q_\varepsilon)} \, |x-y|^{1-\frac{N-1}{p}}, \tag{3.3}$$

for every $x, y \in Q_\varepsilon$, where $Q_\varepsilon \subset Q$ is an arbitrary parallel cube of side $2\varepsilon > 0$.

In fact, by scaling it suffices to establish (3.3) when $Q_\varepsilon = Q$ is the cube $(-1,1)^N$. We then apply Theorem 3.1 and write u as in (i). Note that the proof of Theorem 3.1 provides the following estimate which is slightly better than (ii):

$$\sum_{j=1}^{N} \|\nabla w_j\|_{L^p(Q)} \leq C\|\nabla u\|_{L^p(Q)}$$

By condition (iii), we have

$$\bigl|v(x) - v(y)\bigr| \leq \int_Q |\partial_1 \cdots \partial_N u| \quad \forall x, y \in Q.$$

Since w_j is independent of the variable x_j, and $w_j \in W^{1,p}(Q)$ with $p > N-1$, it follows from Morrey's inequality in dimension $N-1$ that

$$\begin{aligned}\left|w_j(x) - w_j(y)\right| &\le C\|\nabla w_j\|_{L^p(Q)} \, |x-y|^{1-\frac{N-1}{p}} \\ &\le C\|\nabla u\|_{L^p(Q)} \, |x-y|^{1-\frac{N-1}{p}} \quad \forall x, y \in Q.\end{aligned}$$

Combining the inequalities for v and w_j, we obtain (3.3) with $\varepsilon = 1$.

4 Proof of Proposition 1.2

The proof of Proposition 1.2 relies on the following two lemmas:

Lemma 4.1 *Let $v \in L^1(Q_2)$ be such that $\partial_1 \cdots \partial_N v$ is a measure. If v is odd with respect to the variables $x_1, \dots, x_{N-1}$, then there exists $\varepsilon > 0$ such that*

$$\left|v(y,s) - v(y,-t)\right| \ge \left|(\partial_1 \cdots \partial_N v)(S)\right| \tag{4.1}$$

for a.e. $y \in (1, 1+\varepsilon)^{N-1}$ and a.e. $s, t \in (0, \varepsilon)$.

Here, we use the following notation

$$Q_2 := (-2,2)^N \quad \text{and} \quad S := [-1,1]^{N-1} \times \{0\}.$$

Proof of Lemma 4.1. By outer regularity of the Radon measure $\mu := \partial_1 \cdots \partial_N v$, there exists $\varepsilon > 0$, such that

$$\left| \int_{Q_2} \varphi \, d\mu \right| \ge \frac{1}{2^{N-1}} \left|\mu(S)\right|, \tag{4.2}$$

for every $\varphi \in C_0^\infty(Q_2)$ such that $0 \le \varphi \le 1$ in Q_2, $\varphi = 1$ on S, and

$$\operatorname{supp} \varphi \subset (-1-2\varepsilon, 1+2\varepsilon)^{N-1} \times (-2\varepsilon, 2\varepsilon).$$

Let (ρ_n) be a sequence of nonnegative radial mollifiers such that $\rho_n \in C_0^\infty(B_{1/n})$, $\forall n \ge 1$. Since v is odd with respect to the variables $x_1, \dots, x_{N-1}$, one has

$$\rho_n * v(y,s) - \rho_n * v(y,-t) = 2^{N-1} \int_{-y_1}^{y_1} \cdots \int_{-y_{N-1}}^{y_{N-1}} \int_{-t}^{s} \rho_n * \mu.$$

Take $n \ge 1$ large so that $\operatorname{supp} \rho_n \subset B_\varepsilon$. By (4.2), one has

$$\left|\rho_n * v(y,s) - \rho_n * v(y,-t)\right| \ge |\mu(S)|$$

for every $y \in (1, 1+\varepsilon)^{N-1}$ and $s, t \in (0, \varepsilon)$. As $n \to \infty$, we obtain (4.1). This establishes Lemma 4.1.

Lemma 4.2 *Assume $v \in W^{1,1}(Q_2)$ satisfies the assumptions of Lemma 4.1. Then,*

$$(\partial_1 \cdots \partial_N v)\big([-1,1]^{N-1} \times \{0\}\big) = 0.$$

Proof. Let $\varepsilon > 0$ as in Lemma 4.1. Given $\delta > 0$, we denote by

$$A_\delta = (1, 1+\varepsilon)^{N-1} \times (-\delta, \delta).$$

Since

$$\int\limits_{(1,1+\varepsilon)^{N-1}} \int_0^\delta \big|v(y,s) - v(y, s-\delta)\big| \, dy \, ds \leq \delta \int_{A_\delta} |\nabla v|,$$

it follows from (4.1) that

$$\varepsilon^{N-1} \delta \, \big|(\partial_1 \cdots \partial_N v)(S)\big| \leq \delta \int_{A_\delta} |\nabla v|.$$

Dividing both sides by δ and letting $\delta \to 0$, we conclude that

$$\big|(\partial_1 \cdots \partial_N v)(S)\big| = 0.$$

We can now establish Proposition 1.2.

Proof of Proposition 1.2. Without loss of generality, we may assume that $t = 0$ and $i = N$. We split the proof in two steps.

Step 1. We show that

$$(\partial_1 \cdots \partial_N u)(S_r) = 0,$$

where $S_r := [-r, r]^{N-1} \times \{0\}$, with $0 < r < \frac{1}{2}$.

Define $v : Q_1 \to \mathbb{R}$ as

$$v(x_1, \ldots, x_N) = \sum_{\substack{\alpha_i \in \{0,1\} \\ 1 \leq i \leq N-1}} (-1)^{|\alpha|} u\big((-1)^{\alpha_1} x_1, \ldots, (-1)^{\alpha_{N-1}} x_{N-1}, x_N\big),$$

where $|\alpha| = \sum_{i=1}^{N-1} \alpha_i$. One immediately checks that v is odd with respect to the variables $x_1, \ldots, x_{N-1}$ and that

$$(\partial_1 \cdots \partial_N v)(S_r) = 2^{N-1} (\partial_1 \cdots \partial_N u)(S_r).$$

Moreover, $v \in W^{1,1}(\Omega)$; thus, the previous lemma gives the conclusion.

Step 2. Proof of the proposition completed.

Let $\tilde{S}$ be any square of the form $\prod_{i=1}^{N-1} [a_i, b_i]^{N-1} \times \{0\}$. By a variant of Step 1, we have

$$(\partial_1 \cdots \partial_N u)(\tilde{S}) = 0.$$

Thus, by inner regularity of $\partial_1 \cdots \partial_N u$,

$$(\partial_1 \cdots \partial_N u)(\omega \times \{0\}) = 0,$$

for every open set $\omega \subset (-1,1)^{N-1}$. This implies that

$$|\partial_1 \cdots \partial_N u|(S) = 0.$$

The proof of the proposition is complete.

5 Proof of Theorem 3.1

We first introduce some notation. Let $v : Q \to \mathbb{R}$ and let $P \subset Q$ be an (oriented) parallelepiped. If $N = 1$ and $P = [a,b]$, then define

$$\Delta_P v = v(b) - v(a).$$

If $N > 1$ and $P = P' \times [a,b]$, then let

$$\Delta_P v = \Delta_{P'} v(\cdot, b) - \Delta_{P'} v(\cdot, a).$$

In particular, if $v \in C^\infty(Q)$, then one has the identity

$$\Delta_P v = \int_P \partial_1 \cdots \partial_N v. \tag{5.1}$$

The proof of Theorem 3.1 is based on the next

Lemma 5.1 *Let $u \in W^{k,p}(Q)$, with $k \geq 1$ integer and $1 \leq p < \infty$. Then, for a.e. $a \in Q$ there exist $v, w_1, \dots, w_N \in W^{k,p}(Q)$ such that*

(i) $u = v + \sum_j w_j$;

(iii′) *for every $x \in Q$,*

$$v(x) = \Delta_{[a_1,x_1]\times\cdots\times[a_N,x_N]} u;$$

(iv) $\partial_j w_j = 0$ *in $\mathcal{D}'(Q)$, for every $j = 1, \dots, N$.*

Moreover, $a \in Q$ can be chosen so that

$$\|v\|_{W^{k,p}} + \sum_{j=1}^N \|w_j\|_{W^{k,p}} \leq C\|u\|_{W^{k,p}}, \tag{5.2}$$

for some $C > 0$ independent of u.

The proof of Lemma 5.1 is based on a standard Fubini-type argument. We present here a sketch of the proof.

Proof of Lemma 5.1. For every $a \in Q$, let

$$v_a(x) = \Delta_{[a_1,x_1]\times\cdots\times[a_N,x_N]} u \quad \text{and} \quad g_a(x) = u(x) - v_a(x).$$

Observe that g_a can be written as a sum of N functions $w_{1,a}, \ldots, w_{N,a}$ depending on at most $(N-1)$ components of x, and each $w_{j,a}$ is obtained by (sums of) restrictions of u with respect to parallel affine subspaces passing through a.
Let (u_n) be a sequence in $C^\infty(\overline{Q})$ such that

$$u_n \to u \quad \text{in } W^{k,p}(Q).$$

For each $a \in Q$, define $v_{n,a}$ and $g_{n,a}$ accordingly. By Fubini, for a.e. $a \in Q$ we have

$$g_{n,a} \to g_a \quad \text{in } W^{k,p}(Q). \tag{5.3}$$

Therefore,

$$v_{n,a} \to v_a \quad \text{in } W^{k,p}(Q).$$

In order to obtain (5.2), note that by Fubini there exists a set of positive measure $A \subset Q$ such that

$$\|w_{j,a}\|_{W^{k,p}} \leq C\|u\|_{W^{k,p}} \quad \forall a \in A,$$

where the constant $C > 0$ depends on the (Lebesgue) measure of A. To obtain estimate (5.2), it suffices to take $a \in A$ for which (5.3) holds.

Proof of Theorem 3.1. Applying Lemma 5.1, we can find $v, w_1, \ldots, w_N \in W^{k,p}(Q)$ such that (i), (ii) and (iv) hold. We now show that

$$v(x) = \int_{a_1}^{x_1} \cdots \int_{a_N}^{x_N} \partial_1 \cdots \partial_N u. \tag{5.4}$$

Recall that, by Proposition 1.2,

$$|\partial_1 \cdots \partial_N u|\big([y_i = t] \cap Q\big) = 0 \quad \forall t \in (-1,1) \quad \forall i = 1, \ldots, N. \tag{5.5}$$

In particular, the right-hand side of (5.4) is well-defined. Let (ρ_n) be a sequence of mollifiers such that $\operatorname{supp} \rho_n \subset B_{1/n}$. By (iii') and (5.1), we have

$$\rho_n * v(x) = \int_{a_1}^{x_1} \cdots \int_{a_N}^{x_N} \partial_1 \cdots \partial_N (\rho_n * u)$$

for every $x \in Q$ such that $d(x, \partial Q) > \frac{1}{n}$. Proceeding as in the proof of Theorem 1.2, we deduce (5.4). It then follows from (5.5) that v is continuous on $\overline{Q}$; moreover, (3.1) holds. This establishes Theorem 3.1.

6 Further results

Most of our results can be extended to fractionary Sobolev spaces $W^{s,p}$ with $s > 0$ and $1 \le p < \infty$.

We first recall the definition of $W^{s,p}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^N$. Given $0 < s < 1$, we say that $u \in W^{s,p}(\Omega)$ if

$$u \in L^p(\Omega) \quad \text{and} \quad \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty.$$

If $s > 1$, $s \notin \mathbb{N}$, then we say that $u \in W^{s,p}(\Omega)$ if

$$u \in W^{k,p}(\Omega) \quad \text{and} \quad D^k u \in W^{s-k,p}(\Omega),$$

where $k \in \mathbb{N}$ is such that $k < s < k + 1$.

The counterparts of Theorems 1.2 and 1.4 for $W^{s,p}$ are

Theorem 6.1 *If $u \in W^{1/p,p}(\mathbb{R}^N)$, $1 < p < \infty$, and $\partial_1 \cdots \partial_N u$ is a measure, then u is continuous and bounded.*

Theorem 6.2 *Let $u \in W^{s,p}(Q)$, $s > 0$ and $1 < p < \infty$, be such that $\partial_1 \cdots \partial_N u$ is a measure. If $sp > N - 1$, then u is continuous on $\overline{Q}$.*

Simple examples show that the conclusion of Theorem 6.1 is no longer true if $sp < 1$ (take for instance $u = \chi_Q$, the characteristic function of Q). The proofs of Theorems 6.1 and 6.2 are based on counterparts of Lemma 5.1 and Proposition 1.2 for $W^{s,p}$. The analogue of Lemma 5.1 can be still established via a Fubini-type argument using the equivalent form of the Gagliardo seminorm in $\mathbb{R}^N$ (see [1]):

$$|u|^p_{W^{s,p}} := \sum_{i=1}^N \int_0^\infty d\tau \int_{\mathbb{R}^N} \frac{|u(x + \tau \mathrm{e}_i) - u(x)|^p}{\tau^{1+sp}} \, d\tau. \tag{6.1}$$

We shall focus on the counterpart of Proposition 1.2:

Proposition 6.1 *Let $u \in W^{s,p}(Q)$ be such that $\partial_1 \cdots \partial_N u$ is a measure. If $sp \ge 1$, then*

$$|\partial_1 \cdots \partial_N u|\big([x_i = t] \cap Q\big) = 0 \quad \forall t \in (-1, 1), \quad \forall i = 1, \ldots, N. \tag{6.2}$$

We need the analogue of Lemma 4.2 in $W^{s,p}$ for $sp \ge 1$:

Lemma 6.1 *Assume $s > 0$, $p > 1$ and $sp \ge 1$. Let $v \in W^{s,p}(Q)$ satisfy the assumptions of Lemma 4.1. Then*

$$(\partial_1 \cdots \partial_N v)(S) = 0,$$

where $S = [-1, 1]^{N-1} \times \{0\}$.

Proof. If $s \geq 1$, then this follows from Lemma 4.2 and the embedding $W^{s,p}(Q) \subset W^{1,1}(Q)$. We now assume $0 < s < 1$. If the conclusion did not hold, then, by Lemma 4.1, there would exist $\varepsilon, \delta > 0$ such that

$$\big|v(y,s) - v(y,-t)\big| \geq \delta > 0$$

for a.e. $y \in (1, 1+\epsilon)^{N-1}$ and a.e. $s, t \in (0, \epsilon)$. Together with $sp \geq 1$, this would yield

$$\int\limits_{(1,1+\epsilon)^{N-1}} \int_0^\varepsilon \int_0^\varepsilon \frac{|v(y,s) - v(y,-t)|^p}{(s+t)^{1+sp}}\, ds\, dt\, dy \geq$$

$$\geq \varepsilon^{N-1} \int_0^\varepsilon \int_0^\varepsilon \frac{\delta^p}{(s+t)^{1+sp}}\, ds\, dt = +\infty. \tag{6.3}$$

In view of the equivalent seminorm (6.2), this would be in contradiction with the hypothesis $u \in W^{s,p}(\Omega)$.

Proof of Proposition 6.1. One proceeds exactly as in the proof of Proposition 1.2, replacing Lemma 4.2 by Lemma 6.1. We leave the details to the reader.

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References

[1] R.A. Adams, Sobolev spaces, Pure and Applied Mathematics Vol. 65, Academic Press, New York-London, 1975.

[2] M. Campos Pinto, Développement et analyse de schémas adaptatifs pour les équations de transport, Ph.D. dissertation, Université Paris 6, 2005.

[3] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[4] J. Souček, Spaces of functions on domain Ω, whose k-th derivatives are measures defined on $\bar{\Omega}$, Časopis Pěst. Mat. 97 (1972), 10–46, 94.

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