# THE SUB-SUPERSOLUTION METHOD FOR WEAK SOLUTIONS 

MARCELO MONTENEGRO AND AUGUSTO C. PONCE


#### Abstract

We extend the method of sub and supersolutions in order to prove existence of $L^{1}$-solutions of the equation $-\Delta u=f(x, u)$ in $\Omega$, where $f$ is a Carathéodory function. The proof is based on Schauder's fixed point theorem.


## 1. Introduction

We consider the following semilinear problem:

$$
\left\{\begin{array}{cll}
-\Delta u=f(x, u) & & \text { in } \Omega,  \tag{1.1}\\
u=0 & & \text { on } \partial \Omega,
\end{array}\right.
$$

where $\Omega \subset \mathbb{R}^{N}$ is a smooth bounded domain and $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function.

The classical method of sub and supersolutions (see, e.g., $[4,9]$ ) asserts that if $f$ is smooth and if one can find smooth sub and supersolutions $v_{1} \leq v_{2}$ of (1.1), then there exists a classical solution $u$ of (1.1) such that $v_{1} \leq u \leq v_{2}$.

The usual proof is based on a monotone iteration scheme; this requires $f$ to be Lipschitz (or locally Lipschitz). The argument also shows that there exist a smallest and a largest solution $u_{1} \leq u_{2}$ in the interval $\left[v_{1}, v_{2}\right]$. Another proof, based on Schauder's fixed point theorem can be found in Akô [1]. In this case, the existence of a smallest and a largest solution is proved separately, via a Perron-type argument. Based on Akô's strategy, Clément-Sweers [6] were able to implement the method of sub-supersolutions when $v_{1}, v_{2} \in C(\bar{\Omega})$ and $f$ is only assumed to be continuous. Other versions can also be found for instance in Deuel-Hess [8] (see also Dancer-Sweers [7]) for $H^{1}$-solutions and in Brezis-Marcus-Ponce [3, Theorem 4] for $L^{1}$-solutions and $f$ continuous, nondecreasing. However, none of these results is contained in the other. We shall compare them in Section 5 below.

In this paper, we extend the method of sub-supersolutions in order to establish existence of $L^{1}$-solutions of (1.1) in the sense of Definition 1.1 below. We follow the strategy of $[1,6,7]$ based on Schauder's fixed point theorem. However, some of the details had to be substantially modified.

We assume throughout that
$\Omega$ is a smooth bounded domain
and

$$
\begin{equation*}
f: \Omega \times \mathbb{R} \rightarrow \mathbb{R} \text { is a Carathéodory function. } \tag{1.3}
\end{equation*}
$$

The notion of solution we will consider is the following:
Definition 1.1. We say that $u$ is an $L^{1}$-solution of (1.1) if

[^0](i) $u \in L^{1}(\Omega)$;
(ii) $f(\cdot, u) \rho_{0} \in L^{1}(\Omega)$;
(iii)
\[

$$
\begin{equation*}
-\int_{\Omega} u \Delta \zeta=\int_{\Omega} f(x, u) \zeta d x \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}) . \tag{1.4}
\end{equation*}
$$

\]

Here, $\rho_{0}(x)=d(x, \partial \Omega), \forall x \in \Omega$, and $C_{0}^{2}(\bar{\Omega})=\left\{\zeta \in C^{2}(\bar{\Omega}) ; \zeta=0\right.$ on $\left.\partial \Omega\right\}$. Note that (1.4) makes sense in view of $(i)$ and (ii).

We also consider $L^{1}$-sub and $L^{1}$-supersolutions in analogy with this definition. For instance, $u$ is an $L^{1}$-subsolution of (1.1) if $u$ satisfies $(i)-(i i i)$ with " $\leq$ " instead of "=" in (1.4). We will systematically omit the term " $L$ " and simply say that $u$ is a solution of (1.1), meaning (1.4); similar convention for sub and supersolutions.

Our main result is
Theorem 1.1. Let $v_{1}$ and $v_{2}$ be a sub and a supersolution of (1.1), respectively. Assume that $v_{1} \leq v_{2}$ a.e. and

$$
\begin{equation*}
f(\cdot, v) \rho_{0} \in L^{1}(\Omega) \quad \text { for every } v \in L^{1}(\Omega) \text { such that } v_{1} \leq v \leq v_{2} \text { a.e. } \tag{1.5}
\end{equation*}
$$

Then, there exist solutions $u_{1} \leq u_{2}$ of (1.1) in $\left[v_{1}, v_{2}\right]$ such that any solution $u$ of (1.1) in the interval $\left[v_{1}, v_{2}\right]$ satisfies

$$
v_{1} \leq u_{1} \leq u \leq u_{2} \leq v_{2} \quad \text { a.e. }
$$

In general, (1.1) need not have a solution if (1.5) fails (see [13]). However, Orsina-Ponce [13] were recently able to prove existence of solutions of (1.1) (and (5.2) below) for some nonlinearities $f$ which need not satisfy (1.5).

The paper is organized as follows. In Section 2, we develop some tools used in the proof of Theorem 1.1. In Section 3, we recall some existence, compactness and comparison results related to the linear equation $-\Delta w=h$ when $h \rho_{0} \in L^{1}(\Omega)$. We then establish Theorem 1.1 in Section 4. In the last section, we recover some known results; we also apply Theorem 1.1 in order to study semilinear problems involving measures.
Added note: After this paper was completed, the authors have been informed of other related works of M. Marcus [12] (assuming $f$ monotone) and M. C. Palmeri [14] (with (1.5) replaced by a stronger assumption on $f$ and without proving the existence of a smallest and a largest solution).

## 2. Boundedness and EQui-Integrability in $L_{\rho_{0}}^{1}$

We begin with the following well-known result. We shall present a proof below for the convenience of the reader.

Theorem 2.1. Let $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$
\begin{equation*}
g(\cdot, v) \rho_{0} \in L^{1}(\Omega) \quad \text { for every } v \in L^{1}(\Omega) \tag{2.1}
\end{equation*}
$$

Then, the Nemytskii operator

$$
\begin{align*}
G: L^{1}(\Omega) & \longrightarrow L^{1}\left(\Omega ; \rho_{0} d x\right) \\
v & \longmapsto g(\cdot, v) \tag{2.2}
\end{align*}
$$

is continuous.
We first prove the

Lemma 2.1. Let $\left(w_{n}\right) \subset L^{1}(\Omega)$ and let $\left(E_{n}\right)$ be a sequence of measurable subsets of $\Omega$ such that

$$
\begin{equation*}
\left|E_{n}\right| \rightarrow 0 \quad \text { and } \quad \int_{E_{n}}\left|w_{n}\right| \geq 1 \quad \forall n \geq 1 \tag{2.3}
\end{equation*}
$$

Then, there exist a subsequence $\left(w_{n_{k}}\right)$ and a sequence of disjoint measurable sets $\left(F_{k}\right)$ such that

$$
\begin{equation*}
F_{k} \subset E_{n_{k}} \quad \text { and } \quad \int_{F_{k}}\left|w_{n_{k}}\right| \geq \frac{1}{2} \quad \forall k \geq 1 \tag{2.4}
\end{equation*}
$$

Proof. Let $n_{1}:=1$ and $A_{1}:=E_{1}$. By induction, we construct an increasing sequence of integers $\left(n_{k}\right)$ and measurable sets $\left(A_{k}\right)$ as follows.
Let $k \geq 2$. Assume we are given integers $n_{1}<\ldots<n_{k-1}$ and sets $A_{1}, \ldots, A_{k-1}$ (not necessarily disjoint) such that $A_{j} \subset E_{j}$ and

$$
\int_{A_{j+1} \cup \cdots \cup A_{k-1}}\left|w_{n_{j}}\right| \leq \frac{1}{2}-\frac{1}{2^{k-1}} \quad \forall j=1, \ldots, k-2 .
$$

(This condition is vacuous when $k=2$.)
Since $\left|E_{n}\right| \rightarrow 0$, then for $n_{k}>n_{k-1}$ sufficiently large we have

$$
\int_{E_{n_{k}}}\left|w_{n_{j}}\right| \leq \frac{1}{2^{k}} \quad \forall j=1, \ldots, k-1 .
$$

Let $A_{k}:=E_{n_{k}}$. Then,

$$
\int_{A_{j+1} \cup \ldots \cup A_{k}}\left|w_{n_{j}}\right| \leq \frac{1}{2}-\frac{1}{2^{k}} \quad \forall j=1, \ldots, k-1 .
$$

Proceeding with this construction, one gets sequences $\left(n_{k}\right)$ and $\left(A_{k}\right)$. We now set

$$
F_{k}:=A_{k} \backslash \bigcup_{i=k+1}^{\infty} A_{i}
$$

Then, the sets $F_{k}$ are disjoint and

$$
\int_{F_{k}}\left|w_{n_{k}}\right|=\lim _{i \rightarrow \infty} \int_{A_{k} \backslash\left(A_{k+1} \cup \ldots \cup A_{i}\right)}\left|w_{n_{k}}\right| \geq \frac{1}{2} .
$$

The proof of the lemma is complete.
As a consequence of Lemma 2.1 we have
Proposition 2.1. Let $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and $v_{1}, v_{2} \in L^{1}(\Omega)$ be such that $v_{1} \leq v_{2}$ a.e. Assume that

$$
\begin{equation*}
g(\cdot, v) \rho_{0} \in L^{1}(\Omega) \quad \text { for every } v \in L^{1}(\Omega) \text { such that } v_{1} \leq v \leq v_{2} \text { a.e. } \tag{2.5}
\end{equation*}
$$

Then, the set

$$
\begin{equation*}
\mathcal{B}=\left\{g(\cdot, v) \in L^{1}\left(\Omega ; \rho_{0} d x\right) ; v \in L^{1}(\Omega) \text { and } v_{1} \leq v \leq v_{2} \text { a.e. }\right\} \tag{2.6}
\end{equation*}
$$

is bounded and equi-integrable in $L^{1}\left(\Omega ; \rho_{0} d x\right)$.

We recall that a set $\mathcal{B} \subset L^{1}\left(\Omega ; \rho_{0} d x\right)$ is equi-integrable if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$
E \subset \Omega \quad \text { and } \quad|E|<\delta \quad \Longrightarrow \quad \int_{E}|g| \rho_{0}<\varepsilon \quad \forall g \in \mathcal{B} .
$$

Here, $|E|$ denotes the Lebesgue measure of $E$.

Proof of Proposition 2.1. Since $\Omega$ is bounded, it suffices to show that $\mathcal{B}$ is equiintegrable. Assume by contradiction that $\mathcal{B}$ is not equi-integrable. Then, there exist $\varepsilon>0,\left(u_{n}\right) \subset L^{1}(\Omega)$ with $v_{1} \leq u_{n} \leq v_{2}$ a.e., and a sequence of measurable sets $\left(E_{n}\right)$ in $\Omega$ such that

$$
\left|E_{n}\right| \rightarrow 0 \quad \text { and } \quad \int_{E_{n}} g\left(x, u_{n}\right) \rho_{0} d x \geq \varepsilon \quad \forall n \geq 1
$$

Applying Lemma 2.1 with $w_{n}=g\left(\cdot, u_{n}\right) \rho_{0} / \varepsilon$, we can extract a subsequence ( $u_{n_{k}}$ ) and a sequence of disjoint measurable sets $\left(F_{k}\right)$ in $\Omega$ such that

$$
\begin{equation*}
\int_{F_{k}}\left|g\left(x, u_{n_{k}}\right)\right| \rho_{0} d x \geq \frac{\varepsilon}{2} \quad \forall k \geq 1 \tag{2.7}
\end{equation*}
$$

Let

$$
v(x)= \begin{cases}u_{n_{k}}(x) & \text { if } x \in F_{k} \text { for some } k \geq 1 \\ v_{1}(x) & \text { otherwise }\end{cases}
$$

Then, $v_{1} \leq v \leq v_{2}$ a.e.; hence, $v \in L^{1}(\Omega)$. Moreover,

$$
\int_{\Omega}|g(x, v)| \rho_{0} d x \geq \sum_{k=1}^{\infty} \int_{F_{k}}\left|g\left(x, u_{n_{k}}\right)\right| \rho_{0} d x=\infty
$$

This contradicts (2.5). Therefore, $\mathcal{B}$ is equi-integrable in $L^{1}\left(\Omega ; \rho_{0} d x\right)$.

We now present the
Proof of Theorem 2.1. Assume $v_{n} \rightarrow v$ in $L^{1}(\Omega)$. Let $\left(v_{n_{k}}\right)$ be a subsequence such that $v_{n_{k}} \rightarrow v$ a.e. and $\left|v_{n_{k}}\right| \leq V$ a.e. for some function $V \in L^{1}(\Omega)$. In particular,

$$
g\left(\cdot, v_{n_{k}}\right) \rightarrow g(\cdot, v) \quad \text { a.e. }
$$

Moreover, by Proposition 2.1 above (applied to $v_{1}=-V$ and $v_{2}=V$ ), the sequence $\left(g\left(\cdot, v_{n_{k}}\right)\right)$ is equi-integrable in $L^{1}\left(\Omega ; \rho_{0} d x\right)$. It then follows from Egorov's theorem that

$$
g\left(\cdot, v_{n_{k}}\right) \rightarrow g(\cdot, v) \quad \text { in } L^{1}\left(\Omega ; \rho_{0} d x\right)
$$

Since the limit is independent of the subsequence $\left(v_{n_{k}}\right)$, we deduce that

$$
G\left(v_{n}\right) \rightarrow G(v) \quad \text { in } L^{1}\left(\Omega ; \rho_{0} d x\right) .
$$

## 3. Standard existence, compactness and comparison results

In this section, we recall some well-known results related to weak solutions of the linear problem:

$$
\left\{\begin{align*}
-\Delta w=h & \text { in } \Omega  \tag{3.1}\\
w=0 & \text { on } \partial \Omega
\end{align*}\right.
$$

We begin with the existence and compactness of weak solutions of (3.1):
Theorem 3.1. Given $h \in L^{1}\left(\Omega ; \rho_{0} d x\right)$, there exists a unique $w \in L^{1}(\Omega)$ such that

$$
\begin{equation*}
-\int_{\Omega} w \Delta \zeta=\int_{\Omega} h \zeta \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}) \tag{3.2}
\end{equation*}
$$

Moreover,
(i) For every $1 \leq p<\frac{N}{N-1}, w \in L^{p}(\Omega)$ and

$$
\begin{equation*}
\|w\|_{L^{p}} \leq C_{p}\left\|h \rho_{0}\right\|_{L^{1}} \tag{3.3}
\end{equation*}
$$

(ii) Given $h_{n} \in L^{1}\left(\Omega ; \rho_{0} d x\right), n \geq 1$, let $w_{n}$ be the solution of (3.2) associated to $h_{n}$. If $\left(h_{n}\right)$ is bounded in $L^{1}\left(\Omega ; \rho_{0} d x\right)$, then $\left(w_{n}\right)$ is relatively compact in $L^{p}(\Omega)$ for every $1 \leq p<\frac{N}{N-1}$.

Proof. We refer the reader to [2] for the existence and uniqueness of $w$. We split the proof of $(i)-(i i)$ in two steps:
Step 1. Proof of (i).
Note that $w$ satisfies

$$
\begin{equation*}
\left|\int_{\Omega} w \Delta \zeta\right| \leq\left\|h \rho_{0}\right\|_{L^{1}}\left\|\zeta / \rho_{0}\right\|_{L^{\infty}} \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}) \tag{3.4}
\end{equation*}
$$

Given $f \in C^{\infty}(\bar{\Omega})$, let $\zeta \in C_{0}^{2}(\bar{\Omega})$ be the solution of

$$
\left\{\begin{aligned}
-\Delta \zeta=f & \text { in } \Omega \\
\zeta=0 & \text { on } \partial \Omega
\end{aligned}\right.
$$

By standard Calderón-Zygmund estimates (see [10]),

$$
\begin{equation*}
\|\zeta\|_{W^{2, p^{\prime}}} \leq C\|f\|_{L^{p^{\prime}}} . \tag{3.5}
\end{equation*}
$$

Since $p^{\prime}>N$, it follows from Morrey's imbedding that

$$
\begin{equation*}
\left\|\zeta / \rho_{0}\right\|_{L^{\infty}} \leq C\left(\|\zeta\|_{L^{\infty}}+\|\nabla \zeta\|_{L^{\infty}}\right) \leq C\|\zeta\|_{W^{2, p^{\prime}}} \tag{3.6}
\end{equation*}
$$

Combining (3.4)-(3.6), we get

$$
\begin{equation*}
\left|\int_{\Omega} w f\right| \leq C_{p}\left\|h \rho_{0}\right\|_{L^{1}}\|f\|_{L^{p^{\prime}}} \quad \forall f \in C^{\infty}(\bar{\Omega}) \tag{3.7}
\end{equation*}
$$

By duality, one deduces that $w \in L^{p}(\Omega)$ and (3.3) holds.
Step 2. Proof of (ii).
Given a smooth domain $U \subset \subset \Omega$, let $v_{n} \in L^{1}(U)$ be the solution of

$$
\left\{\begin{aligned}
-\Delta v_{n}=h_{n} & \text { in } U, \\
v_{n}=0 & \text { on } \partial U
\end{aligned}\right.
$$

By standard elliptic estimates (see [15]),

$$
\begin{equation*}
\left\|v_{n}\right\|_{W^{1, p}(U)} \leq C_{p}\left\|h_{n}\right\|_{L^{1}(U)} \leq C_{p} \tag{3.8}
\end{equation*}
$$

for every $1 \leq p<\frac{N}{N-1}$. On the other hand, since $w_{n}-v_{n}$ is harmonic in $U$, for every $\omega \subset \subset U$ we have

$$
\begin{equation*}
\left\|w_{n}-v_{n}\right\|_{C^{1}(\bar{\omega})} \leq C_{\omega}\left\|w_{n}-v_{n}\right\|_{L^{1}(U)} \leq C_{\omega}\left\|h_{n} \rho_{0}\right\|_{L^{1}} \leq C_{\omega} . \tag{3.9}
\end{equation*}
$$

Thus, by a standard diagonalization argument there exists a subsequence $\left(w_{n_{k}}\right)$ such that $w_{n_{k}} \rightarrow w$ a.e. in $\Omega$. On the other hand, by $(i)$ the sequence $\left(w_{n}\right)$ is bounded in $L^{p}(\Omega)$ for every $1 \leq p<\frac{N}{N-1}$. The conclusion then follows from Egorov's theorem.

We also need the following version of Kato's inequality (see [3, Proposition B.5]):
Proposition 3.1. Let $w \in L^{1}(\Omega)$ and $h \in L^{1}\left(\Omega ; \rho_{0} d x\right)$ be such that

$$
-\int_{\Omega} w \Delta \zeta \leq \int_{\Omega} h \zeta \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}), \zeta \geq 0 \text { in } \Omega
$$

Then,

$$
-\int_{\Omega} w^{+} \Delta \zeta \leq \int_{[w \geq 0]} h \zeta \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}), \zeta \geq 0 \text { in } \Omega
$$

Corollary 3.1. If $u, v$ are solutions of (1.1), then $\max \{u, v\}$ is a subsolution.
Proof. Apply Proposition 3.1 with $w=v-u$ and $h=f(\cdot, v)-f(\cdot, u)$. Then,

$$
-\int_{\Omega}(v-u)^{+} \Delta \zeta \leq \int_{[v \geq u]}[f(x, v)-f(x, u)] \zeta d x \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}), \zeta \geq 0 \text { in } \Omega
$$

Since $\max \{u, v\}=u+(v-u)^{+}$, the result follows.

## 4. Proof of Theorem 1.1

We split the proof in two steps:
Step 1. Problem (1.1) has a solution $u$ such that $v_{1} \leq u \leq v_{2}$ a.e.
Given $(x, t) \in \Omega \times \mathbb{R}$, let

$$
g(x, t):= \begin{cases}v_{1}(x) & \text { if } t<v_{1}(x) \\ t & \text { if } v_{1}(x) \leq t \leq v_{2}(x) \\ v_{2}(x) & \text { if } v_{2}(x)<t\end{cases}
$$

Then, $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and, by (1.5), we have

$$
g(\cdot, v) \rho_{0} \in L^{1}(\Omega) \quad \text { for every } v \in L^{1}(\Omega)
$$

We now consider

$$
\begin{aligned}
G: L^{1}(\Omega) & \longrightarrow L^{1}\left(\Omega ; \rho_{0} d x\right) \\
v & \longmapsto g(\cdot, v)
\end{aligned}
$$

and

$$
\begin{aligned}
K: L^{1}\left(\Omega ; \rho_{0} d x\right) & \longrightarrow L^{1}(\Omega) \\
h & \longmapsto w
\end{aligned}
$$

where $w$ is the unique solution of

$$
\left\{\begin{align*}
-\Delta w=h & \text { in } \Omega  \tag{4.1}\\
w=0 & \text { on } \partial \Omega
\end{align*}\right.
$$

By Theorems 2.1 and 3.1, $K G: L^{1}(\Omega) \rightarrow L^{1}(\Omega)$ is continuous. Moreover, by Proposition 2.1 (applied to $f$ ), $G\left(L^{1}(\Omega)\right)$ is a bounded subset of $L^{1}\left(\Omega ; \rho_{0} d x\right)$. Hence, by Theorem 3.1, $K G$ is compact and there exists $C>0$ such that

$$
\|K(G(v))\|_{L^{1}} \leq C_{1}\left\|G(v) \rho_{0}\right\|_{L^{1}} \leq C \quad \forall v \in L^{1}(\Omega)
$$

It then follows from Schauder's fixed point theorem that $K G$ has a fixed point $u \in L^{1}(\Omega)$. In other words, $u$ satisfies

$$
\left\{\begin{array}{cl}
-\Delta u=g(x, u) &  \tag{4.2}\\
\text { in } \Omega, \\
u=0 & \\
\text { on } \partial \Omega .
\end{array}\right.
$$

We claim that $u$ is a solution of (1.1) and

$$
\begin{equation*}
v_{1} \leq u \leq v_{2} \quad \text { a.e. } \tag{4.3}
\end{equation*}
$$

It suffices to prove (4.3). We show that $u \leq v_{2}$ a.e.; the proof of the inequality $v_{1} \leq u$ a.e. is similar. Note that

$$
g(\cdot, u)=g\left(\cdot, v_{2}\right) \quad \text { a.e. on the set }\left[u \geq v_{2}\right] .
$$

Thus, applying Proposition 3.1 to $w=u-v_{2}$, we get

$$
-\int_{\Omega} w^{+} \Delta \zeta \leq \int_{\left[u \geq v_{2}\right]}\left[g(x, u)-g\left(x, v_{2}\right)\right] \zeta d x=0 \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega}), \zeta \geq 0 \text { in } \Omega
$$

We easily deduce that $w^{+} \leq 0$ a.e.; hence, $w^{+}=0$ a.e. This implies $u \leq v_{2}$ a.e. The proof of Step 1 is complete.

Step 2. There exist a smallest and a largest solution $u_{1} \leq u_{2}$ of (1.1) in the interval [ $v_{1}, v_{2}$ ].
(In [7], the proof of this step is based on Zorn's lemma. We could have followed their approach, but we present a different argument.)

We prove the existence of the largest solution $u_{2}$; the existence of $u_{1}$ is similar. Let

$$
A=\sup \left\{\int_{\Omega} w ; v_{1} \leq w \leq v_{2} \text { a.e. and } w \text { is a solution of (1.1) }\right\}
$$

Clearly, $A<\infty$. Before we proceed, let us prove the following
Claim. If $w_{1}, w_{2}$ are two solutions of (1.1) such that $v_{1} \leq w_{1}, w_{2} \leq v_{2}$ a.e., then (1.1) has a solution $w$ such that

$$
\begin{equation*}
v_{1} \leq \max \left\{w_{1}, w_{2}\right\} \leq w \leq v_{2} \quad \text { a.e. } \tag{4.4}
\end{equation*}
$$

Indeed, by Corollary 3.1, $\max \left\{w_{1}, w_{2}\right\}$ is a subsolution of (1.1). Applying Step 1 above with $\max \left\{w_{1}, w_{2}\right\}$ and $v_{2}$, one finds a solution $w$ of (1.1) satisfying (4.4). This establishes the claim.

It follows from the claim above that one can find a nondecreasing sequence of solutions $\left(w_{n}\right)$ of (1.1) such that

$$
\begin{equation*}
v_{1} \leq w_{n} \leq v_{2} \text { a.e. and } \int_{\Omega} w_{n} \rightarrow A \tag{4.5}
\end{equation*}
$$

By monotone convergence, there exists $w_{0} \in L^{1}(\Omega)$ such that $w_{n} \rightarrow w_{0}$ a.e.,

$$
v_{1} \leq w_{0} \leq v_{2} \text { a.e. and } \quad \int_{\Omega} w_{n} \rightarrow \int_{\Omega} w_{0}=A
$$

On the other hand, by Proposition 2.1 the sequence $\left(f\left(\cdot, w_{n}\right)\right)$ is equi-integrable in $L^{1}\left(\Omega ; \rho_{0} d x\right)$. It then follows from Egorov's theorem that

$$
f\left(\cdot, w_{n}\right) \rightarrow f\left(\cdot, w_{0}\right) \quad \text { in } L^{1}\left(\Omega ; \rho_{0} d x\right)
$$

Thus, $w_{0}$ is a solution of (1.1) and

$$
\int_{\Omega} w_{0}=A .
$$

By the claim above, $w_{0}$ is the largest solution of (1.1) in the interval $\left[v_{1}, v_{2}\right]$. The proof of Theorem 1.1 is complete.

## 5. Some consequences and further results

In this section, we discuss some consequences of Theorem 1.1. In what follows, we denote by $v_{1}, v_{2}$ sub and supersolutions of (1.1) satisfying $v_{1} \leq v_{2}$ a.e. Using standard regularity theory, in each case one shows that the solution provided in Theorem 1.1 lies in a better space.

Corollary 5.1. If $v_{1}, v_{2} \in C^{2}(\bar{\Omega})$ and $f \in C^{1}(\bar{\Omega} \times \mathbb{R})$, then (1.1) has a classical solution $u \in C^{2}(\bar{\Omega})$ such that $v_{1} \leq u \leq v_{2}$ in $\Omega$.
Corollary 5.2. If $v_{1}, v_{2} \in L^{\infty}(\Omega)$ and $f \in C(\bar{\Omega} \times \mathbb{R})$, then (1.1) has a solution $u \in C^{1, \alpha}(\bar{\Omega}), \forall \alpha \in(0,1)$, such that $v_{1} \leq u \leq v_{2}$ in $\Omega$.

Corollary 5.2 is established in [6] for functions $v_{1}, v_{2} \in C(\bar{\Omega})$.
Corollary 5.3. If $v_{1}, v_{2} \in L^{1}(\Omega)$ and $f$ is a Carathéodory function satisfying

$$
\begin{equation*}
f(\cdot, v) \in L^{\frac{2 N}{N+2}}(\Omega) \quad \text { for every } v \in L^{1}(\Omega) \text { such that } v_{1} \leq v \leq v_{2} \text { a.e., } \tag{5.1}
\end{equation*}
$$

then (1.1) has a solution $u \in H_{0}^{1}(\Omega)$ such that $v_{1} \leq u \leq v_{2}$ a.e.
Corollary 5.3 is stated in [7] under the assumption that $f(\cdot, v) \in L^{p}$ for some $p>\frac{2 N}{N+2}$.

We now apply Theorem 1.1 to study semilinear problems with measure data, namely

$$
\left\{\begin{align*}
-\Delta u & =f(x, u)+\mu & & \text { in } \Omega,  \tag{5.2}\\
u & =\nu & & \text { on } \partial \Omega,
\end{align*}\right.
$$

where $\mu$ is a bounded measure in $\Omega$ and $\nu$ is a bounded measure on $\partial \Omega$. We say that $u$ is a solution of (5.2) if $u \in L^{1}(\Omega), f(\cdot, u) \rho_{0} \in L^{1}(\Omega)$ and

$$
-\int_{\Omega} u \Delta \zeta=\int_{\Omega} f(x, u) \zeta+\int_{\Omega} \zeta d \mu-\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} d \nu \quad \forall \zeta \in C_{0}^{2}(\bar{\Omega})
$$

where $n$ denotes the outward unit norm on $\partial \Omega$. This type of problem has been extensively studied in $[3,5]$ when $f$ is of the form $f(x, t)=-g(t)$, where $g$ is continuous, nondecreasing and $g(0)=0$.

Applying Theorem 1.1 we deduce the following corollary which extends results in $[3,5]$ :

Corollary 5.4. If $v_{1} \leq v_{2}$ are sub and supersolutions of (5.2) and $f$ is a Carathéodory function satisfying (1.5), then (5.2) has a solution $u \in L^{1}(\Omega)$ such that $v_{1} \leq u \leq v_{2}$ a.e.

Proof. Let $w \in L^{1}(\Omega)$ be the unique solution of (see $[11,15]$ )

$$
\left\{\begin{aligned}
-\Delta w=\mu & \text { in } \Omega \\
w=\nu & \text { on } \partial \Omega
\end{aligned}\right.
$$

Let $\tilde{u}:=u-w$. The resulting equation in terms of $\tilde{u}$ satisfies the assumptions of Theorem 1.1. Hence, (5.2) has a solution.

## Acknowledgments

Both authors were partially supported by Fapesp. The second author (ACP) gratefully acknowledges the invitation and warm hospitality of the Math Department of Unicamp.

## References

[1] K. Akô, On the Dirichlet problem for quasi-linear elliptic differential equations of the second order. J. Math. Soc. Japan 13 (1961), 45-62.
[2] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, Blow up for $u_{t}-\Delta u=g(u)$ revisited. Adv. Differential Equations 1 (1996), 73-90.
[3] H. Brezis, M. Marcus, and A. C. Ponce, Nonlinear elliptic equations with measures revisited. In: Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, and S. Klainerman, eds.), Annals of Mathematics Studies, 163, Princeton University Press, Princeton, NJ, 2007. Part of the results were announced in a note by the same authors: $A$ new concept of reduced measure for nonlinear elliptic equations, C. R. Acad. Sci. Paris, Ser. I 339 (2004), 169-174.
[4] H. Brezis and L. Nirenberg, Nonlinear functional analysis. In preparation.
[5] H. Brezis and A. C. Ponce, Reduced measures on the boundary. J. Funct. Anal. 229 (2005), 95-120.
[6] P. Clément and G. Sweers, Getting a solution between sub- and supersolutions without monotone iteration. Rend. Istit. Mat. Univ. Trieste 19 (1987), 189-194.
[7] E. N. Dancer and G. Sweers, On the existence of a maximal weak solution for a semilinear elliptic equation. Differential Integral Equations 2 (1989), 533-540.
[8] J. Deuel and P. Hess, A criterion for the existence of solutions of non-linear elliptic boundary value problems. Proc. Roy. Soc. Edinburgh Sect. A 74A (1974/75), 49-54.
[9] L. C. Evans, Partial differential equations. Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.
[10] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983.
[11] A. Gmira and L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations. Duke Math. J. 64 (1991), 271-324.
[12] M. Marcus. Nonlinear second order elliptic equations with measure data. To appear in Handbook of Differential Equations 2 (M. Chipot, ed.)
[13] L. Orsina and A. C. Ponce, Semilinear elliptic equations and systems with diffuse measures. In preparation.
[14] M. C. Palmeri, Entropy subsolutions and supersolutions for nonlinear elliptic equations in $L^{1}$. Ricerche Mat. 53 (2004), 183-212.
[15] G. Stampacchia, Équations elliptiques du second ordre à coefficients discontinus. Séminaire de Mathématiques Supérieures, no. 16 (été, 1965), Les Presses de l'Université de Montréal, Montréal, Québec, 1966.

Universidade Estadual de Campinas, IMECC, Departamento de Matemática,
Caixa Postal 6065, CEP 13083-970, Campinas, SP, Brasil
E-mail address: msm@ime.unicamp.br
Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 6083)
Fédération Denis Poisson, Université François Rabelais
37200 Tours, France
E-mail address: ponce@lmpt.univ-tours.fr


[^0]:    1991 Mathematics Subject Classification. 35D05, 35J60.
    Key words and phrases. Method of sub-supersolutions; Schauder's fixed point theorem; semilinear elliptic problems; weak solutions.

