A SYSTEM OF ELLIPTIC EQUATIONS ARISING IN CHERN-SIMONS FIELD THEORY

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Abstract. We prove the existence of topological vortices in a relativistic self-dual Abelian Chern-Simons theory with two Higgs particles and two gauge fields through a study of a coupled system of two nonlinear elliptic equations over $\mathbb{R}^2$. We present two approaches to prove existence of solutions on bounded domains: via minimization of an indefinite functional and via a fixed point argument. We then show that we may pass to the full $\mathbb{R}^2$ limit from the bounded-domain solutions to obtain a topological solution in $\mathbb{R}^2$.

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1. Introduction

In this paper, we study the nonlinear elliptic system

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \mathbb{R}^2, \\ -\Delta v + \lambda \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a given real number and μ, ν are finite measures on $\mathbb{R}^2$. System (1.1) arises in a relativistic Abelian Chern-Simons model involving two Higgs scalar fields and two gauge fields, in which case μ and ν are measures of the form $-4\pi \sum_s \delta_{p_s}$. An interesting feature of this problem is that, although (1.1) comprises as two special limiting cases the well-understood Abelian Higgs vortex equation [44] and the Abelian Chern-Simons vortex equation [20–22,57,58,60], it cannot be directly solved using the same methods. We establish in the existence of topological solutions for an arbitrarily prescribed distribution of point vortices.

In fact, one of our main results is the following

Theorem 1.1. *Given points $p'_1, \ldots, p'_{N'}, p''_1, \ldots, p''_{N''} \in \mathbb{R}^2$ (not necessarily distinct), then for every $\lambda > 0$ the system*

$$\begin{cases} \Delta u = \lambda \mathrm{e}^v(\mathrm{e}^u - 1) + 4\pi \displaystyle\sum_{s=1}^{N'} \delta_{p'_s} & \text{in } \mathbb{R}^2, \\ \Delta v = \lambda \mathrm{e}^u(\mathrm{e}^v - 1) + 4\pi \displaystyle\sum_{s=1}^{N''} \delta_{p''_s} & \text{in } \mathbb{R}^2, \end{cases} \tag{1.2}$$

has a solution $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$ decaying exponentially fast at infinity. Moreover,

$$\|u\|_{L^1} + \|v\|_{L^1} \le \frac{C}{\lambda}\big(N' + N''\big)^3, \tag{1.3}$$

$$\|\mathrm{e}^u - 1\|_{L^1} + \|\mathrm{e}^v - 1\|_{L^1} \le \frac{C}{\lambda}\big(N' + N''\big)^2. \tag{1.4}$$

The link between the Chern-Simons equations and (1.2) will be discussed in Sections 2 and 3 below. The counterpart of Theorem 1.1 for (1.1), concerning general finite measures μ and ν, is presented in Section 7.

There has been recently a great amount of activity in the study of field theory models governed by Chern-Simons type dynamics. For example, in particle physics, Chern-Simons terms allow one to generate dually (electrically and magnetically) charged vortex-like solitons [48, 55, 62] known as dyons [56, 68, 69]; in condensed matter physics, Chern-Simons terms are necessary ingredients in various anyon models [49, 64] describing many-fermion systems such as electron-pairing in high-temperature superconductors and the integral and fractional quantum Hall effect [45, 67].

Mathematically, the equations of motion of various Chern-Simons models are hard to approach even in the radially symmetric static cases [48, 55, 62]. However, since the discovery of the self-dual structure in the Abelian Chern-Simons model [41, 43] in 1990, there came a burst of fruitful works on self-dual Chern-Simons equations, nonrelativistic and relativistic, Abelian and non-Abelian [27, 28]. It is now well understood that nonrelativistic self-dual Chern-Simons equations (Chern-Simons electromagnetism or its generalized forms coupled with a scalar particle governed by a gauged Schrödinger equation) are often related to integrable systems such as the Liouville equation [42], sinh-Gordon equation and Toda systems [29]. On the other hand, relativistic self-dual Chern-Simons equations usually are not integrable, and an understanding of any of these equations often presents new challenges. For example, for the relativistic Abelian self-dual Chern-Simons vortex equation, solutions are richly classified into topological solutions [58, 63] giving rise to integer values of charges and energy, nontopological solutions [21, 22, 57] giving rise to continuous ranges of charges and energy [23], and lattice condensate solutions characterized as spatially doubly periodic solutions [20, 60].

Various tools including absolute, min-max, and constrained variational methods, dynamic shooting methods, perturbation and weighted function space methods, etc., have been developed to study these different types of solutions. For the general relativistic non-Abelian self-dual Chern-Simons vortex equations of the form of a perturbed Toda system assuming a nonintegrable structure, the existence of topological solutions is established based on variational methods and a Cholesky decomposition technique [65].

In short, the study of self-dual Chern-Simons equations of various physical models brings into light a great wealth of interesting nonlinear elliptic equations, in particular, coupled systems of nonlinear elliptic equations. However, as in the case of relativistic non-Abelian Chern-Simons equations [65], the issues of existence and complete characterization of nontopological solutions and spatially periodic solutions of system (1.2) (or (2.9)) have not been understood yet.

The paper is organized as follows. In Section 2, we introduce the relativistic two-Higgs Chern-Simons model, the associated equations of motion, and the self-dual equations to be studied. We then state our main result about the existence of multivortex solutions induced by the two Higgs scalar fields; see Theorem 2.1. In Section 3, we transform the renormalized self-dual Chern-Simons equations into (1.2) and state our existence theorems for bounded-domain solutions and for solutions over the full plane, respectively; we explain how to use a full-space solution to obtain a multivortex solution of the self-dual Chern-Simons equations. In Section 4, we provide the existence of bounded-domain solutions via constrained minimization of an indefinite action functional. In Section 5, we study the domain expansion process of the single Chern-Simons equation and we describe some important properties of its solutions. As a result, we prove the convergence of the domain expansion process for the single equation case. In Section 6, we show that the domain expansion process can be carried over to the case of system (1.2). In Section 7, we turn our attention to system (1.1) concerning general measures μ and ν; the main result is Theorem 7.1. The counterpart of Theorem 7.1 on bounded domains is presented in Section 13; the proof is based on Schauder's fixed point theorem. In order to apply Schauder's fixed point theorem, we need some “stability” results spanning over Sections 8–11. In Section 12, we prove some a priori estimates which imply in particular (1.3)–(1.4). Theorem 7.1 is established in Section 14. In Section 15, we discuss assumptions (i)–(iii) of Theorem 7.1. In Section 16, we show that if both measures μ and ν have compact supports in $\mathbb{R}^2$, then the solution (u, v) provided by Theorem 7.1 has exponential decay. In Appendix A, we present some known existence, uniqueness and compactness results which are used in some of our proofs. Finally, in Appendix B we give a short proof of existence of solutions of the scalar Chern-Simons equation.

2. The self-dual Chern-Simons equations with two Higgs particles

Let ϕ and χ be two complex scalar fields in $\mathbb{R}^2$ representing two Higgs particles of charges q_1 and q_2, and let $A_r^{(1)}$ and $A_r^{(2)}$ be two associated gauge fields with the induced electromagnetic fields $F_{rs}^{(I)} = \partial_r A_s^{(I)} - \partial_s A_r^{(I)}$ on the $(2+1)$-dimensional Minkowski space $\mathbb{R}^{2,1}$ of metric tensor (g_{rs}) =diag$(1,-1,-1)$, where $r,s = 0,1,2$ and $I = 1,2$. The Chern-Simons action density (Lagrangian) $\mathcal{L}$ studied in [31, 47] takes the form

$$\mathcal{L} = -\frac{1}{4}\kappa\varepsilon^{rst}A_r^{(1)}F_{st}^{(2)} - \frac{1}{4}\kappa\varepsilon^{rst}A_r^{(2)}F_{st}^{(1)} + \overline{D_r\phi}D^r\phi + \overline{D_r\chi}D^r\chi - V(\phi,\chi), \tag{2.1}$$

where $\kappa > 0$ is a coupling parameter,

$$D_r\phi = \partial_r\phi - \mathrm{i}q_1A_r^{(1)}\phi, \quad D_r\chi = \partial_r\chi - \mathrm{i}q_2A_r^{(2)}\chi \tag{2.2}$$

are the covariant derivatives, and $V(\phi,\chi)$ is the Higgs potential density defined by

$$V(\phi,\chi) = \frac{q_1^2q_2^2}{\kappa^2}\Big(|\phi|^2(|\chi|^2 - c_2^2)^2 + |\chi|^2(|\phi|^2 - c_1^2)^2\Big). \tag{2.3}$$

Note that the special numerical factor in front of the expression of V ensures that self-duality can be achieved for static field configurations and the positive vacuum states $\langle\phi\rangle = c_1 > 0$ and $\langle\chi\rangle = c_2 > 0$ lead to spontaneously broken symmetries.

The equations of motion of the action density (2.1) are the Chern-Simons equations

$$\begin{aligned}
\frac{1}{2}\kappa\varepsilon^{rs\alpha}F_{s\alpha}^{(2)} &= -q_1\mathrm{i}(\phi\overline{D^r\phi} - \overline{\phi}D^r\phi),\\
\frac{1}{2}\kappa\varepsilon^{rs\alpha}F_{s\alpha}^{(1)} &= -q_2\mathrm{i}(\chi\overline{D^r\chi} - \overline{\chi}D^r\chi),\\
D_rD^r\phi &= -\frac{q_1^2q_2^2}{\kappa^2}\Big(2|\chi|^2(|\phi|^2 - c_1^2) + (|\chi|^2 - c_2^2)^2\Big)\phi,\\
D_rD^r\chi &= -\frac{q_1^2q_2^2}{\kappa^2}\Big(2|\phi|^2(|\chi|^2 - c_2^2) + (|\phi|^2 - c_1^2)^2\Big)\chi.
\end{aligned} \tag{2.4}$$

Note that $(j^{(1)r}) = (\rho^{(1)}, \mathbf{j}^{(1)}) = -q_1\mathrm{i}(\phi\overline{D^r\phi} - \overline{\phi}D^r\phi)$ and $(j^{(2)r}) = (\rho^{(2)}, \mathbf{j}^{(2)}) = -q_2\mathrm{i}(\chi\overline{D^r\chi} - \overline{\chi}D^r\chi)$ are the conserved matter current densities. The $r = 0$ components of the first two equations in (2.4) in the static case are

$$\begin{aligned}
\kappa F_{12}^{(2)} &= \rho^{(1)} = 2q_1^2A_0^{(1)}|\phi|^2,\\
\kappa F_{12}^{(1)} &= \rho^{(2)} = 2q_2^2A_0^{(2)}|\chi|^2,
\end{aligned} \tag{2.5}$$

which are simply the Chern-Simons versions of the Gauss laws and give us the mixed flux-charge relations as follows:

$$\begin{aligned}
\kappa\Phi^{(2)} &= \kappa\int_{\mathbb{R}^2} F_{12}^{(2)}\,\mathrm{d}x = \int_{\mathbb{R}^2}\rho^{(1)}\,\mathrm{d}x = Q^{(1)},\\
\kappa\Phi^{(1)} &= \kappa\int_{\mathbb{R}^2} F_{12}^{(1)}\,\mathrm{d}x = \int_{\mathbb{R}^2}\rho^{(2)}\,\mathrm{d}x = Q^{(2)}.
\end{aligned} \tag{2.6}$$

For static field configurations, it is standard that the Hamiltonian (energy) density $\mathcal{H}$ is given by

(2.7)
$$\begin{aligned}\mathcal{H} &= -\mathcal{L} \quad \text{(up to a total divergence)}\\ &= \kappa A_0^{(1)} F_{12}^{(2)} + \kappa A_0^{(2)} F_{12}^{(1)} - q_1^2 (A_0^{(1)})^2 |\phi|^2 - q_2^2 (A_0^{(2)})^2 |\chi|^2 + |D_j\phi|^2 + |D_j\chi|^2 + V,\\ &= \frac{\kappa^2 (F_{12}^{(2)})^2}{4q_1^2|\phi|^2} + \frac{\kappa^2 (F_{12}^{(1)})^2}{4q_2^2|\chi|^2} + |D_j\phi|^2 + |D_j\chi|^2 + V(\phi,\chi),\end{aligned}$$

where we have used the Gauss laws (2.5). Besides, applying the identities

$$|D_j\phi|^2 = |D_1\phi \pm \mathrm{i}D_2\phi|^2 \pm \mathrm{i}\big(\partial_1[\phi\overline{D_2\phi}] - \partial_2[\phi\overline{D_1\phi}]\big) \pm q_1 F_{12}^{(1)}|\phi|^2,$$
$$|D_j\chi|^2 = |D_1\chi \pm \mathrm{i}D_2\chi|^2 \pm \mathrm{i}\big(\partial_1[\chi\overline{D_2\chi}] - \partial_2[\chi\overline{D_1\chi}]\big) \pm q_2 F_{12}^{(2)}|\chi|^2,$$

we have, legitimately neglecting boundary terms after integration, the energy lower bound

(2.8)
$$\begin{aligned}E &= \int_{\mathbb{R}^2} \mathcal{H}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^2} \mathrm{d}x\bigg\{\left(\frac{\kappa F_{12}^{(1)}}{2q_2|\chi|} \pm \frac{q_1q_2}{\kappa}|\chi|(|\phi|^2 - c_1^2)\right)^2 + \left(\frac{\kappa F_{12}^{(2)}}{2q_1|\phi|} \pm \frac{q_1q_2}{\kappa}|\phi|(|\chi|^2 - c_2^2)\right)^2\\ &\qquad + |D_1\phi \pm \mathrm{i}D_2\phi|^2 + |D_1\chi \pm \mathrm{i}D_2\chi|^2 \pm c_1^2 q_1 F_{12}^{(1)} \pm c_2^2 q_2 F_{12}^{(2)}\bigg\}\\ &\geq \pm c_1^2 q_1 \Phi^{(1)} \pm c_2^2 q_2 \Phi^{(2)} = c_1^2 q_1 |\Phi^{(1)}| + c_2^2 q_2 |\Phi^{(2)}|.\end{aligned}$$

Here the signs are chosen so that $\pm\Phi^{(I)} = |\Phi^{(I)}|$ $(I = 1, 2)$. Hence, it is seen that the energy lower bound stated in (2.8) is attained if and only if the field configuration $(\phi, \chi, A_r^{(1)}, A_r^{(2)})$ satisfies the following elegant equations

$$\begin{cases} D_1\phi \pm \mathrm{i}D_2\phi = 0,\\ D_1\chi \pm \mathrm{i}D_2\chi = 0,\\ F_{12}^{(1)} \pm \dfrac{2q_1q_2^2}{\kappa^2}|\chi|^2(|\phi|^2 - c_1^2) = 0,\\ F_{12}^{(2)} \pm \dfrac{2q_1^2q_2}{\kappa^2}|\phi|^2(|\chi|^2 - c_2^2) = 0. \end{cases} \tag{2.9}$$

The first two equations of (2.9) indicate that the complex fields ϕ and χ are holomorphic or antiholomorphic with respect to the gauge-covariant derivatives. Hence, these fields may be viewed as "extended" harmonic maps [4], whereas the last two equations are "vortex" equations, relating "curvatures" to the "strength" of scalar particles. Equations of such characteristics are sometimes called Hitchin's equations [40]. The four equations in (2.9), supplemented with the Gauss law equations (2.5), are the self-dual Chern-Simons equations involving two Higgs particles and two Abelian (electromagnetic) gauge fields. It can be readily checked that a solution of these equations is automatically a solution of the full Chern-Simons equations of motion (2.4). Therefore, the self-dual Chern-Simons equations, which will be our focus of this paper, are a reduction of the full Chern-Simons equations of motion. In what follows, we will only consider the case of (2.9) with the (upper) plus sign because the case with the (lower) minus sign may then be recovered by a simple transformation (e.g. $A_j^{(1)} \mapsto -A_j^{(1)}$ and $\phi \mapsto \overline{\phi}$).

From the form of the potential energy density (2.3), we see that the finite-energy condition imposes the following boundary conditions at infinity:

$$|\phi(x)| \to c_1, \quad |\chi(x)| \to c_2 \quad \text{as } |x| \to \infty, \tag{2.10}$$

or

$$|\phi(x)| \to 0, \quad |\chi(x)| \to 0 \quad \text{as } |x| \to \infty. \tag{2.11}$$

Solutions satisfying (2.10) are called topological; solutions satisfying (2.11) are called nontopological.

In this paper, we are interested in the existence of topological solutions of (2.9) realizing a prescribed distribution of point vortices, characterized as the zeroes of the Higgs fields ϕ and χ. We establish the main theorem:

Theorem 2.1. *For any prescribed points $p'_1, \dots, p'_{k'}, p''_1, \dots, p''_{k''}$ in $\mathbb{R}^2$ and nonnegative integers $n'_1, \dots, n'_{k'}, n''_1, \dots, n''_{k''}$, the self-dual Chern-Simons equations* (2.9) *have a topological multivortex solution $(\phi, \chi, A_j^{(1)}, A_j^{(2)})$ satisfying the boundary condition* (2.10) *exponentially fast so that $p'_{s'}$ and $p''_{s''}$ are the zeroes of the fields ϕ and χ with corresponding algebraic multiplicities $n'_{s'}$ and $n''_{s''}$, respectively. Moreover,*

$$\int_{\mathbb{R}^2} \left| \frac{|\phi|^2}{c_1^2} - 1 \right| \mathrm{d}x + \int_{\mathbb{R}^2} \left| \frac{|\chi|^2}{c_2^2} - 1 \right| \mathrm{d}x \le \frac{C\kappa^2}{c_1^2 c_2^2 q_1^2 q_2^2} (N' + N'')^2, \tag{2.12}$$

where N' and N'' are the total vortex numbers defined by

$$N' = \sum_{s=1}^{k'} n'_s, \quad N'' = \sum_{s=1}^{k''} n''_s. \tag{2.13}$$

Both $D_j\phi$ and $D_j\chi$ ($j = 1, 2$) vanish at infinity exponentially fast; the magnetic fluxes, electric charges, and energy are all quantized and assume the values

$$\begin{gathered} \Phi^{(1)} = 2\pi N', \quad \Phi^{(2)} = 2\pi N'', \quad Q^{(1)} = 2\pi\kappa N'', \quad Q^{(2)} = 2\pi\kappa N', \\ E = 2\pi\big(c_1^2 q_1 N' + c_2^2 q_2 N''\big). \end{gathered} \tag{2.14}$$

Using the change of variables $q_I A_j^{(I)} \mapsto A_j^{(I)}$ ($I = 1, 2$), $\phi \mapsto c_1\phi$, $\chi \mapsto c_2\chi$, and the suppressed parameter $\lambda = 4c_1^2 c_2^2 q_1^2 q_2^2/\kappa^2$, we can simplify (2.9) (with the upper sign) as

$$\begin{cases} D_1\phi + \mathrm{i}D_2\phi = 0, \\ D_1\chi + \mathrm{i}D_2\chi = 0, \\ F_{12}^{(1)} + \dfrac{\lambda}{2}|\chi|^2(|\phi|^2 - 1) = 0, \\ F_{12}^{(2)} + \dfrac{\lambda}{2}|\phi|^2(|\chi|^2 - 1) = 0, \end{cases} \tag{2.15}$$

where now $D_j\phi = \partial_j\phi - \mathrm{i}A_j^{(1)}\phi$ and $D_j\chi = \partial_j\chi - \mathrm{i}A_j^{(2)}\chi$ ($j = 1, 2$). We note that system (2.15) has two interesting limiting cases:

(i) when $N'' = 0$, we may choose $A_j^{(2)} = 0$ and $|\chi| = 1$, which renders (2.15) into

$$D_1\phi + \mathrm{i}D_2\phi = 0, \quad F_{12} + \frac{\lambda}{2}(|\phi|^2 - 1) = 0; \tag{2.16}$$

(ii) when $p'_s = p''_s$ and $n'_s = n''_s$ for $s = 1, 2, \dots, k'$, with $k' = k''$, we may take $\phi = \chi$ and $A_j^{(1)} = A_j^{(2)}$ $(j = 1, 2)$ which renders (2.15) into

$$D_1\phi + \mathrm{i}D_2\phi = 0, \quad F_{12} + \frac{\lambda}{2}|\phi|^2(|\phi|^2 - 1) = 0. \tag{2.17}$$

System (2.16) is the familiar self-dual Ginzburg-Landau equations [11, 12, 44, 52, 54], while system (2.17) is the well-studied single-particle self-dual Abelian Chern-Simons equations [20, 21, 27, 41, 43, 57, 58, 60]; see also Appendix B below. In the general situation, no such reduction can be made and the full system (2.15) has to be solved, which is the goal of this paper.

3. Equivalence between (1.2) and the self-dual Chern-Simons equations

Let ϕ and χ be two complex functions with the prescribed zeroes stated in Theorem 2.1. Then, with the substitutions $u = \ln|\phi|^2$ and $v = \ln|\chi|^2$, we can transform (2.15) into the equivalent form

$$\begin{cases} \Delta u = \lambda \mathrm{e}^v(\mathrm{e}^u - 1) + 4\pi \displaystyle\sum_{s=1}^{N'} \delta_{p'_s}, \\ \Delta v = \lambda \mathrm{e}^u(\mathrm{e}^v - 1) + 4\pi \displaystyle\sum_{s=1}^{N''} \delta_{p''_s} \end{cases} \tag{3.1}$$

over $\mathbb{R}^2$, where we have incorporated multiplicities in order to save notation. The topological boundary condition, translated in terms of u and v, reads

$$\lim_{|x|\to\infty} u(x) = 0, \quad \lim_{|x|\to\infty} v(x) = 0. \tag{3.2}$$

Due to some technical issues, it is hard to pursue a solution of (3.1) over the full space $\mathbb{R}^2$ subject to (3.2). Instead, we will first consider (3.1) over a bounded domain Ω containing all points p'_s $(s = 1, 2, \dots, N')$ and p''_s $(s = 1, 2, \dots, N'')$, subject to the homogeneous boundary condition

$$u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \tag{3.3}$$

Concerning (3.1), the following result is of independent interest:

Theorem 3.1. *Let Ω be a bounded domain containing $p'_1, \dots, p'_{N'}, p''_1, \dots, p''_{N''}$. Then, system* (3.1) *over Ω subject to the homogeneous boundary condition* (3.3) *has a solution $(u, v) \in L^1(\Omega) \times L^1(\Omega)$. Moreover,*

$$\|u\|_{L^1} + \|v\|_{L^1} \le \frac{C}{\lambda}\big(N' + N''\big)^3, \tag{3.4}$$

$$\|\mathrm{e}^u - 1\|_{L^1} + \|\mathrm{e}^v - 1\|_{L^1} \le \frac{C}{\lambda}\big(N' + N''\big)^2. \tag{3.5}$$

Remark 3.1. In view of standard comparison results (see Proposition A.1 in Appendix A), we know that every solution of (3.1) under (3.3) satisfies

$$u, v \le 0 \quad \text{a.e.}$$

We will show that we can use the bounded-domain solutions constructed in Theorem 3.1 and take the limit as Ω tends to $\mathbb{R}^2$ to get a solution of (3.1) over the full space $\mathbb{R}^2$ subject to the topological boundary condition (3.2):

Theorem 3.2. *On the full plane $\mathbb{R}^2$, system* (3.1) *has a solution pair $(u,v) \in L^1(\mathbb{R}^2)\times L^1(\mathbb{R}^2)$ satisfying the boundary condition* (3.2) *and estimates* (3.4)–(3.5)*. Moreover, this boundary condition is achieved exponentially fast at infinity; more precisely,*

$$|u(x)| + |v(x)| \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}}, \tag{3.6}$$

$$|\nabla u(x)| + |\nabla v(x)| \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}}, \tag{3.7}$$

for every $|x|$ sufficiently large.

The proof of the existence of a solution will be carried out in Section 5. Here we only sketch the proofs for the decay estimates. Indeed, near infinity the linearized equations of (3.1) are $\Delta u = \lambda u$ and $\Delta v = \lambda v$. Hence, u and v decay exponentially fast at infinity and (3.6) holds. Furthermore, using L^p-estimates in (3.1) in a neighborhood of infinity we deduce that u and v belong to $W^{2,p}$ (again in a neighborhood of infinity) for any $p > 2$. Hence, $|\nabla u| \to 0$ and $|\nabla v| \to 0$ as $|x| \to \infty$. Differentiating (3.1), we see that the components of ∇u and ∇v satisfy the same linearized equation. Therefore, the estimate for $|\nabla u| + |\nabla v|$ stated in (3.7) is valid. The detailed proof is presented in Section 16 below.

Using the solution pair (u,v) over $\mathbb{R}^2$, we can follow a standard path to construct a solution $(\phi,\chi,A_j^{(1)},A_j^{(2)})$ of system (2.9). For example, using the complex variable $z = x^1 + \mathrm{i}x^2$ and setting $\partial = (\partial_1 - \mathrm{i}\partial_2)/2$, we get

$$\begin{aligned} \theta(z) &= -\sum_{s=1}^{N'} \arg(z - p_s'),\\ \phi(z) &= \exp\left(\frac{1}{2}u(z) + \mathrm{i}\theta(z)\right),\\ A_1^{(1)}(z) = -\mathrm{Re}\big\{2\mathrm{i}\overline{\partial}\ln\phi(z)\big\}&, \quad A_2^{(1)}(z) = -\mathrm{Im}\big\{2\mathrm{i}\overline{\partial}\ln\phi(z)\big\}. \end{aligned} \tag{3.8}$$

These relations allow us to calculate the gauge-covariant derivatives explicitly:

$$\begin{aligned} D_1\phi &= (\partial + \overline{\partial})\phi - \left(\frac{\overline{\partial}\phi}{\phi} - \frac{\partial\overline{\phi}}{\overline{\phi}}\right)\phi = \phi\partial u,\\ D_2\phi &= \mathrm{i}(\partial - \overline{\partial})\phi + \mathrm{i}\left(\frac{\overline{\partial}\phi}{\phi} + \frac{\partial\overline{\phi}}{\overline{\phi}}\right)\phi = \mathrm{i}\phi\partial u. \end{aligned} \tag{3.9}$$

Consequently, we obtain

$$|D_1\phi|^2 + |D_2\phi|^2 = \frac{1}{2}\mathrm{e}^u|\nabla u|^2. \tag{3.10}$$

Identities (3.8) and (3.10), and Theorem 3.2 imply that both $1 - |\phi|^2$ and $|D_j\phi|$ ($j = 1,2$) vanish at infinity exponentially fast as stated in Theorem 2.1. Similarly, we can derive the decay estimates for $1 - |\chi|^2$ and $|D_j\chi|$ ($j = 1,2$). With such decay estimates, the quantum numbers for the fluxes, charges, and energy stated in Theorem 2.1 can be easily computed. Estimate (2.12) follows from (3.5).

4. Variational solutions of system (1.2) on bounded domains

In this section, we prove Theorem 3.1 by a variational method. Our strategy is as follows. First, in order to overcome the difficulty associated with the vortex points $p'_1, \dots, p'_{N'}, p''_1, \dots, p''_{N''}$, we consider a regularized version of the equations so that the Dirac masses δ_{p_s} are replaced by smooth functions labeled by a small positive parameter ε. We then introduce another change of dependent variables so that the regularized equations have a variational principle. The solutions of the new system are critical points of an indefinite action functional. We shall formulate a constrained variational problem and prove the existence of a solution to this problem. We then show that the solution we obtain for the constrained variational problem is in fact a critical point of the indefinite action functional, hence a classical solution of the original system of the ε-regularized nonlinear equations. As $\varepsilon \to 0$, we recover a solution of the two-Higgs Chern-Simons multivortex equations over a bounded domain, which establishes the theorem.

Proof of Theorem 3.1. Given $\varepsilon > 0$, let us replace (3.1) by a regularized form

$$\begin{cases} \Delta u = \lambda \mathrm{e}^v(\mathrm{e}^u - 1) + \displaystyle\sum_{s=1}^{N'} \frac{4\varepsilon}{(\varepsilon + |x - p'_s|^2)^2} \quad \text{in } \Omega, \\ \Delta v = \lambda \mathrm{e}^u(\mathrm{e}^v - 1) + \displaystyle\sum_{s=1}^{N''} \frac{4\varepsilon}{(\varepsilon + |x - p''_s|^2)^2} \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

subject to the boundary condition (3.3). It is clear that

$$\frac{4\varepsilon}{(\varepsilon + |x-p|^2)^2} \overset{*}{\rightharpoonup} 4\pi\delta_p \quad \text{as } \varepsilon \to 0.$$

Introduce the background functions

$$u_0^\varepsilon(x) = \sum_{s=1}^{N'} \ln\left(\frac{\varepsilon + |x - p'_s|^2}{1 + |x - p'_s|^2}\right), \quad v_0^\varepsilon(x) = \sum_{s=1}^{N''} \ln\left(\frac{\varepsilon + |x - p''_s|^2}{1 + |x - p''_s|^2}\right). \tag{4.2}$$

Then,

$$\Delta u_0^\varepsilon = -h_1 + \sum_{s=1}^{N'} \frac{4\varepsilon}{(\varepsilon + |x - p'_s|^2)^2}, \quad \Delta v_0^\varepsilon = -h_2 + \sum_{s=1}^{N''} \frac{4\varepsilon}{(\varepsilon + |x - p''_s|^2)^2},$$

where $h_1, h_2 \in W^{1,2}(\Omega)$ do not depend on $\varepsilon > 0$. Set $u = u_0^\varepsilon + f$ and $v = v_0^\varepsilon + g$ in (4.1). We get

$$\begin{cases} \Delta f = \lambda \mathrm{e}^{v_0^\varepsilon + g}(\mathrm{e}^{u_0^\varepsilon + f} - 1) + h_1 \quad \text{in } \Omega, \\ \Delta g = \lambda \mathrm{e}^{u_0^\varepsilon + f}(\mathrm{e}^{v_0^\varepsilon + g} - 1) + h_2 \quad \text{in } \Omega. \end{cases} \tag{4.3}$$

In order to fulfill the homogeneous boundary condition, we write $f = U_0^\varepsilon + f'$ and $g = V_0^\varepsilon + g'$ where U_0^ε and V_0^ε are harmonic functions on Ω satisfying

$$U_0^\varepsilon = -u_0^\varepsilon, \quad V_0^\varepsilon = -v_0^\varepsilon \quad \text{on } \partial\Omega. \tag{4.4}$$

In view of these modifications, system (4.3) becomes

$$\begin{cases} \Delta f' = \lambda \mathrm{e}^{v_0^\varepsilon + V_0^\varepsilon + g'}(\mathrm{e}^{u_0^\varepsilon + U_0^\varepsilon + f'} - 1) + h_1 \quad \text{in } \Omega, \\ \Delta g' = \lambda \mathrm{e}^{u_0^\varepsilon + U_0^\varepsilon + f'}(\mathrm{e}^{v_0^\varepsilon + V_0^\varepsilon + g'} - 1) + h_2 \quad \text{in } \Omega, \\ f' = 0, \quad g' = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Set
$$f_0^\varepsilon = u_0^\varepsilon + U_0^\varepsilon, \quad g_0^\varepsilon = v_0^\varepsilon + V_0^\varepsilon, \quad f' + g' = F, \quad f' - g' = G.$$
Then, (4.5) becomes

$$\begin{cases} \Delta F = 2\lambda \mathrm{e}^{f_0^\varepsilon + g_0^\varepsilon + F} - \lambda \mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} - \lambda \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)} + (h_1 + h_2) \quad \text{in } \Omega, \\ \Delta G = \lambda \mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} - \lambda \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)} + (h_1 - h_2) \quad \text{in } \Omega, \\ \ F = 0, \quad G = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

It is clear that the equations in (4.6) are the Euler-Lagrange equations of the action functional

$$\begin{aligned} I(F,G) = \int_\Omega \mathrm{d}x \Big\{ \frac{1}{2}|\nabla F|^2 - \frac{1}{2}|\nabla G|^2 + 2\lambda \mathrm{e}^{f_0^\varepsilon + g_0^\varepsilon + F} - 2\lambda \mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} \\ - 2\lambda \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)} + (h_1 + h_2)F - (h_1 - h_2)G \Big\} \end{aligned} \tag{4.7}$$

which is indefinite. The study of critical points of such indefinite functionals was initiated by Benci-Rabinowitz [6]; see also [33].
We consider the following constrained minimization problem:

$$\min \big\{ I(F,G);\ (F,G) \in \mathcal{C} \big\}; \tag{4.8}$$

the admissible class $\mathcal{C}$ is defined by

$$\mathcal{C} = \Big\{ (F,G);\ F, G \in W_0^{1,2}(\Omega),\ F \text{ and } G \text{ satisfy (E)} \Big\}, \tag{4.9}$$

where

$$\int_\Omega \Big\{ \nabla G \cdot \nabla H + \lambda \big[\mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} - \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)} \big] H + (h_1 - h_2)H \Big\} \, \mathrm{d}x = 0, \quad \forall H \in W_0^{1,2}(\Omega). \tag{E}$$

Lemma 4.1. *Definition* (E) *is well-posed. More precisely, for any* $F \in W_0^{1,2}(\Omega)$, *there is a unique* $G \in W_0^{1,2}(\Omega)$ *satisfying* (E)*;* G *is the global minimizer of the functional*

$$J_F(G) = \int_\Omega \Big\{ \frac{1}{2}|\nabla G|^2 + 2\lambda \mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} + 2\lambda \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)} + (h_1 - h_2)G \Big\} \, \mathrm{d}x \tag{4.10}$$

in $W_0^{1,2}(\Omega)$.

Proof. Using the Trudinger-Moser inequality [2], we know that $J_F(\cdot)$ is weakly lower semicontinuous over $W_0^{1,2}(\Omega)$. Next, since the Poincaré inequality implies the coerciveness

$$J_F(G) \geq \frac{1}{4}\|\nabla G\|_{L^2}^2 - C_1 \tag{4.11}$$

where $C_1 > 0$ depends only on h_1 and h_2, we see that (4.10) has a global minimizer. The existence of a critical point follows. Since the functional (4.10) is convex, its critical point must be unique. ∎

Note that $I(F,G)$ defined in (4.7) can be rewritten as

$$I(F,G) = \frac{1}{2}\|\nabla F\|_{L^2}^2 + 2\lambda \int_\Omega \mathrm{e}^{f_0^\varepsilon + g_0^\varepsilon + F} \, \mathrm{d}x + \int_\Omega (h_1 + h_2)F \, \mathrm{d}x - J_F(G). \tag{4.12}$$

For any $(F,G)\in\mathcal{C}$, since G minimizes J_F, we have, in particular, $J_F(G)\le J_F(0)$. Hence,

$$\begin{aligned}I(F,G)&\ge\frac{1}{2}\|\nabla F\|_{L^2}^2+2\lambda\int_\Omega \mathrm{e}^{f_0^\varepsilon+g_0^\varepsilon+F}\,\mathrm{d}x+\int_\Omega(h_1+h_2)F\,\mathrm{d}x-J_F(0)\\ &=\frac{1}{2}\|\nabla F\|_{L^2}^2+\int_\Omega(h_1+h_2)F\,\mathrm{d}x\\ &\quad+2\lambda\int_\Omega\Big(\mathrm{e}^{f_0^\varepsilon+g_0^\varepsilon+F}-\mathrm{e}^{f_0^\varepsilon+\frac{1}{2}F}-\mathrm{e}^{g_0^\varepsilon+\frac{1}{2}F}\Big)\,\mathrm{d}x.\end{aligned}\tag{4.13}$$

Consider the function $\sigma(t)=abt^2-at-bt$. It is seen that the global minimum of $\sigma(\cdot)$ is attained at $t_0=(a+b)/2ab$. Hence, $\sigma(t)\ge\sigma(t_0)=-(a+b)^2/4ab$. As a consequence, we have

$$\mathrm{e}^{f_0^\varepsilon+g_0^\varepsilon+F}-\mathrm{e}^{f_0^\varepsilon+\frac{1}{2}F}-\mathrm{e}^{g_0^\varepsilon+\frac{1}{2}F}\ge-\frac{1}{4}\mathrm{e}^{-f_0^\varepsilon-g_0^\varepsilon}(\mathrm{e}^{f_0^\varepsilon}+\mathrm{e}^{g_0^\varepsilon})^2.\tag{4.14}$$

Inserting (4.14) into (4.13), we see that there holds a partial coerciveness inequality:

$$I(F,G)\ge\frac{1}{4}\|\nabla F\|_{L^2}^2-C(\varepsilon),\tag{4.15}$$

where $C(\varepsilon)>0$ is a constant depending on the parameter ε. In particular, $I(F,G)$ is bounded from below.

Let $\big((F_n,G_n)\big)_{n\ge1}$ be a minimizing sequence of (4.8). We may assume that

$$I(F_1,G_1)\ge I(F_2,G_2)\ge\ldots\ge I(F_n,G_n)\ge\ldots$$

Denote by

$$\eta_0:=\inf\big\{I(F,G);\ (F,G)\in\mathcal{C}\big\}=\lim_{n\to\infty}I(F_n,G_n).$$

By (4.15), (F_n) is bounded in $W_0^{1,2}(\Omega)$. On the other hand, using (4.11) we get

$$\begin{aligned}\frac{1}{4}\|\nabla G_n\|_{L^2}^2&\le C_1+J_{F_n}(G_n)\\ &\le C_1+J_{F_n}(0)=C_1+2\lambda\int_\Omega(\mathrm{e}^{f_0^\varepsilon}+\mathrm{e}^{g_0^\varepsilon})\mathrm{e}^{\frac{1}{2}F_n}\,\mathrm{d}x.\end{aligned}\tag{4.16}$$

The boundedness of the integral on the right-hand side of (4.16) is a consequence of the Trudinger-Moser inequality and the boundedness of $\big(\|F_n\|_{W_0^{1,2}}\big)$. Hence, (G_n) is also bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, we may then assume that

$$F_n\rightharpoonup F,\quad G_n\rightharpoonup G\quad\text{weakly in }W_0^{1,2}(\Omega).\tag{4.17}$$

In order to show that the weak limit (F,G) is a solution to (4.8), we need to strengthen (4.17):

Lemma 4.2. *The functions F and G defined in* (4.17) *satisfy $(F,G)\in\mathcal{C}$ and $G_n\to G$ strongly in $W_0^{1,2}(\Omega)$ as $n\to\infty$.*

Proof. The pair (F_n,G_n) satisfies

$$\int_\Omega\Big\{\nabla G_n\cdot\nabla H+\lambda\big[\mathrm{e}^{f_0^\varepsilon+\frac{1}{2}(F_n+G_n)}-\mathrm{e}^{g_0^\varepsilon+\frac{1}{2}(F_n-G_n)}\big]H+(h_1-h_2)H\Big\}\,\mathrm{d}x=0,\quad\forall H\in W_0^{1,2}(\Omega).\tag{4.18}$$

We may assume that $F_n \to F$ and $G_n \to G$ strongly in $L^2(\Omega)$. Hence, the Trudinger-Moser inequality implies that $\mathrm{e}^{F_n} \to \mathrm{e}^F$ and $\mathrm{e}^{G_n} \to \mathrm{e}^G$ strongly in $L^2(\Omega)$. Taking $n \to \infty$ in (4.18), we get (E). In other words, $(F, G) \in \mathcal{C}$.
Choose $H = G_n - G$ in (E) and (4.18). Subtracting the resulting relations we obtain

$$\begin{aligned}\int_\Omega |\nabla G_n - \nabla G|^2 \,\mathrm{d}x &= \\ = \lambda \int_\Omega \Big\{ \mathrm{e}^{f_0^\varepsilon} \big[\mathrm{e}^{\frac{1}{2}(F+G)} - \mathrm{e}^{\frac{1}{2}(F_n+G_n)}\big](G_n - G) &+ \mathrm{e}^{g_0^\varepsilon}\big[\mathrm{e}^{\frac{1}{2}(F_n-G_n)} - \mathrm{e}^{\frac{1}{2}(F-G)}\big](G_n - G)\Big\} \,\mathrm{d}x \\ &\longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, $G_n \to G$ strongly in $W_0^{1,2}(\Omega)$ as claimed. □

Lemma 4.3. *The pair* (F, G) *defined in* (4.17) *is a solution of the minimization problem* (4.8).

Proof. Since $F_n \rightharpoonup F$ weakly and $G_n \to G$ strongly in $W_0^{1,2}(\Omega)$, we have

$$\lim_{n\to\infty} J_{F_n}(G_n) = J_F(G). \tag{4.19}$$

Hence, using (4.12) and (4.19) we arrive at

$$\begin{aligned}\eta_0 &= \lim_{n\to\infty} I(F_n, G_n) \\ &\geq \frac{1}{2}\|\nabla F\|_{L^2}^2 + 2\lambda \int_\Omega \mathrm{e}^{f_0^\varepsilon + g_0^\varepsilon + F} \,\mathrm{d}x + \int_\Omega (h_1 + h_2) F \,\mathrm{d}x - J_F(G) = I(F, G).\end{aligned}$$

Since $(F, G) \in \mathcal{C}$, we see that (F, G) solves (4.8). □

Lemma 4.4. *The pair* (F, G) *defined in* (4.17) *is a solution of the system* (4.6).

Proof. The second equation (for G) in (4.6) is already valid because its weak form is the constraint defined in (E). In what follows, we only need to verify the first equation in (4.6).
Let $\tilde{F} \in W_0^{1,2}(\Omega)$ be any test function and set $F_t = F + t\tilde{F}$. The unique minimizer of $J_{F_t}(\cdot)$ is denoted by G_t. Then, G_t depends on t smoothly. Set

$$\tilde{G} = \left(\frac{\mathrm{d}}{\mathrm{d}t} G_t\right)_{t=0}.$$

Since $I(F_t, G_t)$ attains its minimum at $t = 0$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} I(F_t, G_t)\right)_{t=0} = 0. \tag{4.20}$$

In view of (4.7), the expression (4.20) can be rewritten as

(4.21)
$$\begin{aligned}\int_\Omega \Big\{\nabla F \cdot \nabla \tilde{F} + \lambda\big[2\mathrm{e}^{f_0^\varepsilon + g_0^\varepsilon + F} - \mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} - \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)}\big]\tilde{F} + (h_1 + h_2)\tilde{F}\Big\} \,\mathrm{d}x \\ = \int_\Omega \Big\{\nabla G \cdot \nabla \tilde{G} + \lambda\big[\mathrm{e}^{f_0^\varepsilon + \frac{1}{2}(F+G)} - \mathrm{e}^{g_0^\varepsilon + \frac{1}{2}(F-G)}\big]\tilde{G} + (h_1 - h_2)\tilde{G}\Big\} \,\mathrm{d}x.\end{aligned}$$

However, in view of (E), the right-hand side of (4.21) vanishes. Since $\tilde{F} \in W_0^{1,2}(\Omega)$ is arbitrary, we obtain the weak form of the first equation in (4.6). So the system (4.6) is fully verified. □

We next go back to the original variables. We see that we have obtained a solution pair, say $(u_\varepsilon, v_\varepsilon)$, of the system (4.1). Using the maximum principle, it is seen that u_ε and v_ε are negative:

$$u_\varepsilon < 0, \quad v_\varepsilon < 0 \quad \text{in } \Omega. \tag{4.22}$$

In order to take the $\varepsilon \to 0$ limit, we also need to bound u_ε and v_ε from below. For this purpose, we add the two equations in (4.1). Using the convexity of e^t, we get

$$\begin{aligned}\Delta(u_\varepsilon + v_\varepsilon) &= 2\lambda \mathrm{e}^{u_\varepsilon + v_\varepsilon} - \lambda(\mathrm{e}^{u_\varepsilon} + \mathrm{e}^{v_\varepsilon}) + \sum_{s=1}^{N'} \frac{4\varepsilon}{(\varepsilon + |x - p'_s|^2)^2} + \sum_{s=1}^{N''} \frac{4\varepsilon}{(\varepsilon + |x - p''_s|^2)^2} \\ &\le 2\lambda(\mathrm{e}^{u_\varepsilon + v_\varepsilon} - \mathrm{e}^{\frac{1}{2}(u_\varepsilon + v_\varepsilon)}) + \sum_{s=1}^{N'} \frac{4\varepsilon}{(\varepsilon + |x - p'_s|^2)^2} + \sum_{s=1}^{N''} \frac{4\varepsilon}{(\varepsilon + |x - p''_s|^2)^2}.\end{aligned}$$

In particular, the "average" $\frac{1}{2}(u_\varepsilon + v_\varepsilon)$ is a supersolution of the (regularized) classical Chern-Simons equation:

$$\begin{cases} \Delta w_\varepsilon = \lambda \mathrm{e}^{w_\varepsilon}(\mathrm{e}^{w_\varepsilon} - 1) + \displaystyle\sum_{s=1}^{N'} \frac{4\varepsilon}{(\varepsilon + |x - p'_s|^2)^2} + \sum_{s=1}^{N''} \frac{4\varepsilon}{(\varepsilon + |x - p''_s|^2)^2} \quad \text{in } \Omega, \\ w_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.23}$$

It is standard (see [58]) that one can start a monotone decreasing iterative scheme from $\frac{1}{2}(u_\varepsilon + v_\varepsilon)$ to get a solution of (4.23). In particular,

$$w_\varepsilon \le \frac{1}{2}(u_\varepsilon + v_\varepsilon) \quad \text{in } \Omega. \tag{4.24}$$

Let

$$w_0^\varepsilon = u_0^\varepsilon + v_0^\varepsilon + W_0^\varepsilon, \tag{4.25}$$

where $u_0^\varepsilon, v_0^\varepsilon$ are given by (4.2) and W_0^ε is a harmonic function chosen so that $w_0^\varepsilon = 0$ on $\partial\Omega$. Note that W_0^ε is uniformly bounded with respect to $\varepsilon > 0$.
In order to get suitable estimates for w_ε, we rewrite (4.23) as

$$\begin{cases} \Delta \tilde{w}_\varepsilon = \lambda \mathrm{e}^{w_0^\varepsilon + \tilde{w}_\varepsilon}(\mathrm{e}^{w_0^\varepsilon + \tilde{w}_\varepsilon} - 1) + h_1 + h_2 \quad \text{in } \Omega, \\ \tilde{w}_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

Using $w_0^\varepsilon + \tilde{w}_\varepsilon \le 0$, we can multiply (4.26) by $\tilde{w}_\varepsilon$ and integrate to get

$$\|\nabla \tilde{w}_\varepsilon\|_{L^2}^2 \le C_1, \tag{4.27}$$

where $C_1 > 0$ is a constant independent of $\varepsilon > 0$.
Combining (4.22), (4.24)–(4.25), and (4.27), we see that (u_ε) and (v_ε) are uniformly bounded in $L^2(\Omega)$. Using this fact with interior elliptic estimates (see [37]), we may assume (passing to a subsequence if necessary) that there are functions

$$u, v \in C^0\big(\Omega \setminus \{p'_1, \dots, p'_{N'}, p''_1, \dots, p''_{N''}\}\big) \cap L^2(\Omega)$$

such that

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in } C^0(K) \quad \text{as } \varepsilon \to 0 \tag{4.28}$$

for any compact subset $K \subset \Omega \setminus \{p'_1, \dots, p'_{N'}, p''_1, \dots, p''_{N''}\}$ and

$$(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v) \quad \text{weakly in } L^2(\Omega). \tag{4.29}$$

Using the Green function to represent the two equations in (4.1) (with $u = u_\varepsilon$ and $v = v_\varepsilon$) in potential integral forms and applying (4.28)–(4.29), we see that, as $\varepsilon \to 0$, (u, v) satisfies the original equations (3.1).
Estimates (3.4)–(3.5) follow from Theorem 12.1 and Proposition 12.2 below. We refer the reader to Section 13 for the details. The proof of Theorem 3.1 is complete. □

5. The limit $\Omega \to \mathbb{R}^2$: the single equation case

Consider the single Higgs particle Chern-Simons vortex equation subject to homogeneous boundary condition:

$$\begin{cases} \Delta u = \lambda \mathrm{e}^u(\mathrm{e}^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j} \quad \text{in } B_R, \\ u = 0 \quad \text{on } \partial B_R, \end{cases} \tag{5.1}$$

where

$$B_R = \{x \in \mathbb{R}^2;\ |x| < R\} \quad \text{and} \quad R > R_0 := \max_{1 \le j \le N} \{|p_j|\}.$$

It is known that (5.1) always has a solution (see Appendix B below). The main goal of this section is to prove the natural result that, as $R \to \infty$, the solutions of (5.1) approach a topological solution of the single Higgs particle Chern-Simons vortex equation on $\mathbb{R}^2$ so that it vanishes at infinity. This result is a preliminary step as we take the large domain limit with the bounded domain solutions obtained in Theorem 3.1. For this purpose, we need to derive some important properties of solutions of the equation in (5.1) over a bounded domain or $\mathbb{R}^2$.

The proof of the next result is based on the method of moving planes of Aleksandrov and Gidas-Ni-Nirenberg [36].

Lemma 5.1. *Every solution of* (5.1) *increases along any radial direction on* $\mathbb{R}^2 \setminus B_{R_0}$.

Proof. We denote by u a solution of (5.1). It suffices to prove the lemma when the point $x = (x_1, x_2)$ changes its position along the x_1-axis.
Given $R > R_0$, let $A_R = \{x;\ R_0 \le |x| \le R\}$. For $R_0 < \sigma < R$, define the set

$$\Sigma_\sigma = \{x \in B_R;\ x_1 > \sigma\} \tag{5.2}$$

and $u_\sigma(x) = u(x^\sigma)$ for $x \in \Sigma_\sigma$ where x^σ is the reflection of x with respect to the line $x_1 = \sigma$. That is, $x^\sigma = (2\sigma - x_1, x_2)$. Since $u < 0$ in B_R (by the maximum principle) and $\Delta u = \lambda c(x) u$ in A_R, where $c(x) = \mathrm{e}^{u(x)+\xi(x)}$ and $u \le \xi \le 0$, we can use the well-known Hopf Boundary Lemma to deduce that

$$\frac{\partial u}{\partial \mathrm{n}}(x) > 0 \quad \text{if } |x| = R, \tag{5.3}$$

where n is the outnormal of B_R at x. In particular,

$$u(x) > u(x^\sigma) = u_\sigma(x) \quad \text{for } x \in \Sigma_\sigma \tag{5.4}$$

if σ is sufficiently close to R. We need to prove (5.4) for all $\sigma \in (R_0, R)$.
Set $w_\sigma(x) = u(x) - u_\sigma(x)$ for $x \in \Sigma_\sigma$. By (5.1), the function w_σ satisfies

$$\begin{cases} \Delta w_\sigma + \lambda c_\sigma(x) w_\sigma = -4\pi \sum_{j} \delta_{p_j^\sigma} \le 0 \quad \text{in } \Sigma_\sigma, \\ w_\sigma \ge 0 \quad \text{on } \partial\Sigma_\sigma, \end{cases} \tag{5.5}$$

where the sum $\sum_j \delta_{p_j^\sigma}$ is computed over all points p_j such that $p_j^\sigma \in \Sigma_\sigma$. Note that c_σ is a bounded in Σ_σ. Indeed, we can write $c_\sigma(x) = f'(\tilde{\xi}(x))$ with $f(t) = \mathrm{e}^t(1-\mathrm{e}^t)$ and $\tilde{\xi}(x)$ lying between $u(x)$ and $u_\sigma(x)$.
Define

$$\begin{aligned} S &= \big\{\rho \in (R_0, R);\ w_\sigma(x) > 0 \text{ in } \Sigma_\sigma \text{ for } \sigma \in (\rho, R)\big\}, \\ \rho_0 &= \inf_{\rho \in S}\{\rho\}. \end{aligned} \tag{5.6}$$

It is clear that $S \neq \emptyset$. If $\rho_0 = R_0$, then the lemma is established.
Suppose by contradiction that $\rho_0 > R_0$. Then, by continuity we have $w_{\rho_0}(x) \geq 0$ in Σ_{ρ_0}.
Note that if $p_j^{\rho_0} \in \Sigma_{\rho_0}$ for some j, then $w_{\rho_0}(x) = u(x) - u(x^{\rho_0}) > 0$ in a neighborhood of $p_j^{\rho_0}$. Thus, if $w_{\rho_0}(x_0) = 0$ for some $x_0 \in \Sigma_{\rho_0}$, then x_0 is not near $p_j^{\rho_0}$ for any $j = 1, 2, \ldots, N$, and by (5.5) and the strong maximum principle, we have $w_{\rho_0} \equiv 0$. This contradicts the fact that $w_{\rho_0}(x) = u(x) - u(x^{\rho_0}) = -u(x^{\rho_0}) > 0$ for every $x \in \partial\Sigma_{\rho_0} \setminus \{x_1 = \rho_0\}$. Therefore,

$$w_{\rho_0}(x) > 0 \quad \forall x \in \Sigma_{\rho_0}.$$

On the other hand, by a maximum principle of Varadhan (see [39, Theorem 2.32]), there exists $\delta > 0$ depending only on λ and $\|c_\sigma\|_{L^\infty}$ such that if ω is a subdomain of Σ_σ, with $|\omega| \leq \delta$, and U satisfies

$$\begin{cases} \Delta U + \lambda c_\sigma(x) U \leq 0 & \text{in } \omega, \\ U \geq 0 & \text{on } \partial\omega, \end{cases} \tag{5.7}$$

then $U(x) \geq 0$ for all $x \in \omega$.
We now choose a compact set $K \subset \Sigma_{\rho_0}$ such that $|\Sigma_{\rho_0} \setminus K| \leq \delta/2$. Since $w_{\rho_0} > 0$ in K, there exists $\varepsilon_0 \in (0, \rho_0 - R_0)$ sufficiently small so that

$$w_\sigma > 0 \quad \text{in } K \quad \text{and} \quad |\Sigma_\sigma \setminus K| \leq \delta \quad \forall \sigma \in (\rho_0 - \varepsilon_0, \rho_0). \tag{5.8}$$

In particular,

$$w_\sigma \geq 0 \quad \text{on } \partial(\Sigma_\sigma \setminus K) \quad \forall \sigma \in (\rho_0 - \varepsilon_0, \rho_0).$$

Thus, by (5.8) and the maximum principle of Varadhan (applied to $\omega = \Sigma_\sigma \setminus K$),

$$w_\sigma \geq 0 \quad \text{in } \Sigma_\sigma \quad \forall \sigma \in (\rho_0 - \varepsilon_0, \rho_0). \tag{5.9}$$

As before, we can strengthen (5.9) by the strong maximum principle to conclude that $w_\sigma(x) > 0$ for every $x \in \Sigma_\sigma$ whenever $\sigma \in (\rho_0 - \varepsilon_0, \rho_0)$, which contradicts the definition of ρ_0. Thus, $\rho_0 = R_0$ and the proof is complete. □

We now consider the single Higgs particle Chern-Simons equation over the full space. Namely,

$$\Delta u + \lambda \mathrm{e}^u(1 - \mathrm{e}^u) = 4\pi \sum_{j=1}^N \delta_{p_j} \quad \text{in } \mathbb{R}^2. \tag{5.10}$$

Lemma 5.2. *Let u be a solution of* (5.10) *satisfying*

$$u(x) < 0 \quad \textit{in } \mathbb{R}^2 \quad \textit{and} \quad \int_{\mathbb{R}^2} \mathrm{e}^u(1 - \mathrm{e}^u)\,\mathrm{d}x < \infty. \tag{5.11}$$

Then, we have the asymptotic estimate

$$u(x) = -\alpha \ln|x| + O(1) \tag{5.12}$$

as $|x| \to \infty$*, where*

$$\alpha = \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^u (1 - \mathrm{e}^u) \, \mathrm{d}x - 2N. \tag{5.13}$$

In the case $\alpha = 0$*, the function* u *vanishes exponentially fast at infinity. In fact,* u *is a topological solution (i.e.* $u = 0$ *at infinity) if and only if*

$$\lambda \int_{\mathbb{R}^2} \mathrm{e}^u (1 - \mathrm{e}^u) \, \mathrm{d}x = 4\pi N. \tag{5.14}$$

On $\mathbb{R}^2 \setminus B_{R_0+1}$, equation (5.10) is of the form

$$\Delta u + K(x) \mathrm{e}^u = 0, \tag{5.15}$$

where $K(x) = \lambda(1 - \mathrm{e}^{u(x)})$. Lemma 5.2 is essentially [24, Theorem 1.1] when $K(x)$ satisfies

$$C_1 \mathrm{e}^{-|x|^\beta} \le K(x) \le C_2 |x|^m \tag{5.16}$$

for $|x|$ large, where $\beta \in (0, 1)$ and $m > 0$ are two constants. In our case, $K(x)$ need not satisfy the lower bound in (5.16); we have to modify the argument in the proof of [24, Theorem 2.1].

Proof of Lemma 5.2. Let u be a solution of (5.15) with $K(x) = \lambda(1 - \mathrm{e}^{u(x)})$ for $|x| \ge R_0 + 1$. We extend u in $\mathbb{R}^2$ as a smooth negative function; we still denote this extended function by u. It is seen that u satisfies

$$\Delta u + K(x) \mathrm{e}^u = 0 \quad \text{in } \mathbb{R}^2, \tag{5.17}$$

where we set $K(x) = -\mathrm{e}^{-u(x)} \Delta u(x)$ for $|x| \le R_0 + 1$. Define the potential

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \left(\frac{|x - y|}{|y|} \right) K(y) \mathrm{e}^{u(y)} \, \mathrm{d}y. \tag{5.18}$$

As in [24, Theorem 2.1], we can show that $u + v$ is in fact a constant. For this purpose, let $\psi(x) = K(x) \mathrm{e}^{u(x)}$ and write

$$\begin{aligned} 2\pi v(x) &= \left(\int_{|y| \le R_0 + 1} + \int_{T_1} + \int_{T_2} \right) \ln \left(\frac{|x - y|}{|y|} \right) \psi(y) \, \mathrm{d}y \\ &=: I_0 + I_1 + I_2, \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} T_1 &= \big\{ y;\ |y - x| \le |x|/2 \text{ and } |y| > R_0 + 1 \big\}, \\ T_2 &= \big\{ y;\ |y - x| > |x|/2 \text{ and } |y| > R_0 + 1 \big\}. \end{aligned}$$

We assume that $|x| \ge 1$. If $y \in T_1$, then we have $|x - y| \le |y|$ and $\psi(y) \ge 0$. Thus,

$$I_1 \le 0. \tag{5.20}$$

It is also clear that there is a constant $C > 0$ such that

$$I_0 \le \ln |x| \int_{|y| \le R_0 + 1} |\psi(y)| \, \mathrm{d}y + C. \tag{5.21}$$

Finally, since $|x - y| \le |x| + |y| \le 2|x||y|$ for $|x|, |y| \ge 1$, we have

$$I_2 \le \ln 2|x| \int_{T_2} \psi(y) \, \mathrm{d}y. \tag{5.22}$$

Inserting (5.20)–(5.22) into (5.19) we get

$$2\pi v(x) \leq \ln|x| \left\{ \int_{|y|\leq R_0+1} |\psi(y)|\,\mathrm{d}y + \int_{T_2} \psi(y)\,\mathrm{d}y \right\} + C \leq C_0 \ln|x| + C, \tag{5.23}$$

for $|x| \geq 1$. Since $u+v$ is a harmonic function and $u<0$ in $\mathbb{R}^2$, we see from (5.23) that

$$u(x) + v(x) \leq C_0 \ln|x| + C \quad \forall x \in \mathbb{R}^2 \setminus B_1. \tag{5.24}$$

Therefore, by the Liouville Theorem (see [66, Lemma 4.6.1]), we conclude that $u+v$ must be a constant as claimed.
The rest of the proof of Lemma 5.2 follows that of [24, Theorem 1.1]. The details are omitted here. □

We now consider a sequence (u_n) where u_n satisfies

$$\begin{cases} \Delta u_n = \lambda \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1) + 4\pi \displaystyle\sum_{j=1}^{N} \delta_{p_j} & \text{in } \Omega_n, \\ u_n = 0 \quad \text{on } \partial\Omega_n, \end{cases} \tag{5.25}$$

where $\Omega_n = B_{R_n}$ and $R_n \to \infty$. We want to prove that (u_n) converges to a solution of (5.10) satisfying the topological boundary condition

$$\lim_{|x|\to\infty} u(x) = 0. \tag{5.26}$$

Lemma 5.3. *Let (u_n) be a sequence of solutions of the equation (5.25), where $\Omega_n = B_{R_n}$ $(n = 1, 2, \ldots)$. Then, there is a subsequence (u_{n_k}) which converges pointwise to a topological solution u of (5.10) satisfying (5.26).*

Proof. Write $u_n = w_0 + v_n$ where

$$w_0(x) = \sum_{j=1}^{N} \ln\left(\frac{|x-p_j|^2}{1+|x-p_j|^2} \right). \tag{5.27}$$

We shall prove that (v_n) is uniformly bounded. This immediately implies that some subsequence (u_{n_k}) converges to a solution u of (5.10); moreover, by Lemma 5.2, u must satisfy (5.25).
Suppose by contradiction that (v_n) is not bounded. Hence, there is a sequence (x_n) in $\mathbb{R}^2$ so that $v_n(x_n)$ tends to $-\infty$ as $n\to\infty$. From this we can infer that $v_n \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2$.
We claim that there is a sequence (x_n) such that

$$u_n(x_n) = -\frac{1}{2} \quad \text{and} \quad \operatorname{dist}(x_n, \partial B_{R_n}) \to \infty. \tag{5.28}$$

Suppose this is not true. Then, there is a constant $K>0$ such that if $u_n(x) \geq -1/2$, then $\operatorname{dist}(x, \partial B_{R_n}) \leq K$. Taking $n \geq 1$ sufficiently large, we may assume that $R_n \geq R_0 + K$. Let

$$\overline{u}_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u_n(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{5.29}$$

Since

$$\overline{u}_n(R_n - K) \leq -\frac{1}{2} \quad \text{and} \quad \overline{u}_n(R_n) = 0,$$

there is some $r_n \in (R_n - K, R_n)$ such that $\overline{u}_n'(r_n) \geq 1/2K$. Recall the identity

$$\overline{u}_n'(r) = \frac{1}{2\pi r} \int_{B_r} \Delta u_n \quad \forall r > R_0.$$

Taking in particular $r = r_n$, we get

$$\frac{r_n}{2K} \leq r_n \overline{u}_n'(r_n) = \frac{\lambda}{2\pi} \int_{B_{r_n}} \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1)\,\mathrm{d}x + 2N \leq 2N,$$

which yields a contradiction as we take $n \to \infty$.

Since $u_n \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2$ as $n \to \infty$, the sequence (x_n) defined in (5.28) satisfies $|x_n| \to \infty$ as $n \to \infty$. Set

$$U_n(x) = u_n(x + x_n). \tag{5.30}$$

Then, $U_n(0) = -1/2$. Clearly, U_n is well defined in a ball B_{ρ_n} with $\rho_n \to \infty$ as $n \to \infty$ and B_{ρ_n} does not contain any of the points $p_1, \ldots, p_N$.

Using Lemma 5.1, we may assume without loss of generality that $U_n(x)$ increases along the positive x_2-axis. For any $0 < r < \rho_n$, we have by integrating the Chern-Simons equation that

$$r\overline{U}_n'(r) = \frac{\lambda}{2\pi} \int_{B_r} \mathrm{e}^{U_n}(\mathrm{e}^{U_n} - 1)\,\mathrm{d}x. \tag{5.31}$$

Since $U_n \leq 0$, this implies that $|\overline{U}_n'(r)| \leq \lambda r/2$. Consequently,

$$|\overline{U}_n(r)| \leq \frac{1}{2} + \frac{\lambda r^2}{4}. \tag{5.32}$$

Using (5.32) and

$$|\overline{U}_n(r)| = \frac{1}{2\pi} \int_0^{2\pi} |U_n(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta \tag{5.33}$$

(recall that U_n does not change sign), we see that the sequence (U_n) has a uniform L^1 bound on ∂B_r. By elliptic estimates, we conclude that (U_n) is uniformly bounded over any compact subset of $\mathbb{R}^2$. From this fact we see that, by extracting a subsequence if necessary, we may assume that (U_n) converges (in any good local topology) to a solution U of the "bare" Chern-Simons equation so that

$$\begin{cases} \Delta U = \lambda \mathrm{e}^U(\mathrm{e}^U - 1) & \text{in } \mathbb{R}^2, \\ U \leq 0 \quad \text{and} \quad U(0) = -\dfrac{1}{2}. \end{cases} \tag{5.34}$$

Recall that u_n satisfies (5.25) with $\Omega_n = B_{R_n}$ and

$$\frac{\partial u_n}{\partial \mathrm{n}} > 0 \quad \text{on } \partial B_{R_n}.$$

Let v_0 be a function with support in B_{R_0}, $v_0(x) = \ln|x - p_j|^2$ for x in a small neighborhood of p_j $(j = 1, \ldots, N)$, and v_0 is smooth away from $p_1, \ldots, p_N$. Set $u_n = v_0 + V_n$. Then, V_n satisfies

$$\begin{cases} \Delta V_n = \lambda \mathrm{e}^{v_0 + V_n}(\mathrm{e}^{v_0 + V_n} - 1) + g(x) & \text{in } B_{R_n}, \\ V_n = 0 \quad \text{and} \quad \dfrac{\partial V_n}{\partial \mathrm{n}} > 0 & \text{on } \partial B_{R_n}, \end{cases} \tag{5.35}$$

for some fixed function g. Integrating (5.35), we obtain

$$\lambda \int_{B_{R_n}} \mathrm{e}^{v_0+V_n}(\mathrm{e}^{v_0+V_n}-1)\,\mathrm{d}x + \int_{B_{R_n}} g(x)\,\mathrm{d}x = \int_{\partial B_{R_n}} \frac{\partial V_n}{\partial \mathrm{n}}\,\mathrm{d}\ell > 0. \tag{5.36}$$

An immediate consequence of (5.36) is the uniform bound

$$\lambda \int_{B_{R_n}} \mathrm{e}^{u_n}(1-\mathrm{e}^{u_n})\,\mathrm{d}x \le \int_{\mathbb{R}^2} |g(x)|\,\mathrm{d}x. \tag{5.37}$$

(Alternatively, one could apply Lemma A.1 to u_n; proceeding as in the proof of Proposition A.3, one gets

$$\lambda \int_{B_{R_n}} \mathrm{e}^{u_n}(1-\mathrm{e}^{u_n})\,\mathrm{d}x \le 4\pi N \quad \forall n \ge 1.)$$

Clearly, (5.37) still holds when u_n is replaced by U_n. In particular, we have

$$\lambda \int_{\mathbb{R}^2} \mathrm{e}^{U}(1-\mathrm{e}^{U})\,\mathrm{d}x < \infty. \tag{5.38}$$

In view of (5.38) and Lemma 5.2, we have

$$\lim_{|x|\to\infty} \frac{U(x)}{\ln|x|} = -\alpha, \quad \alpha = \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{U}(1-\mathrm{e}^{U})\,\mathrm{d}x > 0; \tag{5.39}$$

the latter follows from $U(0) = -1/2$. However, since U_n is nondecreasing along the positive x_2-axis, the same property holds for U. This contradicts the established nontopological boundary condition

$$\lim_{|x|\to\infty} U(x) = -\infty \tag{5.40}$$

stated in Lemma 5.2. Therefore Lemma 5.3 is proved. $\square$

We remark that in the proof of Lemma 5.3, we only use a very special part of Lemma 5.2, namely the asymptotic characterization of the "bare" Chern-Simons equation (i.e., the differential equation in (5.34)). In fact, this bare case may be seen more transparently by an earlier result obtained in [57]:

Lemma 5.4. *Let U be a solution of the bare Chern-Simons equation over the full space $\mathbb{R}^2$. Then, either $U \equiv 0$ or $U < 0$ everywhere. If U is a solution so that $U \not\equiv 0$ and satisfies the finite-energy condition* (5.38)*, then*

$$\lim_{|x|\to\infty} \frac{U(x)}{\ln|x|} = -\alpha, \quad \alpha = \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{U}(1-\mathrm{e}^{U})\,\mathrm{d}x, \tag{5.41}$$

and $rU_r \equiv x_j\partial_j U \to -\alpha$, uniformly as $r = |x| \to \infty$. Moreover, U must be radially symmetric about some point in $\mathbb{R}^2$ and U is decreasing along all radial directions about this point.

With this lemma, the proof of Lemma 5.3 may be carried out in a similar way (with Lemma 5.4 replacing Lemma 5.2). In particular, we can arrive at the contradiction (5.40) as before.

6. The limit $\Omega \to \mathbb{R}^2$: the full system case

In this section, we prove the existence of a solution stated in Theorems 1.1 and 3.2 by taking the large domain limit of the solutions obtained in Theorem 3.1. We apply the preliminary results obtained in the previous section for a single Higgs particle Chern-Simons vortex equation.

Proofs of Theorems 1.1 and 3.2. Let (R_n) be a sequence such that

$$R_n > \max\left\{|p'_s|, |p''_s|\right\} \quad \text{and} \quad R_n \to \infty.$$

Consider the equation

$$\begin{cases} \Delta u = \lambda \mathrm{e}^v(\mathrm{e}^u - 1) + 4\pi \displaystyle\sum_{s=1}^{N'} \delta_{p'_s}(x) & \text{in } B_{R_n}, \\ \Delta v = \lambda \mathrm{e}^u(\mathrm{e}^v - 1) + 4\pi \displaystyle\sum_{s=1}^{N''} \delta_{p''_s}(x) & \text{in } B_{R_n}, \\ u = v = 0 \quad \text{on } \partial B_{R_n}. \end{cases} \tag{6.1}$$

By Theorem 3.1, (6.1) has a solution (u_n, v_n). Proceeding as in Section 4, we deduce that $\frac{1}{2}(u_n + v_n)$ is a nonpositive supersolution of the single Higgs particle Chern-Simons equation

$$\begin{cases} \Delta w = \lambda \mathrm{e}^w(\mathrm{e}^w - 1) + 4\pi \displaystyle\sum_{s=1}^{N'} \delta_{p'_s}(x) + 4\pi \sum_{s=1}^{N''} \delta_{p''_s}(x) & \text{in } B_{R_n}, \\ w = 0 \quad \text{on } \partial B_{R_n}. \end{cases} \tag{6.2}$$

In view of the construction in [58], we can use $\frac{1}{2}(u_n + v_n)$ as an initial function to iterate monotonically to obtain a solution w_n of (6.2). In particular,

$$w_n \le \frac{1}{2}(u_n + v_n) \quad \text{in } B_{R_n}. \tag{6.3}$$

By Lemma 5.3, passing to a subsequence if necessary we may assume that the sequence (w_n) converges pointwise to a solution of the problem

$$\begin{cases} \Delta w = \lambda \mathrm{e}^w(\mathrm{e}^w - 1) + 4\pi \displaystyle\sum_{s=1}^{N'} \delta_{p'_s}(x) + 4\pi \sum_{s=1}^{N''} \delta_{p''_s}(x) & \text{in } \mathbb{R}^2, \\ \displaystyle\lim_{|x|\to\infty} w(x) = 0, \quad w < 0 \quad \text{a.e.} \end{cases} \tag{6.4}$$

Taking a further subsequence, (u_n) and (v_n) converge pointwise to u and v on $\mathbb{R}^2$, respectively. It is clear that u and v are both negative and satisfy the two-Higgs particle system (3.1). Since (6.3) implies

$$2w(x) \le u(x) < 0 \quad \text{and} \quad 2w(x) \le v(x) < 0 \quad \text{for all } x \in \mathbb{R}^2, \tag{6.5}$$

we see that the desired topological boundary condition (3.2) is achieved. Estimates (1.3)–(1.4) follow from (3.4)–(3.5). Using a well-known ODE-result (see Proposition 16.1 below), one deduces the decay estimates (3.6)–(3.7). The complete argument is carried out in Section 16 for equation (1.1) in the case of measures μ and ν with compact supports. The proof of Theorem 3.2 is complete. □

Following the procedure described in Section 3, all the statements made in Theorem 2.1 are established.

7. Existence of solutions of system (1.1)

Let $\mathcal{M}(\omega)$ denote the space of (finite) Radon measures μ on an open set $\omega \subset \mathbb{R}^2$. We equip $\mathcal{M}(\omega)$ with the standard norm

$$\|\mu\|_{\mathcal{M}} = |\mu|(\omega) = \int_\omega \mathrm{d}|\mu|.$$

We now consider equation (1.1) for finite measures μ, ν on $\mathbb{R}^2$. Our goal is to prove the following

Theorem 7.1. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^2)$ be such that*

(i) $\mu^+(\{x\}) + \nu^+(\{x\}) \leq 4\pi$, $\forall x \in \mathbb{R}^2$;

(ii) $\nu(\{x\}) = 0$ *whenever* $\mu(\{x\}) = 4\pi$;

(iii) $\mu(\{x\}) = 0$ *whenever* $\nu(\{x\}) = 4\pi$.

Then, for every $\lambda > 0$ the system

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \mathbb{R}^2, \\ -\Delta v + \lambda \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \mathbb{R}^2, \end{cases} \tag{7.1}$$

has a solution $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$ in the sense of distributions such that

$$\|u\|_{L^1} + \|v\|_{L^1} \leq \frac{C}{\lambda}\big(1 + \|\mu\|_{\mathcal{M}}^2 + \|\nu\|_{\mathcal{M}}^2\big)\big(\|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}\big), \tag{7.2}$$

$$\|\mathrm{e}^u - 1\|_{L^1} + \|\mathrm{e}^v - 1\|_{L^1} \leq \frac{C}{\lambda}\big(1 + \|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}\big)\big(\|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}\big). \tag{7.3}$$

Note that if μ and ν are nonpositive measures, then assumptions (i)–(iii) are always satisfied. Taking in particular $\mu = -4\pi \sum_{s=1}^{N'} \delta_{p'_s}$ and $\nu = -4\pi \sum_{s=1}^{N''} \delta_{p''_s}$, one deduces Theorem 1.1 as a corollary; the exponential decay of the solutions is provided by Theorem 16.1 below. The requirement of (i)–(iii) will be discussed in Section 15.

The strategy to prove Theorem 7.1 is the following. We first study the existence of solutions of the scalar equation

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

where v is a *given* function and $\Omega \subset \mathbb{R}^2$ is any smooth bounded domain. We show that solutions of (7.4) are "stable" with respect to suitable perturbations of the data v and μ (see Proposition 8.1). A useful tool is the notion of *reduced measure* μ^*, recently introduced in [14].

Applying Schauder's fixed point theorem, we are then able to prove existence of solutions for the counterpart of (7.1) on *bounded* domains, namely

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ -\Delta v + \lambda \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

We also show that every solution of (7.5) satisfies (7.2)–(7.3). The main ingredient in the proof of (7.2) is the following inequality (see Proposition 12.1)

$$\int_{[|\varphi| \geq 3]} |\varphi| \, \mathrm{d}x \leq C \|\Delta\varphi\|_{L^1}^2 \big|[1 < |\varphi| < 2]\big| \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2), \tag{7.6}$$

where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^2$

Remark 7.1. A more elementary estimate of the L^1-norm of solutions (u, v) of (7.5) is

$$\|u\|_{L^1} + \|v\|_{L^1} \leq C_\Omega\big(\|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}\big). \tag{7.7}$$

This follows from (see Proposition A.3 below)

$$\|\Delta u\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}} \quad \text{and} \quad \|\Delta v\|_{\mathcal{M}} \leq 2\|\nu\|_{\mathcal{M}}, \tag{7.8}$$

combined with the well-known elliptic estimate (see [59] and also [14, Theorem B.1])

$$\|w\|_{L^1} \leq C_\Omega \|\mu\|_{\mathcal{M}}, \tag{7.9}$$

where w is the unique solution of

$$\begin{cases} -\Delta w = \mu & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

Note however that estimates (7.7) and (7.9) *depend* on Ω, while (7.2) is true for *any* solution of (7.5), regardless of the domain Ω.

In order to obtain a solution of (7.1), let (Ω_n) denote an increasing sequence of smooth bounded domains such that $\bigcup_n \Omega_n = \mathbb{R}^2$. Denote by (u_n, v_n) a solution of (7.5) on Ω_n. By (7.2) and elliptic estimates, one deduces that (u_n) and (v_n) are relatively compact in $L^1_{\text{loc}}(\mathbb{R}^2)$ and

$$(u_{n_k}, v_{n_k}) \to (u, v) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^2) \times L^1_{\text{loc}}(\mathbb{R}^2),$$

for some $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$. By the stability of (7.4), (u, v) satisfies (7.1).

Remark 7.2. Our strategy to prove Theorem 7.1 can presumably be adapted to study system (7.1) in $\mathbb{R}^N$ for $N \geq 3$. A useful tool should be some estimates recently proved by Bartolucci-Leoni-Orsina-Ponce [3], which are the counterpart in dimension $N \geq 3$ of a result of Brezis-Merle [15]. In view of the results in [3], one expects to have assumptions (i)–(iii) stated in terms of the $(N-2)$-dimensional Hausdorff measure $\mathcal{H}^{N-2}$.

8. Study of the scalar problem (7.4)

We shall assume that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Let $\mu, \nu \in \mathcal{M}(\Omega)$ be two measures such that

(a_1) $\nu(\{x\}) \leq 4\pi$, $\forall x \in \Omega$;
(a_2) $\mu(\{x\}) \leq 4\pi - \nu(\{x\})$, $\forall x \in \Omega$;
(a_3) $\mu(\{x\}) = 0$ whenever $\nu(\{x\}) = 4\pi$.

We then prove the following

Theorem 8.1. *Suppose that μ and ν satisfy (a_1)–(a_3) above. Then, for every $\lambda > 0$ the equation*

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

has a unique solution for every $v \in L^1(\Omega)$ such that $v \leq V$ a.e., where $V \in L^1(\Omega)$ satisfies $-\Delta V = \nu$ in $\mathcal{D}'(\Omega)$.

We say that u is a solution of (8.1) if $u \in L^1(\Omega)$, $e^v(e^u - 1)\rho_0 \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta \, dx + \lambda \int_\Omega e^v(e^u - 1)\zeta \, dx = \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),$$

where $\rho_0(x) = \operatorname{dist}(x, \partial\Omega)$, $\forall x \in \Omega$, and

$$C_0^2(\overline{\Omega}) = \big\{\zeta \in C^2(\overline{\Omega});\ \zeta = 0 \text{ on } \partial\Omega\big\}.$$

Our proof of Theorem 8.1 is based on a "stability" property satisfied by equation (8.1) under assumptions (a_1)–(a_3); see Proposition 8.1 below.

In order to state our next result, let $u_n, v_n \in L^1(\Omega)$ and $\mu_n \in \mathcal{M}(\Omega)$ be such that

$$-\Delta u_n + \lambda e^{v_n}(e^{u_n} - 1) = \mu_n \quad \text{in } \mathcal{D}'(\Omega), \tag{8.2}$$

where

- (b_1) $u_n \to u$ in $L^1(\Omega)$;
- (b_2) $v_n \to v$ in $L^1(\Omega)$ and $v_n \le V_n$ a.e., where (V_n) is a bounded sequence in $L^1(\Omega)$ such that $-\Delta V_n = \nu_n$ in $\mathcal{D}'(\Omega)$ and the sequence $(\nu_n) \subset \mathcal{M}(\Omega)$ satisfies $\nu_n \overset{*}{\rightharpoonup} \nu$ weak* in $\mathcal{M}(\Omega)$;
- (b_3) $\mu_n^+ \overset{*}{\rightharpoonup} \mu^+$ and $\mu_n^- \overset{*}{\rightharpoonup} \mu^-$ weak* in $\mathcal{M}(\Omega)$;
- (b_4) $(\theta\mu_n^+ + \nu_n)^+ \overset{*}{\rightharpoonup} (\theta\mu^+ + \nu)^+$ weak* in $\mathcal{M}(\Omega)$, $\forall \theta \in [0, 1]$.

We then have the following

Proposition 8.1. *Let $\lambda > 0$ and $\mu, \nu \in \mathcal{M}(\Omega)$ be such that (a_1)–(a_3) hold. Assume that*

$$-\Delta u_n + \lambda e^{v_n}(e^{u_n} - 1) = \mu_n \quad \textit{in } \mathcal{D}'(\Omega),$$

where u_n, v_n, μ_n satisfy (b_1)–(b_4). Then,

$$e^{v_n}(e^{u_n} - 1) \to e^v(e^u - 1) \quad \textit{in } L^1(\omega), \tag{8.3}$$

for every $\omega \subset\subset \Omega$. In particular,

$$-\Delta u + \lambda e^v(e^u - 1) = \mu \quad \textit{in } \mathcal{D}'(\Omega). \tag{8.4}$$

Examples of sequences of measures (μ_n) and (ν_n) satisfying (b_3)–(b_4) are:

1. $\mu_n = \mu$ and $\nu_n = \nu$, $\forall n \ge 1$;
2. $\mu_n = \rho_n * \mu$ and $\nu_n = \rho_n * \nu$, where (ρ_n) is a sequence of nonnegative mollifiers; μ and ν are extended to $\mathbb{R}^2$ as identically zero outside Ω.

Let us prove that (b_3)–(b_4) hold in case (2). We recall the easy inequality

$$\rho_n * \mu \le (\rho_n * \mu)^+ \le \rho_n * \mu^+. \tag{8.5}$$

A standard argument then implies

$$\mu_n^+ = (\rho_n * \mu)^+ \overset{*}{\rightharpoonup} \mu^+ \quad \text{weak* in } \mathcal{M}(\Omega).$$

Replacing μ by $-\mu$, one deduces (b_3). Condition (b_4) follows from a similar argument based on the estimate

$$\theta\mu_n^+ + \nu_n \le (\theta\mu_n^+ + \nu_n)^+ \le \rho_n * (\theta\mu^+ + \nu)^+ \quad \forall \theta \in [0, 1].$$

An important ingredient to establish Proposition 8.1 is the next

Lemma 8.1. *Let $v \in L^1(\Omega)$ be such that $v \le V$ a.e. for some $V \in L^1(\Omega)$ with $\Delta V \in \mathcal{M}(\Omega)$. Given $\lambda > 0$, assume that*

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{8.6}$$

has a solution for some $\mu \in \mathcal{M}(\Omega)$ and $f \in L^1(\partial\Omega)$. Then, (8.6) *also has a solution with data (μ^+, f^+) and $(-\mu^-, -f^-)$.*

Given $u \in L^1(\Omega)$, we say that $\Delta u \in \mathcal{M}(\Omega)$ if

$$\left| \int_\Omega u \Delta\varphi \, \mathrm{d}x \right| \le C \|\varphi\|_{L^\infty} \quad \forall \varphi \in C_0^\infty(\Omega); \tag{8.7}$$

we denote by $\|\Delta u\|_{\mathcal{M}}$ the smallest constant $C \ge 0$ for which (8.7) holds. By the Riesz Representation Theorem (see e.g. [34]), $\Delta u \in \mathcal{M}(\Omega)$ if and only if there exists $\sigma \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega u \Delta\varphi \, \mathrm{d}x = \int_\Omega \varphi \, \mathrm{d}\sigma \quad \forall \varphi \in C_0^\infty(\Omega),$$

in which case $\|\Delta u\|_{\mathcal{M}} = \|\sigma\|_{\mathcal{M}}$. Note that σ, whenever exists, is uniquely determined; we systematically identify the distribution Δu with σ.

Lemma 8.1 is established in Section 10 below. The proof relies on the existence of the reduced measure; see Section 9. Theorem 8.1 and Proposition 8.1 are proved in Section 11.

9. Existence of the reduced measure μ^*

Let us consider the following equation

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$

where $\mu \in \mathcal{M}(\Omega)$, $h \in L^1(\partial\Omega)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We say that u is a solution of (9.1) if $u \in L^1(\Omega)$, $g(\cdot, u)\rho_0 \in L^1(\Omega)$ and

$$-\int_\Omega u \Delta\zeta \, \mathrm{d}x + \int_\Omega g(x,u)\zeta \, \mathrm{d}x = \int_\Omega \zeta \, \mathrm{d}\mu - \int_{\partial\Omega} h \frac{\partial\zeta}{\partial \mathrm{n}} \, \mathrm{d}\ell \quad \forall \zeta \in C_0^2(\overline{\Omega}),$$

where n denotes the outward normal on $\partial\Omega$.

The following theorem will be established using ideas from [14, 18, 19]:

Theorem 9.1. *Assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying*

- (A_1) *$g(x, \cdot)$ is nondecreasing for a.e. $x \in \Omega$;*
- (A_2) *$g(x,t) = 0$ for a.e. $x \in \Omega$, $\forall t \le 0$;*
- (A_3) *$g(\cdot, t)$ is quasifinite $\forall t \in \mathbb{R}$.*

Then, for every $\mu \in \mathcal{M}(\Omega)$ there exists $\mu^* \in \mathcal{M}(\Omega)$, with $\mu^* \le \mu$, such that the following holds:

- (I) (9.1) *has a solution with data (μ^*, h) for every $h \in L^1(\partial\Omega)$;*
- (II) *If* (9.1) *has a solution with data $(\tilde\mu, \tilde h)$ for some $\tilde\mu \le \mu$ and $\tilde h \in L^1(\partial\Omega)$, then $\tilde\mu \le \mu^*$.*

Theorem 9.1 will be proved in the next section. The notion of *reduced measure* μ^* was introduced by Brezis-Marcus-Ponce [14] in the case where $g(x,t) = g(t)$ has no dependence with respect to x.

A measurable function $G : \Omega \to \mathbb{R}$ is quasifinite if for every $\varepsilon > 0$ and $K \subset \Omega$ compact there exist $M > 0$ and an open set $\omega \subset \Omega$ such that $\operatorname{cap}(\omega) < \varepsilon$ and $|G| \leq M$ a.e. on $K \setminus \omega$. We say that G is quasicontinuous if for every $\varepsilon > 0$ there exists an open set $\omega \subset \Omega$ such that $\operatorname{cap}(\omega) < \varepsilon$ and G is continuous on $\Omega \setminus \omega$. In particular, every quasicontinuous function is quasifinite. Throughout the paper, we denote by $\operatorname{cap}(E)$ the Newtonian (H^1) capacity of a Borel set $E \subset \Omega$, with respect to some large ball $B_R \supset\supset \Omega$; although the capacity "cap" depends on R, the notions of quasifiniteness and quasicontinuity do not.

If $u \in L^1(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then one shows (see e.g. [1, 17]) that there exists a quasicontinuous function $\tilde{u} : \Omega \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. We shall systematically identify such functions u with their quasicontinuous representative $\tilde{u}$ and simply say that u is quasicontinuous, meaning $\tilde{u}$.

We conclude this section with some tools which will be used in the proof of Theorem 9.1.

We recall that any Radon measure μ in $\mathbb{R}^N$ can be decomposed as a sum $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$, where μ_{a} and μ_{s} are the absolutely continuous and the singular parts of μ with respect to the Lebesgue measure. There are several other possible decompositions of μ however. For instance, (see [10] and also [35])

$$\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}},$$

where

$$\begin{aligned} \mu_{\mathrm{d}}(E) = 0 \qquad & \text{for any Borel set } E \subset \Omega \text{ such that } \operatorname{cap}(E) = 0, \\ |\mu_{\mathrm{c}}|(\Omega \backslash E_0) = 0 \qquad & \text{for some Borel set } E_0 \subset \Omega \text{ such that } \operatorname{cap}(E_0) = 0. \end{aligned}$$

In particular, the Radon measures μ_{d} and μ_{c} are mutually singular.

Using the above notation, one proves the

Theorem 9.2 (Inverse Maximum Principle [30])**.** *Assume $u \in L^1(\Omega)$ is such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \geq 0$ a.e. in Ω, then*

$$(-\Delta u)_{\mathrm{c}} \geq 0. \tag{9.2}$$

In order to prove Theorem 9.1 we will also need the following

Lemma 9.1. *Let (g_k) be a bounded sequence in $L^1(\Omega)$ such that*

$$g_k \overset{*}{\rightharpoonup} \sigma \quad \textit{weak}^* \textit{ in } \mathcal{M}(\Omega).$$

Assume that

(B_1) *$g_k \to g$ a.e.;*
(B_2) *There exists a quasifinite function $G : \Omega \to \mathbb{R}$ such that $|g_k| \leq G$ a.e.;*
(B_3) *For every $\varepsilon > 0$, there exist $\delta > 0$ and an open set $\omega_0 \subset \Omega$, with $\operatorname{cap}(\overline{\omega}_0) < \varepsilon$, such that*

$$\int_{A \backslash \overline{\omega}_0} |g_k| \,\mathrm{d}x < \varepsilon \quad \forall k \geq 1,$$

for every open set $A \subset \Omega$ such that $\operatorname{cap}(A) < \delta$.

Then,

$$\sigma = g + \gamma \quad \textit{in } \Omega, \tag{9.3}$$

for some measure γ concentrated on a set of zero capacity; in other words, $\sigma_{\mathrm{d}} = g$.

Proof of Lemma 9.1. Given $\varepsilon > 0$, take $\delta > 0$ and $\omega_0 \subset \Omega$ as in assumption (B_3). Since G is quasifinite, for every open set $A \subset\subset \Omega$, there exist $M > 0$ and an open set $\omega_1 \subset \Omega$ such that $\operatorname{cap}(\omega_1) < \delta$ and $|G(x)| \leq M$, $\forall x \in \overline{A} \setminus \omega_1$. Thus, by (B_1), (B_2), and dominated convergence,

$$g_k\, \chi_{A\setminus\omega_1} \to g\, \chi_{A\setminus\omega_1} \quad \text{in } L^1(\Omega),$$

where $\chi_{A\setminus\omega_1}$ denotes the characteristic function of $A \setminus \omega_1$. Moreover, since we have $\operatorname{cap}(\omega_1) < \delta$,

$$\int_{\omega_1\setminus\overline{\omega}_0} |g_k| \, \mathrm{d}x < \varepsilon \quad \forall k \geq 1.$$

Thus,

$$\limsup_{k\to\infty} \int_{A\setminus\overline{\omega}_0} |g_k - g| \, \mathrm{d}x \leq \varepsilon + \int_{\omega_1\setminus\overline{\omega}_0} |g| \, \mathrm{d}x.$$

We then deduce that (see e.g. [32, Theorem 1, p.54])

$$\int_{A\setminus\overline{\omega}_0} |g - \sigma| \leq \varepsilon + \int_{\omega_1\setminus\overline{\omega}_0} |g| \, \mathrm{d}x.$$

Since $A \subset\subset \Omega$ was arbitrary,

$$\int_{\Omega\setminus\overline{\omega}_0} |g - \sigma| \leq \varepsilon + \int_{\omega_1\setminus\overline{\omega}_0} |g| \, \mathrm{d}x.$$

Recall that σ_{d} and σ_{c} are singular with respect to each other; hence,

$$|g - \sigma_{\mathrm{d}}| \leq |g - \sigma_{\mathrm{d}}| + |\sigma_{\mathrm{c}}| = |g - \sigma|.$$

Therefore,

$$\begin{aligned} \int_\Omega |g - \sigma_{\mathrm{d}}| &= \int_{\Omega\setminus\overline{\omega}_0} |g - \sigma_{\mathrm{d}}| + \int_{\overline{\omega}_0} |g - \sigma_{\mathrm{d}}| \\ &\leq \varepsilon + \int_{\overline{\omega}_0\cup\omega_1} |g| \, \mathrm{d}x + \int_{\overline{\omega}_0} |\sigma_{\mathrm{d}}|. \end{aligned} \tag{9.4}$$

As $\varepsilon \to 0$, we have $|\overline{\omega}_0 \cup \omega_1| \to 0$ and $\operatorname{cap}(\overline{\omega}_0) \to 0$. Thus, the right-hand side of (9.4) converges to 0. We then conclude that $g = \sigma_{\mathrm{d}}$. In other words, $\gamma := \sigma - g = \sigma_{\mathrm{c}}$ is concentrated on a set of zero capacity. □

10. Proofs of Theorem 9.1 and Lemma 8.1

Proof of Theorem 9.1. Given $k \geq 1$, let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation operator at k; more precisely,

$$T_k(t) = \begin{cases} t & \text{if } t \leq k, \\ k & \text{if } t > k. \end{cases} \tag{10.1}$$

We then let $g_k : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function given by

$$g_k(x,t) = T_k\big(g(x,t)\big).$$

In particular, g_k is bounded. For every $k \geq 1$, let $u_k \in L^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta u_k + g_k(x, u_k) = \mu & \text{in } \Omega, \\ \hfill u_k = h & \text{on } \partial\Omega. \end{cases} \tag{10.2}$$

The existence of u_k was originally proved by Bénilan-Brezis [7] (see also [61]); alternatively, one can apply Theorem A.1 in Appendix A below.
By Proposition A.1, the sequence (u_k) is non-increasing and bounded from below by $\hat{U}$, where $\hat{U}$ is the solution of

$$\begin{cases} -\Delta \hat{U} = -\mu^- & \text{in } \Omega, \\ \quad \hat{U} = h & \text{on } \partial\Omega. \end{cases}$$

Let u^* denote the pointwise limit of the sequence (u_k). Then,

$$u_k \to u^* \quad \text{in } L^1(\Omega).$$

By Proposition A.2,

$$\int_\Omega g_k(x, u_k)\zeta_0 \,\mathrm{d}x \leq \int_\Omega \zeta_0 \,\mathrm{d}|\mu| - \int_{\partial\Omega} |h| \frac{\partial \zeta_0}{\partial \mathrm{n}} \,\mathrm{d}\ell, \tag{10.3}$$

where $\zeta_0 \in C_0^2(\overline{\Omega})$ denotes the solution of

$$\begin{cases} -\Delta \zeta_0 = 1 & \text{in } \Omega, \\ \quad \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, $\big(g_k(\cdot, u_k)\zeta_0\big)$ is bounded in $L^1(\Omega)$. Passing to a subsequence, one finds non-negative measures $\sigma \in \mathcal{M}(\Omega)$ and $\tau \in \mathcal{M}(\partial\Omega)$ such that

$$\int_\Omega g_{k_j}(x, u_{k_j})\zeta \,\mathrm{d}x = \int_\Omega g_{k_j}(x, u_{k_j})\zeta_0 \frac{\zeta}{\zeta_0} \,\mathrm{d}x \xrightarrow{j\to\infty} \int_\Omega \zeta \,\frac{\mathrm{d}\sigma}{\zeta_0} - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathrm{n}} \,\mathrm{d}\tau, \tag{10.4}$$

for every $\zeta \in C_0^2(\overline{\Omega})$. On the other hand, by Fatou's lemma,

$$\int_\Omega g(x, u^*)\varphi \,\mathrm{d}x \leq \liminf_{j\to\infty} \int_\Omega g(x, u_{k_j})\varphi \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega. \tag{10.5}$$

Comparison between (10.4) and (10.5) implies

$$g(x, u^*) \leq \frac{\sigma}{\zeta_0} \quad \text{in } \Omega. \tag{10.6}$$

Let

$$\mu^* = \mu - \left(\frac{\sigma}{\zeta_0} - g(x, u^*)\right) \quad \text{and} \quad h^* = h - \tau.$$

Then, $\mu^* \leq \mu$ and $h^* \leq h$. Moreover, u^* satisfies

$$\begin{cases} -\Delta u^* + g(x, u^*) = \mu^* & \text{in } \Omega, \\ \hfill u^* = h^* & \text{on } \partial\Omega. \end{cases} \tag{10.7}$$

In particular, μ^* and h^* are well-defined, *independently* of the subsequence (u_{k_j}). Thus, (10.4) holds for the entire sequence (u_k). We claim that

- *(a)* If w is a subsolution of (9.1), then $w \leq u^*$ a.e.;
- *(b)* $h^* \in L^1(\partial\Omega)$ and $h^* = h$ a.e. on $\partial\Omega$;
- *(c)* $(\mu^*)_{\mathrm{d}} = \mu_{\mathrm{d}}$;
- *(d)* $0 \leq \mu_{\mathrm{c}} - (\mu^*)_{\mathrm{c}} \leq (\mu^+)_{\mathrm{c}}$.

Assertion (a) is proved as in [14, Proposition 1]. We now split the proof in 3 steps:

Step 1. Proof of (b).

Let v_1, v_2 be the solutions of (9.1) with data $(-\mu^-, -h^-)$ and $(0, h)$, respectively. The existence of v_1 is trivial since $g(x,t) = 0$ if $t \le 0$; the existence of v_2 is established as in [38, Proposition 6.6]. By Proposition A.1, we have $v_1 \le v_2$ a.e. Applying the method of sub and supersolutions (see Theorem A.1), we conclude that there exists a solution v of (9.1) with data $(-\mu^-, h)$. Since v is a subsolution of (9.1), it follows from (a) above that $v \le u^*$ a.e. Thus, by [19, Lemma 1],

$$h \le h^* \quad \text{on } \partial\Omega.$$

Since $h^* \le h$, we conclude that $h^* \in L^1(\partial\Omega)$ and $h^* = h$ a.e.

Step 2. Proof of (c).

We show that the sequence $\big(g_k(\cdot, u_k)\big)$ satisfies the assumptions of Lemma 9.1 on every subdomain $\tilde{\Omega} \subset\subset \Omega$. We first note that

$$g_k(x, u_k) \to g(x, u^*) \quad \text{a.e.}$$

Let $\tilde{U}$ be the solution of

$$\begin{cases} -\Delta \tilde{U} = \mu^+ & \text{in } \Omega, \\ \quad \tilde{U} = h & \text{on } \partial\Omega. \end{cases}$$

Then, by Proposition A.1 we have

$$u_k \le \tilde{U} \quad \text{a.e.}, \quad \forall k \ge 1.$$

Thus,

$$0 \le g(x, u_k) \le g(x, \tilde{U}) \quad \text{a.e.}, \quad \forall k \ge 1.$$

Since $\tilde{U}$ is quasicontinuous and $g(\cdot, t)$ is quasifinite for all t, one easily checks that $g(\cdot, \tilde{U})$ is quasifinite. Hence, (B_2) holds.

It remains to prove (B_3). Given $\varepsilon > 0$, let $F \subset \tilde{\Omega}$ be a compact set such that $\operatorname{cap}(F) = 0$ and $|\mu_{\mathrm{c}}|(\tilde{\Omega} \setminus F) < \varepsilon$. Let $\omega_0 \subset \tilde{\Omega}$ be an open set containing F such that $\operatorname{cap}(\overline{\omega}_0) < \varepsilon$. Applying [18, Lemma 3] (although Lemma 3 in [18] deals with homogeneous boundary condition, the conclusion for equation (9.1) remains unchanged on every subdomain $\tilde{\Omega} \subset\subset \Omega$), it follows that there exist $\delta > 0$ and $k_0 \ge 1$ such that

$$\int_{A \setminus \overline{\omega}_0} g_k(x, u_k)\, \mathrm{d}x < 2\varepsilon \quad \forall k \ge k_0, \tag{10.8}$$

for every open set $A \subset \tilde{\Omega}$ such that $\operatorname{cap}(A) < \delta$. Taking $\delta > 0$ smaller if necessary, we can assume that (10.8) is true for every $k \ge 1$. Thus, (B_3) holds.

Applying Lemma 9.1, we conclude that

$$g_k(x, u_k) \overset{*}{\rightharpoonup} g(x, u^*) + \gamma \quad \text{weak}^* \text{ in } \mathcal{M}(\tilde{\Omega})$$

for some concentrated measure γ. Hence,

$$\mu^* = -\Delta u^* + g(x, u^*) = \mu - \gamma.$$

Comparing the diffuse parts from both sides, we deduce that

$$(\mu^*)_{\mathrm{d}} = \mu_{\mathrm{d}}.$$

This concludes the proof of (c).

Step 3. Proof of (d).

It suffices to show that

$$0 \leq (\mu^*)_{\mathrm{c}}^{+} \leq \mu_{\mathrm{c}}^{+} \quad \text{and} \quad (\mu^*)_{\mathrm{c}}^{-} = \mu_{\mathrm{c}}^{-}. \tag{10.9}$$

The estimates for $(\mu^*)_{\mathrm{c}}^{+}$ just follows from $0 \leq (\mu^*)_{\mathrm{c}}^{+}$ and $\mu^* \leq \mu$. Similarly, we also have

$$(\mu^*)_{\mathrm{c}}^{-} \geq \mu_{\mathrm{c}}^{-}.$$

In order to prove the reverse inequality, let v be the solution of (9.1) with data $(-\mu^-, h)$ (the existence of v is established in Step 1 above). Since v is a subsolution of (8.6), we get $v \leq u^*$. It then follows from the Inverse Maximum Principle (see Theorem 9.2) that

$$\mu_{\mathrm{c}}^{-} = (\Delta v)_{\mathrm{c}} \geq (\Delta u^*)_{\mathrm{c}} = -(\mu^*)_{\mathrm{c}}.$$

Comparing the positive parts from both sides,

$$\mu_{\mathrm{c}}^{-} \geq \big[-(\mu^*)_{\mathrm{c}}\big]^{+} = (\mu^*)_{\mathrm{c}}^{-}.$$

This establishes the reverse inequality. Therefore, (10.9) holds.

We have proved that (a)–(d) are satisfied. In particular, (c)–(d) imply that $\mu^* \in \mathcal{M}(\Omega)$. In addition, by (b) we conclude that u^* solves (9.1) with data (μ^*, h). This shows that (I) holds.

It remains to prove (II). Let us first show a special case of (II):

Claim 1. If (9.1) has a solution with data $(\tilde{\mu}, \tilde{h})$ for some $\tilde{\mu} \leq \mu$ *and* $\tilde{h} \leq h$, then $\tilde{\mu} \leq \mu^*$.

Assume (9.1) has a solution $\tilde{u}$ with data $(\tilde{\mu}, \tilde{h})$, where $\tilde{\mu} \leq \mu$ and $\tilde{h} \leq h$. In particular, by (c) we have

$$(\tilde{\mu})_{\mathrm{d}} \leq \mu_{\mathrm{d}} = (\mu^*)_{\mathrm{d}}. \tag{10.10}$$

Since $\tilde{u}$ is a subsolution of (9.1), it follows from (a) that $\tilde{u} \leq u^*$. Thus, by the Inverse Maximum Principle,

$$(\tilde{\mu})_{\mathrm{c}} = (-\Delta \tilde{u})_{\mathrm{c}} \leq (-\Delta u^*)_{\mathrm{c}} = (\mu^*)_{\mathrm{c}}. \tag{10.11}$$

Combining (10.10)–(10.11) we deduce that

$$\tilde{\mu} \leq \mu^*.$$

This establishes Claim 1.

In order to construct the measure μ^*, we used the sequence (u_k) of solutions of (10.2); thus u_k depends on μ but also on h. As we shall see, the reduced measure μ^* itself *does not* depend on h:

Claim 2. Given $h_0 \in L^1(\partial\Omega)$, let μ_0^* be the reduced measure associated to (μ, h_0). Then, $\mu_0^* = \mu^*$.

It suffices to prove Claim 2 for $h_0 = 0$. Let u_0 and v be the solutions of (9.1) with data $(\mu_0^*, 0)$ and $(-\mu^-, -h^-)$, respectively. In particular, $u_0 \geq v$ a.e. (note that by (c) and (d) the reduced measure is always $\geq -\mu^-$). By the method of sub and supersolutions, there exists a solution of (9.1) with data $(\mu_0^*, -h^-)$. By Claim 1 above, $\mu_0^* \leq \mu^*$. A similar argument shows that (9.1) also has a solution corresponding to $(\mu^*, -h^-)$; hence, $\mu^* \leq \mu_0^*$. We conclude that $\mu_0^* = \mu^*$.

Assertion (II) now follows from (I) and Claims 1 and 2 above. Indeed, assume (9.1) has a solution associated to $(\tilde{\mu}, \tilde{h})$, where $\tilde{\mu} \leq \mu$ and $\tilde{h} \in L^1(\partial\Omega)$. By (I) and

Claim 2, (9.1) has a solution with data $(\mu^*, \tilde{h})$. Thus, by Claim 1, $\tilde{\mu} \le \mu^*$. The proof of Theorem 9.1 is complete. □

Proof of Lemma 8.1. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be given by

$$g(x,t) = \lambda \mathrm{e}^{v(x)}(\mathrm{e}^t - 1)^+. \tag{10.12}$$

Since $v \le V$ a.e. and V is quasicontinuous, g satisfies the assumptions of Theorem 9.1. Applying Theorem 9.1 with data (μ^+, f^+), we obtain a measure $(\mu^+)^* \le \mu^+$ such that (9.1) has a solution with $((\mu^+)^*, f^+)$. Note that $(\mu^+)^* \ge 0$. Indeed, it suffices to observe that (9.1) has a solution for $(0,0)$; thus, by (II), $(\mu^+)^* \ge 0$. We now show that $\mu \le (\mu^+)^*$. We claim that (9.1) has a solution v with data (μ, f). Indeed, let u_0 be the solution of (9.1) with data $(-\mu^-, -f^-)$; u_0 is a subsolution of (9.1). Denote by u the solution of (8.6). Since $u \ge u_0$ is a supersolution of (9.1), it follows from the method of sub and supersolutions (see Theorem A.1) that (9.1) has a solution v with data (μ, f). By (II), we conclude that $\mu \le (\mu^+)^*$. Therefore,

$$\mu^+ = \sup\{0, \mu\} \le (\mu^+)^* \le \mu^+.$$

In other words, $(\mu^+)^* = \mu^+$ and (9.1) has a solution u^* associated to (μ^+, f^+). Since $u^* \ge 0$ a.e., we deduce that u^* solves the corresponding problem (8.6) for (μ^+, f^+). Similarly, one shows that (8.6) has a solution with data $(-\mu^-, -f^-)$. □

11. Proofs of Proposition 8.1 and Theorem 8.1

Proof of Proposition 8.1. Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$ in Ω, and $\omega \subset\subset \Omega$ be a smooth domain such that $\operatorname{supp}\varphi \subset \omega$. By (b_1) and Fubini's theorem, we can choose ω so that

$$u_n \to u \quad \text{in } L^1(\partial\omega).$$

It then follows from Proposition A.2 that $\big(\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1)\zeta\big)$ is bounded in $L^1(\omega)$ for every $\zeta \in C_0^2(\overline{\omega})$. Thus, $\big(\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1)\varphi\big)$ is bounded in $L^1(\Omega)$. Passing to a subsequence, we have

$$u_{n_k} \to u \quad \text{and} \quad v_{n_k} \to v \quad \text{a.e.}, \tag{11.1}$$

$$\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi \to \mathrm{e}^{v}(\mathrm{e}^{u} - 1)\varphi \quad \text{a.e.}, \tag{11.2}$$

$$\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi \overset{*}{\rightharpoonup} \sigma \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega), \tag{11.3}$$

for some $\sigma \in \mathcal{M}(\Omega)$. We claim that

$$\sigma = \mathrm{e}^{v}(\mathrm{e}^{u} - 1)\varphi \quad \text{in } \Omega \tag{11.4}$$

equi-integrable in $L^1(\Omega)$ and

$$\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1)\varphi \to \mathrm{e}^{v}(\mathrm{e}^{u} - 1)\varphi \quad \text{in } L^1(\Omega). \tag{11.5}$$

In order to prove (11.4)–(11.5), we split the proof of Proposition 8.1 in three main steps:

Step 1. Proof of (11.4)–(11.5) if $u_n \ge 0$ a.e. and $\mu_n \ge 0$, $\forall n \ge 1$.

We first establish Step 1 under a stronger assumption on μ:

Step 1A. Proof of Step 1 assuming in addition that

(a_4) $\mu(\{x\}) = 0$ whenever $\mu(\{x\}) = 4\pi - \nu(\{x\})$.

Since $u_n \geq 0$ a.e. and $\mu_n \geq 0$, we have $u \geq 0$ a.e., $\mu \geq 0$, and $\sigma \geq 0$. In order to prove (11.4)–(11.5), we first show that

$$\sigma = \mathrm{e}^v(\mathrm{e}^u - 1)\varphi + \gamma \quad \text{in } \Omega, \tag{11.6}$$

where γ is a nonnegative measure supported on the set

$$A = \Big\{x \in \Omega;\ \mu(\{x\}) + \nu(\{x\}) \geq 4\pi\Big\}. \tag{11.7}$$

Since $\mu+\nu$ is a bounded measure, A has at most finitely many points. In particular, A is closed. Now let $x_0 \in \Omega \setminus A$; thus,

$$\mu(\{x_0\}) + \nu(\{x_0\}) < 4\pi.$$

By outer regularity of Radon measures, there exist $r_0, \varepsilon > 0$ sufficiently small so that $B_{3r_0}(x_0) \subset \Omega \setminus A$ and

$$\int_{B_{3r_0}(x_0)} \mathrm{d}(\mu + \nu)^+ \leq 4\pi - \varepsilon.$$

Since $(\mu_n + \nu_n)^+ \overset{*}{\rightharpoonup} (\mu + \nu)^+$ weak* in $\mathcal{M}(\Omega)$ (this is (b_4) with $\theta = 1$), one can find $n_0 \geq 1$ such that

$$\int_{B_{2r_0}(x_0)} \mathrm{d}(\mu_n + \nu_n)^+ \leq 4\pi - \frac{\varepsilon}{2} \quad \text{for every } n \geq n_0. \tag{11.8}$$

Recall that $u_n \geq 0$ a.e.; thus, $\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) \geq 0$ a.e., from which we deduce that

$$-\Delta(u_n + V_n) \leq \mu_n + \nu_n \quad \text{in } \mathcal{D}'(\Omega).$$

By (b_1)–(b_2), the sequence $(u_n + V_n)$ is bounded in $L^1(\Omega)$. A result of Brezis-Merle [15] (see Theorem 15.2 below) implies that $(\mathrm{e}^{u_n+V_n})$ is bounded in $L^p(B_{r_0}(x_0))$ for some $p > 1$. Since

$$0 \leq \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) \leq \mathrm{e}^{u_n + V_n} \quad \text{a.e.},$$

it follows that $\big(\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi\big)$ is an equi-integrable sequence in $L^1\big(B_{r_0}(x_0)\big)$ that converges a.e. to $\mathrm{e}^v(\mathrm{e}^u - 1)\varphi$. By Egorov's theorem, we deduce that

$$\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi \to \mathrm{e}^v(\mathrm{e}^u - 1)\varphi \quad \text{in } L^1\big(B_{r_0}(x_0)\big).$$

Therefore, $\gamma = \sigma - \mathrm{e}^v(\mathrm{e}^u - 1)\varphi$ is a nonnegative measure supported on A. It remains to show that $\gamma = 0$. Note that u satisfies

$$-\Delta u + \mathrm{e}^v(\mathrm{e}^u - 1) = \mu - \gamma \quad \text{in } \mathcal{D}'(\Omega).$$

In particular, Δu is a measure in Ω. Denoting by "c" the concentrated part of the measure with respect to (Newtonian) capacity (see Section 9 above), we get

$$(-\Delta u)_{\mathrm{c}} = \mu_{\mathrm{c}} - \gamma_{\mathrm{c}}.$$

Since $u \geq 0$ a.e., it follows from the Inverse Maximum Principle (see Theorem 9.2) that

$$0 \leq (-\Delta u)_{\mathrm{c}} = \mu_{\mathrm{c}} - \gamma_{\mathrm{c}}.$$

On the other hand, since A is finite, it has zero capacity; thus, $\gamma_{\mathrm{c}} = \gamma$. By (a_2) and (a_4), $\mu = 0$ on A. We deduce that

$$\gamma = 0.$$

Therefore, σ satisfies (11.4). In particular,

$$\int_\Omega \mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi \to \int_\Omega \mathrm{e}^v(\mathrm{e}^u - 1)\varphi. \tag{11.9}$$

We apply the Brezis-Lieb Lemma (see [13]) to the sequence $\big(\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}}-1)\varphi\big)$. In view of (11.2) and (11.9), we deduce that

$$\mathrm{e}^{v_n}(\mathrm{e}^{u_n}-1)\varphi \to \mathrm{e}^{v}(\mathrm{e}^{u}-1)\varphi \quad \text{in } L^1(\Omega).$$

Since the limit does not depend on the subsequences (u_{n_k}) and (v_{n_k}), (11.5) follows.

Step 1B. Proof of Step 1 completed.

We now drop assumption (a_4). For this purpose, for every $\theta \in (0,1)$ we denote by $u_{n,\theta}$ the solution of

$$\begin{cases} -\Delta u_{n,\theta} + \lambda \mathrm{e}^{v_n}(\mathrm{e}^{u_{n,\theta}}-1) = \theta\mu_n & \text{in } \omega,\\ u_{n,\theta} = u_n & \text{on } \partial\omega, \end{cases} \tag{11.10}$$

where $\omega \subset\subset \Omega$ is chosen as in the beginning of the proof of the lemma. The existence of $u_{n,\theta}$ follows from the method of sub and supersolutions (see Theorem A.1) applied with 0 and u_n; recall that $u_n \ge 0$ a.e. by hypothesis.
We next observe that assumptions (a_1)–(a_3) are satisfied by $(\theta\mu,\nu)$ for every $\theta \in (0,1)$. Assumptions (b_1)–(b_4) also hold. Let us check (a_4). Recall that

$$\mu(\{x\}) \le 4\pi - \nu(\{x\}) \quad \forall x \in \Omega.$$

If

$$\theta\mu(\{x_0\}) = 4\pi - \nu(\{x_0\}) \quad \text{for some } x_0 \in \Omega,$$

then since $\mu \ge 0$ it follows that $\mu(\{x_0\}) = 0$. Thus, (a_4) holds for $(\theta\mu,\nu)$.
Since $(\theta\mu_n)$ is bounded in $\mathcal{M}(\omega)$ and (u_n) is bounded in $L^1(\partial\omega)$, the sequence $(u_{n,\theta})$ is relatively compact in $L^1(\omega)$. Passing to a subsequence if necessary, we may assume that

$$u_{n,\theta} \to u_\theta \quad \text{in } L^1(\omega).$$

By Step 1A, we have

$$\mathrm{e}^{v_n}(\mathrm{e}^{u_{n,\theta}}-1)\varphi \to \mathrm{e}^{v}(\mathrm{e}^{u_\theta}-1)\varphi \quad \text{in } L^1(\omega).$$

On the other hand, by Proposition A.2,

$$\lambda\int_\omega \big|\mathrm{e}^{v_n}(\mathrm{e}^{u_{n,\theta}}-1) - \mathrm{e}^{v_n}(\mathrm{e}^{u_n}-1)\big|\varphi\,\mathrm{d}x \le C(1-\theta)\int_\omega \mathrm{d}|\mu_n| \le \tilde{C}(1-\theta).$$

A standard argument implies that $\big(\mathrm{e}^{v_n}(\mathrm{e}^{u_n}-1)\varphi\big)$ is a Cauchy sequence in $L^1(\Omega)$. In view of (11.2), we deduce that (11.4)–(11.5) hold. The proof of Step 1 is complete.

Step 2. Proof of (11.4)–(11.5) if $u_n \le 0$ a.e. and $\mu_n \le 0$, $\forall n \ge 1$.

In this case, $u \le 0$ a.e., $\mu \le 0$, and $\sigma \le 0$. As in Step 1A, we first show that

$$\sigma = \mathrm{e}^{v}(\mathrm{e}^{u}-1) + \gamma \quad \text{in } \Omega, \tag{11.11}$$

where γ is a nonpositive measure supported on

$$\tilde{A} = \Big\{x \in \Omega;\ \nu(\{x\}) \ge 4\pi\Big\}. \tag{11.12}$$

Let $x_0 \in \Omega \setminus \tilde{A}$. By assumption, we have

$$\nu(\{x_0\}) < 4\pi.$$

Take $\varepsilon, r_0 > 0$ sufficiently small so that $B_{3r_0}(x_0) \subset \Omega \setminus \tilde{A}$ and

$$\int_{B_{3r_0}(x_0)} \mathrm{d}\nu^+ \le 4\pi - \varepsilon.$$

For $n_0 \geq 1$ sufficiently large, it follows from (b_4) with $\theta = 0$ that $\nu_n^+ \overset{*}{\rightharpoonup} \nu^+$ weak* in $\mathcal{M}(\Omega)$; hence,

$$\int_{B_{2r_0}(x_0)} \mathrm{d}\nu_n^+ \leq 4\pi - \frac{\varepsilon}{2} \quad \forall n \geq n_0.$$

Thus, by Theorem 15.2, (e^{V_n}) is bounded in $L^p(B_{r_0}(x_0))$ for some $p > 1$. Since

$$0 \leq \mathrm{e}^{v_n}(1 - \mathrm{e}^{u_n}) \leq \mathrm{e}^{V_n} \quad \text{a.e.},$$

it follows that $\big(\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\big)$ is an equi-integrable sequence in $L^1\big(B_{r_0}(x_0)\big)$ that converges a.e. to $\mathrm{e}^v(\mathrm{e}^u - 1)$. By Egorov's theorem, we deduce that

$$\mathrm{e}^{v_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi \to \mathrm{e}^v(\mathrm{e}^u - 1)\varphi \quad \text{in } L^1\big(B_{r_0}(x_0)\big).$$

We then conclude that $\gamma = \sigma - \mathrm{e}^v(\mathrm{e}^u - 1)\varphi$ is a nonpositive measure supported on $\tilde{A}$. Proceeding as in Step 1A (where (a_4) is replaced by (a_3)), we deduce that $\gamma = 0$. Thus, (11.4) holds. Applying the Brezis-Lieb Lemma as in Step 1A, we obtain (11.5). This concludes the proof of Step 2.

Step 3. Proof of (11.4)–(11.5) completed.

By Lemma 8.1, both problems

$$\begin{cases} -\Delta\overline{u}_n + \lambda\mathrm{e}^{v_n}(\mathrm{e}^{\overline{u}_n} - 1) = \mu_n^+ & \text{in } \omega,\\ \overline{u}_n = u_n^+ & \text{on } \partial\omega, \end{cases}$$

and

$$\begin{cases} -\Delta\underline{u}_n + \lambda\mathrm{e}^{v_n}(\mathrm{e}^{\underline{u}_n} - 1) = -\mu_n^- & \text{in } \omega,\\ \underline{u}_n = -u_n^- & \text{on } \partial\omega, \end{cases}$$

have a solution for every $n \geq 1$. In addition, by Proposition A.1, $\underline{u}_n$ and $\overline{u}_n$ satisfy

$$\underline{u}_n \leq u_n \leq \overline{u}_n \quad \text{a.e. in } \omega;$$

thus,

$$\mathrm{e}^{v_n}(\mathrm{e}^{\underline{u}_n} - 1) \leq \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) \leq \mathrm{e}^{v_n}(\mathrm{e}^{\overline{u}_n} - 1) \quad \text{a.e. in } \omega. \tag{11.13}$$

By Propositions A.2 and A.5, $(\underline{u}_n)$ and $(\overline{u}_n)$ are relatively compact in $L^1(\omega)$. We then deduce from Steps 1 and 2 above that both sequences

$$\big(\mathrm{e}^{v_n}(\mathrm{e}^{\underline{u}_n} - 1)\varphi\big) \quad \text{and} \quad \big(\mathrm{e}^{v_n}(\mathrm{e}^{\overline{u}_n} - 1)\varphi\big)$$

are also relatively compact in $L^1(\omega)$. In view of (11.13), it follows from dominated convergence that for some subsequence we have

$$\mathrm{e}^{v_{n_{k_j}}}(\mathrm{e}^{u_{n_{k_j}}} - 1)\varphi \to \mathrm{e}^v(\mathrm{e}^u - 1)\varphi \quad \text{in } L^1(\Omega).$$

Since the limit does not depend on the subsequence, (11.4)–(11.5) hold.

We have thus proved (11.5) for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω, from which assertions (8.3)–(8.4) follow. The proof of Proposition 8.1 is complete. □

Proof of Theorem 8.1. Let (ρ_n) be a sequence of nonnegative mollifiers such that $\operatorname{supp}\rho_n \subset B_{1/n}$, $\forall n \geq 1$. We take $\mu_n = \rho_n * \mu$, $\nu_n = \rho_n * \nu$, and $v_n = \rho_n * v$. For each $n \geq 1$, the equation

$$\begin{cases} -\Delta u_n + \lambda\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) = \mu_n & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

has a (unique) solution $u_n \in W_0^{1,2}(\Omega)$ (the existence of u_n can be obtained for instance via standard minimization). Applying Proposition A.3, we have

$$\|\Delta u_n\|_{\mathcal{M}} \le 2\|\mu_n\|_{\mathcal{M}} \le 2\|\mu\|_{\mathcal{M}}.$$

Thus, by standard elliptic estimates (see [59]), (u_n) is bounded in $W_0^{1,p}(\Omega)$ for every $1 \le p < 2$. Passing to a subsequence if necessary, we may assume that

$$u_n \to u \quad \text{in } L^1(\Omega), \tag{11.14}$$

for some $u \in W_0^{1,1}(\Omega)$.

Take $\omega \subset\subset \Omega$. Assumptions (a_1)–(a_3), (b_1), and (b_3)–(b_4) are all satisfied in Ω; hence in ω as well. Note that for $n \ge 1$ sufficiently large, we have $-\Delta V_n = \nu_n$ in $\mathcal{D}'(\omega)$, where $V_n = \rho_n * V$. Thus, (b_2) holds in ω. By Proposition 8.1, u satisfies

$$-\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu \quad \text{in } \mathcal{D}'(\omega),$$

for every $\omega \subset\subset \Omega$. Therefore,

$$-\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu \quad \text{in } \mathcal{D}'(\Omega).$$

Since $u \in W_0^{1,1}(\Omega)$, we apply [14, Proposition B.1] to deduce that

$$-\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu \quad \text{in } \big[C_0^2(\overline{\Omega})\big]^*.$$

In other words, u is a solution of (8.1). The uniqueness follows from Proposition A.1. □

12. Some a priori estimates

In this section we present some tools in order to establish estimates (1.3)–(1.4) and, more generally, (7.2)–(7.3). Our main goal is the next

Theorem 12.1. *Let $u, v \in L^1(\mathbb{R}^2)$ be such that*

$$\mathrm{e}^v(\mathrm{e}^u - 1),\ \mathrm{e}^u(\mathrm{e}^v - 1) \in L^1(\mathbb{R}^2).$$

If $\Delta u \in \mathcal{M}(\mathbb{R}^2)$, then

$$\|u\|_{L^1} \le C\big(1 + \|\Delta u\|_{\mathcal{M}}^2\big)\Big\{\big\|\mathrm{e}^v(\mathrm{e}^u - 1)\big\|_{L^1} + \big\|\mathrm{e}^u(\mathrm{e}^v - 1)\big\|_{L^1}\Big\}, \tag{12.1}$$

$$\|\mathrm{e}^u - 1\|_{L^1} \le C\big(1 + \|\Delta u\|_{\mathcal{M}}\big)\Big\{\big\|\mathrm{e}^v(\mathrm{e}^u - 1)\big\|_{L^1} + \big\|\mathrm{e}^u(\mathrm{e}^v - 1)\big\|_{L^1}\Big\}. \tag{12.2}$$

Theorem 12.1 bears some similarity with some global L^1-estimates of Bénilan-Brezis-Crandall [8] (see e.g. Lemma 12.2 below). Our case is slightly different in view of the degeneracy of the nonlinear terms at $-\infty$:

$$\lim_{s,t \to -\infty} \big(\mathrm{e}^s|\mathrm{e}^t - 1| + \mathrm{e}^t|\mathrm{e}^s - 1|\big) = 0.$$

The main ingredient in the proof of Theorem 12.1 is the next

Proposition 12.1. *Let $u \in L^1(\mathbb{R}^2)$ be such that $\Delta u \in \mathcal{M}(\mathbb{R}^2)$. Then,*

$$\big|\big[|u| \ge 2\big]\big| \le C\|\Delta u\|_{\mathcal{M}}\big|\big[1 < |u| < 2\big]\big| \tag{12.3}$$

and

$$\int\limits_{[|u| \ge 3]} |u|\,\mathrm{d}x \le C\|\Delta u\|_{\mathcal{M}}^2\big|\big[1 < |u| < 2\big]\big|. \tag{12.4}$$

Before establishing Proposition 12.1, we first present some preliminary estimates:

Lemma 12.1. *For every $u \in W^{1,2}(\mathbb{R}^2)$, we have*

$$\big|[|u| \geq 2]\big| \leq C \|\nabla u\|_{L^2}^2 \big|[1 < |u| < 2]\big|. \tag{12.5}$$

Proof. Let $S : \mathbb{R} \to \mathbb{R}$ be given by

$$S(t) = \begin{cases} 0 & \text{if } |t| \leq 1, \\ |t| - 1 & \text{if } 1 < |t| < 2, \\ 1 & \text{if } |t| \geq 2. \end{cases}$$

Then, $S(u) \in W^{1,2}(\mathbb{R}^2)$. Moreover,

$$|\nabla S(u)| = \begin{cases} |\nabla u| & \text{a.e. on } \big[1 < |u| < 2\big], \\ 0 & \text{otherwise.} \end{cases}$$

Since the set $[1 < |u| < 2]$ has finite measure, it then follows from Hölder's inequality that $\nabla S(u) \in L^1(\mathbb{R}^2)$ and

$$\|\nabla S(u)\|_{L^1} = \int\limits_{[1<|u|<2]} |\nabla u| \, \mathrm{d}x \leq \|\nabla u\|_{L^2} \big|[1 < |u| < 2]\big|^{1/2}. \tag{12.6}$$

On the other hand, by the Gagliardo-Nirenberg inequality (see [53]), we have

$$\|S(u)\|_{L^2} \leq C \|\nabla S(u)\|_{L^1}. \tag{12.7}$$

Also, by the Tchebychev inequality,

$$\big|[|u| \geq 2]\big|^{1/2} = \big|[S(u) \geq 1]\big|^{1/2} \leq \|S(u)\|_{L^2}. \tag{12.8}$$

Combining (12.6)–(12.8), we deduce (12.5). □

We also recall the following (see [8])

Lemma 12.2. *Let $u \in L^1(\mathbb{R}^2)$ be such that $\Delta u \in \mathcal{M}(\mathbb{R}^2)$. Then,*

$$\int\limits_{[|u|\geq 3]} |u| \, \mathrm{d}x \leq C \|\Delta u\|_{\mathcal{M}} \big|[|u| \geq 2]\big| \tag{12.9}$$

Proof of Proposition 12.1. We split the proof in two steps:

Step 1. Proof of (12.3).

Let $\tilde{T}_2 : \mathbb{R} \to \mathbb{R}$ be the truncation operator at levels ± 2; more precisely,

$$\tilde{T}_2(t) = \begin{cases} -2 & \text{if } t \leq -2, \\ t & \text{if } |t| < 2, \\ 2 & \text{if } t \geq 2. \end{cases} \tag{12.10}$$

We then write $v = \tilde{T}_2(u)$. We claim that $\nabla v \in L^2(\mathbb{R}^2)$ and

$$\|\nabla v\|_{L^2}^2 = \int_{\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x \leq 2 \|\Delta u\|_{\mathcal{M}}. \tag{12.11}$$

(This inequality amounts to a formal integration by parts using the identity $|\nabla v|^2 = \nabla v \cdot \nabla u$ a.e.)

Indeed, given $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^2$, $\varphi = 1$ on B_1, and $\operatorname{supp} \varphi \subset B_2$, let $\varphi_n(x) = \varphi(\frac{x}{n})$. By [17, Lemma 1], we know that $v \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} |\nabla v|^2 \varphi_n \, \mathrm{d}x \leq 2 \big(\|\Delta u\|_{\mathcal{M}} + \|\Delta \varphi_n\|_{L^\infty} \|u\|_{L^1}\big).$$

As $n \to \infty$, we have $\varphi_n \to 1$ a.e. and $\|\Delta\varphi_n\|_{L^\infty} \to 0$. Thus, $\nabla v \in L^2(\mathbb{R}^2)$ and (12.11) holds. Since

$$\big[|v| \geq 2\big] = \big[|u| \geq 2\big] \quad \text{and} \quad \big[1 < |v| < 2\big] = \big[1 < |u| < 2\big],$$

we obtain (12.3) by applying Lemma 12.1 to v and using (12.11) to estimate $\|\nabla v\|_{L^2}$.

Step 2. Proof of (12.4).

It suffices to combine estimates (12.3) and (12.9). The proof of the proposition is complete. □

In the proof of Theorem 12.1, we also need the following elementary

Lemma 12.3. *There exists $C > 0$ such that if $s > -3$, then*

$$|\mathrm{e}^s - 1| \leq C\left(\mathrm{e}^t|\mathrm{e}^s - 1| + \mathrm{e}^s|\mathrm{e}^t - 1|\right) \quad \forall t \in \mathbb{R}. \tag{12.12}$$

Proof. Let $s > -3$. Note that for every $t \in \mathbb{R}$, we have

$$\begin{aligned} &\mathrm{e}^t|\mathrm{e}^s - 1| \geq \mathrm{e}^{-3}\,|\mathrm{e}^s - 1| && \text{if } t > -3,\\ &\mathrm{e}^s|\mathrm{e}^t - 1| \geq (1 - \mathrm{e}^{-3})\,\mathrm{e}^s \geq \mathrm{e}^{-3}\,|\mathrm{e}^s - 1| && \text{if } t \leq -3. \end{aligned}$$

Thus, for any such $s, t \in \mathbb{R}$,

$$|\mathrm{e}^s - 1| \leq \mathrm{e}^3\left(\mathrm{e}^t|\mathrm{e}^s - 1| + \mathrm{e}^s|\mathrm{e}^t - 1|\right).$$

We then obtain (12.12) with $C = \mathrm{e}^3$. □

Corollary 12.1. *There exists $C > 0$ such that*

$$\mathrm{e}^s + \mathrm{e}^{s+t} \leq C\left(\mathrm{e}^t|\mathrm{e}^s - 1| + \mathrm{e}^s|\mathrm{e}^t - 1| + 1\right) \quad \forall s, t \in \mathbb{R}. \tag{12.13}$$

Proof. The estimate for e^s easily follows from (12.12). In order to deduce (12.13), it remains to observe that

$$\mathrm{e}^{s+t} \leq \mathrm{e}^s|\mathrm{e}^t - 1| + \mathrm{e}^s \quad \forall s, t \in \mathbb{R}.$$

□

We now present the

Proof of Theorem 12.1. We split the proof in two steps:

Step 1. $(\mathrm{e}^u - 1) \in L^1(\mathbb{R}^2)$ and (12.2) holds.

By Lemma 12.3, we have

$$\int_{[u>-3]} |\mathrm{e}^u - 1|\,\mathrm{d}x \leq C\left(\int_{\mathbb{R}^2} \mathrm{e}^v|\mathrm{e}^u - 1|\,\mathrm{d}x + \int_{\mathbb{R}^2} \mathrm{e}^u|\mathrm{e}^v - 1|\,\mathrm{d}x\right). \tag{12.14}$$

On the other hand, applying (12.3),

$$\int_{[u\leq-3]} |\mathrm{e}^u - 1|\,\mathrm{d}x \leq \big|[u \leq -3]\big| \leq C\|\Delta u\|_{\mathcal{M}}^2\big|\big[1 < |u| < 2\big]\big|. \tag{12.15}$$

By the Tchebychev inequality and (12.14), we also have

$$\begin{aligned} \big|\big[1 < |u| < 2\big]\big| &\leq \frac{1}{1 - \mathrm{e}^{-1}} \int_{[1<|u|<2]} |\mathrm{e}^u - 1|\,\mathrm{d}x\\ &\leq C\left(\int_{\mathbb{R}^2} \mathrm{e}^v|\mathrm{e}^u - 1|\,\mathrm{d}x + \int_{\mathbb{R}^2} \mathrm{e}^u|\mathrm{e}^v - 1|\,\mathrm{d}x\right). \end{aligned} \tag{12.16}$$

Combining (12.14)–(12.16), we deduce (12.2). In particular, $(\mathrm{e}^u - 1) \in L^1(\mathbb{R}^2)$.

Step 2. Proof of (12.1).

Since $|\mathrm{e}^t - 1| \geq C|t|$ for every $t > -3$, we deduce from (12.14) that

$$\int_{[u>-3]} |u| \,\mathrm{d}x \leq C \left(\int_{\mathbb{R}^2} \mathrm{e}^v |\mathrm{e}^u - 1| \,\mathrm{d}x + \int_{\mathbb{R}^2} \mathrm{e}^u |\mathrm{e}^v - 1| \,\mathrm{d}x \right). \tag{12.17}$$

On the other hand, by (12.4),

$$\int_{[u\leq -3]} |u| \,\mathrm{d}x \leq \int_{[|u|\geq 3]} |u| \,\mathrm{d}x \leq C \|\Delta u\|_{\mathcal{M}}^2 \big| \big[1 < |u| < 2\big] \big|. \tag{12.18}$$

Estimate (12.1) then follows from (12.16) and (12.17)–(12.18). The proof is complete. $\square$

In order to apply Theorem 12.1 in the sequel, we shall need the following extension result:

Proposition 12.2. *Let $u \in L^1(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ be such that*

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.19}$$

Let $\bar{u} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{12.20}$$

Then, $\Delta \bar{u} \in \mathcal{M}(\mathbb{R}^2)$ and

$$\|\Delta \bar{u}\|_{\mathcal{M}(\mathbb{R}^2)} \leq 2 \|\mu\|_{\mathcal{M}(\Omega)}. \tag{12.21}$$

We refer the reader to [16] for a proof of Proposition 12.2.

13. Study of system (1.1) on bounded domains

In this section, we consider the counterpart of (1.1) on bounded domains:

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v (\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ -\Delta v + \lambda \mathrm{e}^u (\mathrm{e}^v - 1) = \nu & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.1}$$

We prove the following

Theorem 13.1. *Assume $\mu, \nu \in \mathcal{M}(\Omega)$ satisfy (i)–(iii). Then, for every $\lambda > 0$ (13.1) has a solution $(u, v) \in L^1(\Omega) \times L^1(\Omega)$. Moreover, every solution of (13.1) satisfies (7.2)–(7.3).*

Proof. Let U and V be given by

$$\begin{cases} -\Delta U = \mu^+ & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta V = \nu^+ & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

To each $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ we associate a pair $(\tilde{u}, \tilde{v})$, where $\tilde{u}$ solves

$$\begin{cases} -\Delta \tilde{u} + \lambda \mathrm{e}^{\min\{v, V\}} (\mathrm{e}^{\tilde{u}} - 1) = \mu & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{13.2}$$

and $\tilde{v}$ solves

$$\begin{cases} -\Delta \tilde{v} + \lambda e^{\min\{u,U\}}(e^{\tilde{v}} - 1) = \nu & \text{in } \Omega, \\ \tilde{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.3}$$

Note that problems (13.2) and (13.3) fulfill the assumptions of Theorem 8.1. Thus, $\tilde{u}$ and $\tilde{v}$ both exist and are uniquely determined. We can now consider the mapping $\mathcal{K}$ from $L^1(\Omega) \times L^1(\Omega)$ into itself, given by

$$\mathcal{K}(u,v) := (\tilde{u}, \tilde{v}).$$

Claim 1. $\mathcal{K}(L^1 \times L^1)$ is a bounded subset of $W_0^{1,p} \times W_0^{1,p}$ for every $1 \le p < 2$.

It suffices to observe that for every $(u,v) \in L^1 \times L^1$ the corresponding pair $(\tilde{u}, \tilde{v})$ satisfies (see Proposition A.3)

$$\int_\Omega |\Delta \tilde{u}| \le 2\|\mu\|_{\mathcal{M}} \quad \text{and} \quad \int_\Omega |\Delta \tilde{v}| \le 2\|\nu\|_{\mathcal{M}}.$$

It then follows from standard elliptic estimates that $\mathcal{K}(L^1 \times L^1)$ is contained in a bounded set of $W_0^{1,p} \times W_0^{1,p}$ for every $1 \le p < 2$, i.e. there exists $C_p > 0$ such that

$$\|\mathcal{K}(u,v)\|_{W_0^{1,p} \times W_0^{1,p}} \le C_p \quad \forall (u,v) \in L^1 \times L^1.$$

Claim 2. $\mathcal{K}$ is continuous.

In fact, assume $(u_n, v_n) \to (u,v)$ in $L^1 \times L^1$. Let us prove for instance that

$$\tilde{u}_n \to \tilde{u} \quad \text{in } L^1(\Omega). \tag{13.4}$$

By the previous claim, the sequence $(\tilde{u}_n)$ is bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence, we have

$$\tilde{u}_{n_k} \to \hat{u} \quad \text{in } L^1(\Omega),$$

for some $\hat{u} \in W_0^{1,1}(\Omega)$. We apply Proposition 8.1 with $V_n = V$, $\mu_n = \mu$ and $\nu_n = \nu^+$, $\forall n \ge 1$. We deduce that

$$-\Delta \hat{u} + \lambda e^{\min\{v,V\}}(e^{\hat{u}} - 1) = \mu \quad \text{in } \mathcal{D}'(\Omega).$$

Since $\hat{u} \in W_0^{1,1}(\Omega)$, then by [14, Proposition B.1] we conclude that $\hat{u}$ is a solution of (13.2). By uniqueness, we must have $\hat{u} = \tilde{u}$ a.e. Since the limit does not depend on the subsequence $(\tilde{u}_{n_k})$, (13.4) holds. Reverting the roles of (u_n) and (v_n), we obtain the counterpart of (13.4) for $(\tilde{v}_n)$. Therefore, $\mathcal{K}$ is continuous.

Applying Schauder's fixed point theorem, we deduce that $\mathcal{K}$ has a fixed point (u_0, v_0). Note that $U \ge 0$ a.e. Thus, by Proposition A.1 we have $u_0 \le U$ a.e. Similarly, $v_0 \le V$ a.e. We then conclude that (u_0, v_0) is a solution of (13.1).

It remains to show that (7.2)–(7.3) hold for every solution (u,v) of (13.1). In fact, let $\bar{u}, \bar{v}$ denote the extensions of u, v as 0 outside Ω, respectively. By Proposition 12.2, we know that $\Delta \bar{u} \in \mathcal{M}(\mathbb{R}^2)$ and

$$\|\Delta \bar{u}\|_{\mathcal{M}} \le 2\|\Delta u\|_{\mathcal{M}}.$$

Applying Theorem 12.1 to $\bar{u}$ and $\bar{v}$, we conclude that (12.1)–(12.2) hold. On the other hand, by Proposition A.3,

$$\|\Delta u\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}},$$
$$\int_{\mathbb{R}^2} \mathrm{e}^{\bar{v}}|\mathrm{e}^{\bar{u}}-1|\,\mathrm{d}x = \int_{\Omega} \mathrm{e}^{v}|\mathrm{e}^{u}-1|\,\mathrm{d}x \leq \frac{\|\mu\|_{\mathcal{M}}}{\lambda},$$
$$\int_{\mathbb{R}^2} \mathrm{e}^{\bar{u}}|\mathrm{e}^{\bar{v}}-1|\,\mathrm{d}x = \int_{\Omega} \mathrm{e}^{u}|\mathrm{e}^{v}-1|\,\mathrm{d}x \leq \frac{\|\nu\|_{\mathcal{M}}}{\lambda}.$$

Therefore,

$$\|u\|_{L^1} = \int_{\mathbb{R}^2} |\bar{u}|\,\mathrm{d}x \leq C\big(1+\|\mu\|_{\mathcal{M}}^2\big)\,\frac{\big(\|\mu\|_{\mathcal{M}}+\|\nu\|_{\mathcal{M}}\big)}{\lambda}, \tag{13.5}$$

$$\|\mathrm{e}^{u}-1\|_{L^1} = \int_{\mathbb{R}^2} |\mathrm{e}^{\bar{u}}-1|\,\mathrm{d}x \leq C\big(1+\|\mu\|_{\mathcal{M}}\big)\,\frac{\big(\|\mu\|_{\mathcal{M}}+\|\nu\|_{\mathcal{M}}\big)}{\lambda}. \tag{13.6}$$

Interchanging the roles of u and v, we obtain a similar estimate for v. This immediately implies (7.2)–(7.3). $\square$

Remark 13.1. The proof of Theorem 13.1 is based on a standard fixed point argument. However, the continuity of $\mathcal{K}$ relies on Proposition 8.1, whose proof is rather technical. If one assumes $\mu, \nu \leq 0$ (this is precisely the setting of Theorem 1.1), then the continuity of $\mathcal{K}$ becomes much easier. Indeed, in this case $U = V = 0$. Assume

$$(u_n, v_n) \to (u, v) \quad \text{in } L^1 \times L^1.$$

Note that by Proposition A.1 we have $\tilde{u}_n, \tilde{v}_n \leq 0$ a.e. If $\tilde{u}_{n_k} \to \hat{u}$ in $L^1(\Omega)$ for some $\hat{u} \in W_0^{1,1}(\Omega)$, then by dominated convergence we get

$$\mathrm{e}^{\min\{v_{n_k},0\}}(\mathrm{e}^{\tilde{u}_{n_k}}-1) \to \mathrm{e}^{\min\{v,0\}}(\mathrm{e}^{\hat{u}}-1) \quad \text{in } L^p(\Omega),$$

for every $1 \leq p < \infty$. Hence, $\hat{u}$ and $\tilde{u}$ are solutions of the same equation. By uniqueness, $\hat{u} = \tilde{u}$ a.e. and

$$\tilde{u}_n \to \tilde{u} \quad \text{in } L^1(\Omega).$$

A similar argument holds for $(\tilde{v}_n)$. Therefore, $\mathcal{K}$ is continuous.

14. Proof of Theorem 7.1

Let (Ω_n) denote an increasing sequence of smooth bounded domains such that $\bigcup_n \Omega_n = \mathbb{R}^2$. Since μ and ν satisfy *(i)*–*(iii)*, it follows from Theorem 13.1 that for every $n \geq 1$ there exists a pair $(u_n, v_n) \in L^1(\Omega_n) \times L^1(\Omega_n)$ satisfying (7.2)–(7.3) such that

$$\begin{cases} -\Delta u_n + \lambda \mathrm{e}^{v_n}(\mathrm{e}^{u_n}-1) = \mu & \text{in } \Omega_n,\\ -\Delta v_n + \lambda \mathrm{e}^{u_n}(\mathrm{e}^{v_n}-1) = \nu & \text{in } \Omega_n,\\ \qquad\qquad u = v = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{14.1}$$

Claim 1. The sequence (u_n, v_n) is bounded in $W^{1,p}_{\mathrm{loc}} \times W^{1,p}_{\mathrm{loc}}$ for every $1 \leq p < 2$ and there exists a subsequence (u_{n_k}, v_{n_k}) such that

$$(u_{n_k}, v_{n_k}) \to (u, v) \quad \text{in } L^1_{\mathrm{loc}} \times L^1_{\mathrm{loc}} \tag{14.2}$$

for some $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$.

We recall that u_n and v_n satisfy (see Proposition A.3)

$$\int_{\Omega_n} |\Delta u_n| \le 2\|\mu\|_{\mathcal{M}} \quad \text{and} \quad \int_{\Omega_n} |\Delta v_n| \le 2\|\nu\|_{\mathcal{M}}. \tag{14.3}$$

Moreover,

$$\int_{\Omega_n} |u_n| \,\mathrm{d}x + \int_{\Omega_n} |v_n| \,\mathrm{d}x \le \frac{C}{\lambda}\big(1 + \|\mu\|_{\mathcal{M}}^2 + \|\nu\|_{\mathcal{M}}^2\big)\big(\|\mu\|_{\mathcal{M}} + \|\nu\|_{\mathcal{M}}\big). \tag{14.4}$$

We deduce from (14.3)–(14.4) that (u_n) and (v_n) are relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ (see Proposition A.4). Passing to a subsequence, we get (14.2) for some $(u, v) \in L^1_{\mathrm{loc}} \times L^1_{\mathrm{loc}}$. By (14.4) and Fatou's lemma, we actually have $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$ and (7.2) holds; similarly, (7.3) is also true. This concludes the proof of Claim 1.

Claim 2. The pair (u, v) given by (14.2) satisfies (7.1).

It suffices to show that (u, v) satisfies (7.1) on B_r, for every $r > 0$. We shall prove that

$$-\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu \quad \text{in } \mathcal{D}'(B_r). \tag{14.5}$$

Let $n_0 \ge 1$ be such that $B_r \subset \Omega_{n_0}$. Clearly, for every $n \ge n_0$ we have

$$-\Delta u_n + \lambda \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) = \mu \quad \text{in } \mathcal{D}'(B_r).$$

Without loss of generality, we may assume that the convergence (14.2) holds for the entire sequence $\big((u_n, v_n)\big)_{n \ge 1}$. Since (v_n) is bounded in $W^{1,p}(B_r)$ for every $1 \le p < 2$, we have from Trace Theory that

$$v_n \to v \quad \text{in } L^1(\partial B_r).$$

Passing to a further sequence if necessary, we may assume there exists $h \in L^1(\partial B_r)$ such that

$$|v_n| \le h \quad \text{a.e. on } \partial B_r \quad \forall n \ge n_0.$$

By Proposition A.1, $v_n \le \tilde{V}$ a.e., $\forall n \ge n_0$, where $\tilde{V} \ge 0$ is the solution of

$$\begin{cases} -\Delta \tilde{V} = \nu^+ & \text{in } B_r, \\ \tilde{V} = h & \text{on } \partial B_r. \end{cases}$$

Applying Proposition 8.1 on B_r with $\mu_n = \mu$, $\nu_n = \nu^+$, and $V_n = \tilde{V}$, $\forall n \ge n_0$, we conclude that u satisfies (14.5). The counterpart for v follows by interchanging the roles of u and v. The proof of Theorem 7.1 is complete. $\square$

15. Study of assumptions (*i*)–(*iii*) of Theorem 7.1

In this section, we use the following results:

Theorem 15.1 (Vázquez [61])**.** *Let $w \in L^1(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ be such that*

$$-\Delta w = \mu \quad \text{in } \mathcal{D}'(\Omega).$$

If $\mathrm{e}^w \in L^1(\Omega)$, then

$$\mu(\{x\}) \le 4\pi \quad \forall x \in \Omega.$$

Theorem 15.2 (Brezis-Merle [15])**.** *Let $w \in L^1(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ be such that*

$$-\Delta w = \mu \quad \text{in } \mathcal{D}'(\Omega).$$

Assume there exist $r > 0$ and $\varepsilon > 0$ such that

$$|\mu|\big(B_r(x) \cap \Omega\big) \le 4\pi - \varepsilon \quad \forall x \in \Omega.$$

Then, for every $\omega \subset\subset \Omega$ there exists $p > 1$ such that

$$\mathrm{e}^w \in L^p(\omega) \quad \text{and} \quad \|\mathrm{e}^w\|_{L^p(\omega)} \le C,$$

for some constant $C > 0$ depending on $\|w\|_{L^1}$, ε, r, ω, and Ω.

Theorem 15.2 is stated in [15] for functions w satisfying, in addition, "$w = 0$ on $\partial\Omega$". The general case above can be easily recovered from [15, Theorem 1].

We then establish the

Proposition 15.1. *Given $\mu, \nu \in \mathcal{M}(\mathbb{R}^2)$, assume there exists $(u, v) \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \times L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that*

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \mathbb{R}^2, \\ -\Delta v + \lambda \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \mathbb{R}^2. \end{cases} \tag{15.1}$$

Then,

$$\mu^+(\{x\}) + \nu^+(\{x\}) \le 4\pi \quad \forall x \in \mathbb{R}^2. \tag{15.2}$$

Proof. Since (u, v) satisfies (15.1), we have

$$\mathrm{e}^v(\mathrm{e}^u - 1),\ \mathrm{e}^u(\mathrm{e}^v - 1) \in L^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Thus, by Corollary 12.1, e^{u+v}, e^u, $\mathrm{e}^v \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Applying Theorem 15.1 with $w = u + v$, u, v, we get

(*i'*) $\mu(\{x\}) + \nu(\{x\}) \le 4\pi$, $\forall x \in \mathbb{R}^2$;
(*ii'*) $\mu(\{x\}) \le 4\pi$, $\forall x \in \mathbb{R}^2$;
(*iii'*) $\nu(\{x\}) \le 4\pi$, $\forall x \in \mathbb{R}^2$.

The conclusion then follows from the identity

$$a^+ + b^+ = \max\big\{0, a, b, a + b\big\} \quad \forall a, b \in \mathbb{R}.$$

□

It follows from Proposition 15.1 that assumption (*i*) in Theorem 7.1 is necessary. We now study assumptions (*ii*)–(*iii*).

Proposition 15.2. *If* (7.1) *has a solution with $\mu = 4\pi\delta_0$ and $\nu = a\delta_0$ for some $a \in \mathbb{R}$, then $a = 0$.*

Proof. It follows from the previous proposition that $a \le 0$. Assume by contradiction that (15.1) has a solution (u, v) with $\mu = 4\pi\delta_0$ and $\nu = a\delta_0$, for some $a < 0$. Since $\mu(\{0\}) + \nu(\{0\}) < 4\pi$, then by Theorem 15.2 we have $\mathrm{e}^{u+v} \in L^p(B_1)$ for some $p > 1$. Let z be the solution of

$$\begin{cases} \Delta z = \lambda \mathrm{e}^{u+v} & \text{in } B_1, \\ z = 0 & \text{on } \partial B_1. \end{cases}$$

Then, by standard elliptic estimates, $z \in C^0(\overline{B}_1)$. On the other hand, we have

$$-\Delta u \ge 4\pi\delta_0 - \lambda \mathrm{e}^{u+v} = -\Delta\Big(2\log\frac{1}{|x|} + z\Big) \quad \text{in } \mathcal{D}'(B_1).$$

Let h be the harmonic function such that $h = u$ on ∂B_1. By the maximum principle,

$$u(x) \geq 2\log\frac{1}{|x|} + z(x) + h(x) \quad \forall x \in B_1.$$

Thus,

$$\mathrm{e}^u \geq \frac{\mathrm{e}^{z+h}}{|x|^2} \quad \text{in } B_1.$$

Since $z+h$ is continuous on B_1, we deduce that $\mathrm{e}^u \notin L^1(B_1)$. This is a contradiction since $\mathrm{e}^u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ by Corollary 12.1. We then must have $a = 0$. □

In view of Proposition 15.2, assumptions *(ii)*–*(iii)* are also necessary in the case of isolated Dirac masses. But as we will see below, equation (7.1) can have solutions for measures μ and ν which do not satisfy *(ii)*–*(iii)*. Indeed,

Proposition 15.3. *For every $a < 0$, there exists $f_a \in L^1(\mathbb{R}^2)$ such that* (7.1) *has a solution for $\mu = 4\pi\delta_0 + f_a$ and $\nu = a\delta_0$.*

Proof. Assume for simplicity that $\lambda = 1$. Given $a < 0$, let $u \geq w$ be the solutions of (see [61])

$$-\Delta u + (\mathrm{e}^u - 1) = 4\pi\delta_0 \quad \text{in } \mathbb{R}^2, \tag{15.3}$$

$$-\Delta w + (\mathrm{e}^w - 1) = (4\pi + a)\delta_0 \quad \text{in } \mathbb{R}^2, \tag{15.4}$$

such that $(\mathrm{e}^u - 1), (\mathrm{e}^w - 1) \in L^1(\mathbb{R}^2)$. Set $v = w - u$ and $f_a = (\mathrm{e}^v - 1)(\mathrm{e}^u - 1)$. Since $v \leq 0$, we have

$$|f_a| \leq |\mathrm{e}^u - 1|.$$

Thus, $f_a \in L^1(\mathbb{R}^2)$ and (u, v) is a solution of (7.1) with data $\mu = 4\pi\delta_0 + f_a$ and $\nu = a\delta_0$. □

16. Asymptotic behavior of (u, v) at infinity

We now study the behavior of solutions of (7.1) when both measures μ and ν have compact supports in $\mathbb{R}^2$. Our main result in this section is the following

Theorem 16.1. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^2)$ and $\lambda > 0$ be such that*

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \mathbb{R}^2, \\ -\Delta v + \lambda \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \mathbb{R}^2, \end{cases} \tag{16.1}$$

has a solution $(u, v) \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$. If μ and ν have compact supports in $\mathbb{R}^2$, then

$$|u(x)| + |v(x)| \leq C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}}, \tag{16.2}$$

$$|\nabla u(x)| + |\nabla v(x)| \leq C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}}, \tag{16.3}$$

for every $|x| \geq R$, where $R > 0$ is such that $\operatorname{supp}\mu \cup \operatorname{supp}\nu \subset B_R$.

We first recall the following well-known (see e.g. [5])

Proposition 16.1. *Let* $\alpha, \lambda, t_0 > 0$ *and let* $\Phi : [t_0, \infty) \to \mathbb{R}$ *be a continuous function such that* $\lim_{t\to\infty} \Phi(t) = 0$. *Then, the equation*

$$\begin{cases} w'' + \dfrac{1}{t} w' - \big(\lambda + \Phi(t)\big) w = 0 & \textit{in } (t_0, \infty), \\ w(t_0) = \alpha \quad \textit{and} \quad \lim_{t\to\infty} w(t) = 0, \end{cases} \tag{16.4}$$

has a unique solution w_0. *If, in addition,* $\int_{t_0}^{\infty} |\Phi(t)| \, \mathrm{d}t < \infty$, *then there exist constants* $C_0, C_1 > 0$ *such that*

$$C_0 \le \frac{w_0(t)}{W_0(t)} \le C_1 \quad \forall t \ge t_0, \tag{16.5}$$

where

$$W_0(t) = \frac{\mathrm{e}^{-\sqrt{\lambda}\, t}}{t^{1/2}}. \tag{16.6}$$

Proof. The substitution $z(t) = t^{1/2} w(t)$ transforms equation (16.4) into

$$z'' - \left(\lambda + \Phi(t) - \frac{1}{4t^2}\right) z = 0 \quad \text{in } (t_0, \infty), \tag{16.7}$$

with initial data $z(t_0) = \alpha t_0^{1/2}$. By [5, pp.125–126], this equation has a unique *bounded* solution z_0; every other solution of (16.5) grows exponentially fast as $t \to \infty$. Thus, the solution of (16.4) exists and is unique. In addition, if $\int_{t_0}^{\infty} |\Phi(t)| \, \mathrm{d}t < \infty$, then z_0 satisfies

$$C_0 \le \frac{z_0(t)}{\mathrm{e}^{-\sqrt{\lambda}\, t}} \le C_1 \quad \forall t \ge t_0.$$

This implies (16.5). □

Proof of Theorem 16.1. We split the proof in four steps:
Step 1. There exists $C > 0$ such that

$$|u(x)| + |v(x)| \le \frac{C}{|x|^2} \quad \forall x \in \mathbb{R}^2 \setminus B_R. \tag{16.8}$$

Let $\varepsilon > 0$ be sufficiently small so that $\operatorname{supp} \mu \cup \operatorname{supp} \nu \subset B_{R-\varepsilon}$. By Kato's inequality (see [46]), we have

$$-\Delta |u| + \lambda \mathrm{e}^{v} |\mathrm{e}^{u} - 1| \le 0 \quad \text{in } \mathcal{D}'(A_R), \tag{16.9}$$

where $A_R = \mathbb{R}^2 \setminus \overline{B_{R-\varepsilon}}$. Thus, $|u|$ is subharmonic in A_R and given $x \in \mathbb{R}^2 \setminus B_R$ we have

$$|u(x)| \le \frac{1}{\pi r^2} \int_{B_r(x)} |u| \, \mathrm{d}y \quad \text{for every } 0 < r \le |x| - R + \varepsilon.$$

In particular, taking $r = |x| - R + \varepsilon$ we deduce that

$$|u(x)| \le \frac{1}{\pi \big(|x| - R + \varepsilon\big)^2} \int_{\mathbb{R}^2} |u| \, \mathrm{d}y \le \frac{C}{|x|^2} \quad \forall x \in \mathbb{R}^2 \setminus B_R.$$

A similar estimate holds for v.

Step 2. For every $r \ge R$, let

$$\Phi(r) = \lambda \min_{|x|=r} \left\{ \mathrm{e}^{v} \frac{|\mathrm{e}^{u} - 1|}{|u|} - 1 \right\}. \tag{16.10}$$

(We use the convention that $\frac{e^t-1}{t} = 1$ if $t = 0$.)
Then, $\Phi : [R, \infty) \to \mathbb{R}$ is continuous, $\lim_{r\to\infty} \Phi(r) = 0$ and

$$\int_R^\infty |\Phi(r)|\,\mathrm{d}r < \infty. \tag{16.11}$$

Since u and v are uniformly bounded on $\mathbb{R}^2 \setminus B_R$, it follows from elliptic estimates that u and v are continuous; thus, Φ is continuous. Moreover, since

$$\left| e^s \frac{|e^t - 1|}{|t|} - 1 \right| \le C\big(|s| + |t|\big) \quad \forall s, t \in [-M, M],$$

for some constant $C > 0$ depending on M, we have by (16.8)

$$|\Phi(r)| \le C \max_{|x|=r} \big\{|u(x)| + |v(x)|\big\} \le \frac{C}{r^2} \quad \forall r \ge R.$$

Thus, $\Phi(r) \to 0$ as $r \to \infty$ and (16.11) holds.

Step 3. Let w_0 be the (unique) radial solution of

$$\begin{cases} -\Delta w_0 + \big(\lambda + \Phi(x)\big) w_0 = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B_R}, \\ w_0 = M & \text{on } \partial B_R, \\ \lim\limits_{|x|\to\infty} w_0(x) = 0, \end{cases} \tag{16.12}$$

where $M := \max_{x\in\mathbb{R}^2\setminus B_R} |u(x)|$ and $\Phi(x) := \Phi(|x|)$ is given by (16.10). Then,

$$|u| \le w_0 \quad \text{in } \mathbb{R}^2 \setminus B_R. \tag{16.13}$$

The existence and uniqueness of w_0 follows from Proposition 16.1. Given $\varepsilon > 0$, take $R' > R$ sufficiently large so that

$$|u(x)| \le \varepsilon \quad \forall x \in \mathbb{R}^2 \setminus B_{R'}.$$

Thus, by (16.9) the function $Z = |u| - w_0 - \varepsilon$ satisfies

$$\begin{cases} -\Delta Z + \lambda e^v |e^u - 1| - \big(\lambda + \Phi(x)\big) w_0 \le 0 & \text{in } B_{R'} \setminus \overline{B_R}, \\ Z \le 0 & \text{on } \partial B_R \cup \partial B_{R'}. \end{cases} \tag{16.14}$$

More precisely,

$$-\int_{B_{R'}\setminus\overline{B_R}} Z \Delta\zeta\,\mathrm{d}x \le \int_{B_{R'}\setminus\overline{B_R}} \Big\{\big(\lambda + \Phi(x)\big) w_0 - \lambda e^v |e^u - 1|\Big\}\zeta\,\mathrm{d}x$$

for every $\zeta \in C_0^2(\overline{B_{R'}} \setminus B_R)$, with $\zeta \ge 0$ in $B_{R'} \setminus \overline{B_R}$. Thus, by [14, Proposition B.5],

$$\begin{aligned} -\int_{B_{R'}\setminus\overline{B_R}} Z^+ \Delta\zeta\,\mathrm{d}x &\le \int_{[|u|\ge w_0+\varepsilon]} \Big\{\big(\lambda + \Phi(x)\big) w_0 - \lambda e^v |e^u - 1|\Big\}\zeta\,\mathrm{d}x \\ &\le \int_{[|u|\ge w_0+\varepsilon]} \Big\{\big(\lambda + \Phi(x)\big) - \lambda e^v \frac{|e^u - 1|}{|u|}\Big\} w_0 \zeta\,\mathrm{d}x \le 0, \end{aligned}$$

since the term in brackets is nonnegative and $w_0, \zeta \ge 0$. Therefore, $Z^+ \le 0$; hence,

$$|u| \le w_0 + \varepsilon \quad \text{in } B_{R'} \setminus \overline{B_R}.$$

As $R' \to \infty$, we get

$$|u| \le w_0 + \varepsilon \quad \text{in } \mathbb{R}^2 \setminus \overline{B_R}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (16.13) holds.

Step 4. Proof of Theorem 16.1 completed.

By (16.5) and (16.13), u satisfies

$$|u(x)| \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}} \quad \forall x \in \mathbb{R}^2 \setminus B_R.$$

A similar estimate holds for v. This implies (16.2). It then follows from (16.2) that

$$\mathrm{e}^v|\mathrm{e}^u - 1| + \mathrm{e}^v|\mathrm{e}^u - 1| \le C\big(|u(x)| + |v(x)|\big) \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}} \quad \forall x \in \mathbb{R}^2 \setminus B_R. \tag{16.15}$$

We now recall the following (see e.g. [9, Lemma A.1])

Lemma 16.1. *Let $u, f \in L^\infty(B_1)$ be such that*

$$-\Delta u = f \quad \text{in } \mathcal{D}'(B_1).$$

Then,

$$\|\nabla u\|^2_{L^\infty(B_{1/2})} \le C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\big)\|u\|_{L^\infty(B_1)}. \tag{16.16}$$

Applying Lemma 16.1 to u and v on balls $B_1(x)$ for $|x| \ge R+1$, we get

$$|\nabla u(x)| + |\nabla v(x)| \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,(|x|-1)}}{(|x|-1)^{1/2}} \le C\frac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{|x|^{1/2}}.$$

The proof of Theorem 16.1 is complete. □

Appendix A. Standard existence, comparison and compactness results

In this appendix we gather some known results related to the equation

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a smooth bounded domain, $\mu \in \mathcal{M}(\Omega)$, $h \in L^1(\partial\Omega)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The statements presented here complement Appendix B in [14].

We begin with the following generalization of the classical method of sub and supersolutions. This theorem extends previous results of Clément-Sweers [25] and Dancer-Sweers [26]:

Theorem A.1 (Montenegro-Ponce [51])**.** *Let $u_1, u_2 \in L^1(\Omega)$ be a sub and a supersolution of* (A.1)*, respectively, such that*

$$u_1 \le u_2 \quad a.e. \tag{A.2}$$

and

$$g(\cdot, v)\rho_0 \in L^1(\Omega) \quad \text{for every } v \in L^1(\Omega) \text{ such that } u_1 \le v \le u_2 \ a.e. \tag{A.3}$$

Then, (A.1) *has a solution u such that*

$$u_1 \le u \le u_2 \quad a.e.$$

Here, $\rho_0(x) = \operatorname{dist}(x, \partial\Omega)$, $\forall x \in \Omega$. We recall that $v \in L^1(\Omega)$ is a subsolution of (A.1) if $g(\cdot, v)\rho_0 \in L^1(\Omega)$ and

$$-\int_\Omega v\Delta\zeta \,\mathrm{d}x + \int_\Omega g(x,v)\zeta \,\mathrm{d}x \le \int_\Omega \zeta \,\mathrm{d}\mu - \int_{\partial\Omega} h \frac{\partial\zeta}{\partial \mathrm{n}} \,\mathrm{d}\ell \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega.$$

The notion of supersolution is defined accordingly.

We next present the following

Lemma A.1. *Let $v \in L^1(\Omega)$, $f \in L^1(\Omega; \rho_0 \,\mathrm{d}x)$, $\mu \in \mathcal{M}(\Omega)$, and $h \in L^1(\partial\Omega)$ be such that*

$$\text{(A.4)} \qquad -\int_\Omega v\Delta\zeta \,\mathrm{d}x + \int_\Omega f\zeta \,\mathrm{d}x = \int_\Omega \zeta \,\mathrm{d}\mu - \int_{\partial\Omega} h \frac{\partial\zeta}{\partial \mathrm{n}} \,\mathrm{d}\ell \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Then, for every $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \ge 0$ in Ω, we have

$$\text{(A.5)} \qquad -\int_\Omega v^+\Delta\zeta \,\mathrm{d}x + \int\limits_{[v\ge 0]} f\zeta \,\mathrm{d}x \le \int_\Omega \zeta \,\mathrm{d}\mu^+ - \int_{\partial\Omega} h^+ \frac{\partial\zeta}{\partial \mathrm{n}} \,\mathrm{d}\ell$$

and thus

$$\text{(A.6)} \qquad -\int_\Omega |v|\Delta\zeta \,\mathrm{d}x + \int_\Omega f \operatorname{sgn}(v)\, \zeta \,\mathrm{d}x \le \int_\Omega \zeta \,\mathrm{d}|\mu| - \int_{\partial\Omega} |h| \frac{\partial\zeta}{\partial \mathrm{n}} \,\mathrm{d}\ell.$$

Proof. Estimate (A.5) is established in [50, Lemma 1.5] when $\mu = 0$. The same strategy can also be used to prove (A.5) for any $\mu \in \mathcal{M}(\Omega)$. Applying (A.5) to v and $-v$, one obtains (A.6). □

Proposition A.1. *Suppose that $g_1, g_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying*

(A_1) *$g(x, \cdot)$ is nondecreasing for a.e. $x \in \Omega$;*
(A_2') *$g(x, 0) = 0$ for a.e. $x \in \Omega$.*

Let u_i be a solution of (A.1) *associated to g_i and (μ_i, h_i), $i = 1, 2$. If*

$$g_1 \ge g_2, \quad \mu_1 \le \mu_2, \quad \text{and} \quad h_1 \le h_2,$$

then

$$\text{(A.7)} \qquad u_1 \le u_2 \quad \text{a.e.}$$

In particular, if g satisfies (A_1) and (A_2'), then (A.1) *has at most one solution.*

Proof. Apply Lemma A.1 to $v = u_1 - u_2$. By (A.5) and (A_1)–(A_2'), we have

$$-\int_\Omega (u_1 - u_2)^+ \Delta\zeta \,\mathrm{d}x \le 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega.$$

Thus, $(u_1 - u_2)^+ \le 0$ a.e.; in other words, $u_1 \le u_2$ a.e. □

Proposition A.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (A_1) *and* (A_2')*. If u solves* (A.1)*, then*

$$\text{(A.8)} \qquad -\int_\Omega |u|\Delta\zeta \,\mathrm{d}x + \int_\Omega |g(x,u)|\zeta \,\mathrm{d}x \le \int_\Omega \zeta \,\mathrm{d}|\mu| - \int_{\partial\Omega} |h| \frac{\partial\zeta}{\partial \mathrm{n}} \,\mathrm{d}\ell,$$

for every $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \ge 0$ in Ω. Let u_i be the solution of (A.1) *associated to (μ_i, h_i), $i = 1, 2$. Then,*

$$\text{(A.9)} \quad \|u_1 - u_2\|_{L^1} + \big\|g(\cdot, u_1) - g(\cdot, u_2)\big\|_{L^1_{\rho_0}} \le C\Big(\|\mu_1 - \mu_2\|_{\mathcal{M}(\Omega)} + \|h_1 - h_2\|_{L^1(\partial\Omega)}\Big).$$

Proof. Estimate (A.8) follows from (A.6) applied to $v = u$ and $f = g(\cdot, u)$. The proof of (A.9) follows along the same lines by taking $\zeta = \zeta_0$, where ζ_0 satisfies

$$\begin{cases} -\Delta\zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

$\square$

Proposition A.3. *Suppose that g satisfies (A_1) and (A_2'). Let u be the solution of* (A.1) *with $h = 0$. Then,*

$$\int_\Omega |g(x,u)| \, \mathrm{d}x \le \|\mu\|_{\mathcal{M}} \quad \text{and} \quad \int_\Omega |\Delta u| \le 2\|\mu\|_{\mathcal{M}}. \tag{A.10}$$

In particular, $g(\cdot, u) \in L^1(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$.

Proof. By (A.8), for every superharmonic function $\zeta \in C_0^2(\overline{\Omega})$, we have

$$\int_\Omega |g(x,u)|\zeta \, \mathrm{d}x \le \int_\Omega \zeta \, \mathrm{d}|\mu|. \tag{A.11}$$

Apply (A.11) to a sequence of superharmonic functions (ζ_n) in $C_0^2(\overline{\Omega})$ such that $0 \le \zeta_n \le 1$ and $\zeta_n \to 1$ in $L^\infty_{\mathrm{loc}}(\Omega)$. As $n \to \infty$, we obtain

$$\int_\Omega |g(x,u)| \, \mathrm{d}x \le \int_\Omega \mathrm{d}|\mu| = \|\mu\|_{\mathcal{M}}.$$

Since

$$-\Delta u = \mu - g(x,u) \quad \text{in } \Omega,$$

we deduce that $\Delta u \in \mathcal{M}(\Omega)$ and (A.10) holds. $\square$

We recall the following compactness result:

Proposition A.4. *Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then, for every $\omega \subset\subset \Omega$ and $1 \le p < \frac{N}{N-1}$,*

$$\|u\|_{W^{1,p}(\omega)} \le C\big(\|u\|_{L^1(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}\big), \tag{A.12}$$

for some constant $C > 0$ depending on ω and p. In particular, if (u_n) is a bounded sequence in $L^1(\Omega)$ such that (Δu_n) is bounded in $\mathcal{M}(\Omega)$, then (u_n) is relatively compact in $L^q(\omega)$ for every $1 \le q < \frac{N}{N-2}$.

Proof. Clearly, it suffices to establish (A.12). Let $v \in L^1(\Omega)$ be the solution of

$$\begin{cases} \Delta v = \Delta u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard elliptic estimates (see [59]),

$$\|v\|_{W^{1,p}(\Omega)} \le C_p \|\Delta u\|_{\mathcal{M}(\Omega)}, \tag{A.13}$$

for every $1 \le p < \frac{N}{N-1}$. On the other hand, since $u - v$ is harmonic in Ω, we have

$$\|u - v\|_{C^1(\overline{\omega})} \le C_\omega \|u - v\|_{L^1(\Omega)} \le C_\omega\big(\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)}\big), \tag{A.14}$$

for every $\omega \subset\subset \Omega$. Combining (A.13)–(A.14), we obtain (A.12). $\square$

We conclude this section with the following "global" companion of Proposition A.4:

Proposition A.5. *Let $(u_n) \subset L^1(\Omega)$ be such that*

$$\left|\int_\Omega u_n \Delta\zeta \,\mathrm{d}x\right| \le K\|\zeta/\rho_0\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}), \tag{A.15}$$

for every $n \ge 1$. Then, (u_n) is relatively compact in $L^p(\Omega)$ for every $1 \le p < \frac{N}{N-1}$.

Proof. We split the proof in two steps:

Step 1. For every $1 < p < \frac{N}{N-1}$, $(u_n) \subset L^p(\Omega)$ and there exists $C_p > 0$ such that

$$\|u_n\|_{L^p} \le C_p K. \tag{A.16}$$

By duality it suffices to show that, for every $w \in C^\infty(\overline{\Omega})$,

$$\left|\int_\Omega u_n w \,\mathrm{d}x\right| \le C_p K \|w\|_{L^{p'}} \quad \forall n \ge 1. \tag{A.17}$$

For this purpose, let $\zeta \in C_0^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta\zeta = w & \text{in } \Omega,\\ \quad\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard Calderón-Zygmund estimates (see [37]),

$$\|\zeta\|_{W^{2,p'}} \le C_p \|w\|_{L^{p'}}. \tag{A.18}$$

Since $p' > N$, it follows from Morrey's imbedding that

$$\|\zeta/\rho_0\|_{L^\infty} \le C\big(\|\zeta\|_{L^\infty} + \|\nabla\zeta\|_{L^\infty}\big) \le C_p \|\zeta\|_{W^{2,p'}}. \tag{A.19}$$

Combining (A.18)–(A.19), one deduces (A.17) for functions $w \in C^\infty(\overline{\Omega})$. A standard argument implies that $u_n \in L^p(\Omega)$, $\forall n \ge 1$, and (A.17) holds for every $w \in L^{p'}(\Omega)$. By duality, (A.16) follows.

Step 2. Proof of the proposition completed.

By Step 1, (u_n) is bounded in $L^p(\Omega)$ for every $1 < p < \frac{N}{N-1}$. In particular, (u_n) is equi-integrable in $L^1(\Omega)$. On the other hand, by (A.15), (Δu_n) is a bounded sequence in $\mathcal{M}_{\mathrm{loc}}(\Omega)$. We deduce from Proposition A.4 that (u_n) is relatively compact in $L^1(\omega)$ for every $\omega \subset\subset \Omega$. Passing to a subsequence, we have $u_{n_k} \to u$ a.e. in Ω. It then follows from from Egorov's theorem that $u_{n_k} \to u$ in $L^1(\Omega)$. Since (u_{n_k}) is bounded in $L^p(\Omega)$ for every $1 < p < \frac{N}{N-1}$, the conclusion follows by interpolation. □

Appendix B. Existence of solutions of the scalar Chern-Simons equation

In this appendix, we present a short proof of existence of solutions of the equation

$$-\Delta u + \lambda \mathrm{e}^u(\mathrm{e}^u - 1) = \mu \quad \text{in } \mathbb{R}^2, \tag{B.1}$$

where $\lambda > 0$ and μ is a given finite measure in $\mathbb{R}^2$. Although (B.1) is a special case of system (1.1), we cannot directly apply Theorem 7.1 here. Indeed, since the proof of Theorem 7.1 is based on a fixed point argument, it is not clear that the solution of (1.1) provided by that theorem satisfies $u = v$ when $\mu = \nu$. In any case, as we shall see below existence of solutions of (B.1) can be established in a much simpler way.

The main result in this section is the next

Theorem B.1. *Let $\lambda > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^2)$. Then,* (B.1) *has a solution $u \in L^1(\mathbb{R}^2)$ in the sense of distributions if and only if*

$$\mu(\{x\}) \le 2\pi \quad \forall x \in \mathbb{R}^2. \tag{B.2}$$

In addition, u satisfies

$$\|u\|_{L^1} \le \frac{C}{\lambda}\big(1 + \|\mu\|_{\mathcal{M}}^2\big)\, \|\mu\|_{\mathcal{M}} \quad \textit{and} \quad \|\mathrm{e}^{2u} - 1\|_{L^1} \le \frac{C}{\lambda}\|\mu\|_{\mathcal{M}}. \tag{B.3}$$

We first consider the counterpart of Theorem B.1 on smooth bounded domains $\Omega \subset \mathbb{R}^2$:

Proposition B.1. *Given $\lambda > 0$ and $\mu \in \mathcal{M}(\Omega)$, then*

$$\begin{cases} -\Delta u + \lambda \mathrm{e}^u(\mathrm{e}^u - 1) = \mu & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{B.4}$$

has a solution $u \in L^1(\Omega)$ if and only if

$$\mu(\{x\}) \le 2\pi \quad \forall x \in \Omega. \tag{B.5}$$

Moreover,

- (B_1) *Every solution of* (B.4) *satisfies* (B.3)*;*
- (B_2) *There exists $U \in L^1(\mathbb{R}^2)$ with $\mathrm{e}^U(\mathrm{e}^U - 1) \in L^1(\mathbb{R}^2)$ such that $u \le U$ a.e. for every solution u of* (B.4)*.*

Proof. Extend the measure μ to $\mathbb{R}^2$ as identically zero outside Ω. Since the function $t \mapsto \mathrm{e}^t(\mathrm{e}^t - 1)$ is increasing for $t \ge 0$, we can apply [61, Theorem 2 and Proposition A.1] to deduce that under assumption (B.2) equation (B.1) with data μ^+ has a solution $U \in L^1(\mathbb{R}^2)$ such that $U \ge 0$ a.e. and $\mathrm{e}^U(\mathrm{e}^U - 1) \in L^1(\mathbb{R}^2)$. Let $v \in L^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta v = -\mu^- & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, v and U are sub and supersolutions of (B.4) such that $v \le 0 \le U$ a.e. Thus, by Theorem A.1 above, (B.4) has a solution $u \in L^1(\Omega)$.
We next note that by Proposition A.3 every solution of (B.4) satisfies

$$\lambda \int_\Omega \mathrm{e}^u|\mathrm{e}^u - 1|\,\mathrm{d}x \le \|\mu\|_{\mathcal{M}} \quad \text{and} \quad \int_\Omega |\Delta u| \le 2\|\mu\|_{\mathcal{M}}.$$

The second estimate in (B.3) then easily follows. In order to obtain the first one it suffices to apply Theorem 12.1 (with $u = v$) and Proposition 12.2. We conclude that (B_1) holds. By Proposition A.1, the supersolution U in the beginning of the proof satisfies (B_2).
It remains to show that if (B.4) has a solution, then μ satisfies (B.5). For this purpose, notice that $\mathrm{e}^{2u} \in L^1(\Omega)$ and then apply Theorem 15.1. This concludes the proof of the proposition. □

Proof of Theorem B.1. Let $(\Omega_n) \subset \mathbb{R}^2$ be an increasing sequence of smooth bounded domains such that $\mathbb{R}^2 = \bigcup_n \Omega_n$. For each $n \ge 1$, let u_n be a solution of (B.4) in Ω_n. Note that, by (B_1) and Proposition A.3,

$$\|u_n\|_{L^1(\Omega_n)} + \|\Delta u_n\|_{\mathcal{M}(\Omega_n)} \le C \quad \forall n \ge 1.$$

Applying Proposition A.4, one can extract a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2).$$

By Fatou's lemma, u satisfies (B.3). Finally, since $u_{n_k} \leq U$ a.e., $\forall k \geq 1$, where $\mathrm{e}^U(\mathrm{e}^U - 1) \in L^1(\mathbb{R}^2)$, it follows from dominated convergence that u is a solution of (B.1).

Conversely, proceeding as in the proof of Proposition B.1, one shows that if (B.1) has a solution, then (B.2) holds. The proof of the theorem is complete. □

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