

Hausdorff dimension of rupture sets and removable singularities

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Abstract

Given $\alpha > 0$ and a bounded domain $\Omega \subset \mathbb{R}^N$, we show that for every finite energy solution $u \geq 0$ of the equation

$$-\Delta u + \frac{1}{u^\alpha} = f(x) \quad \text{in } \Omega, \quad (1)$$

the set $[u = 0]$ has Hausdorff dimension at most $N - 2 + \frac{2}{\alpha+1}$. The proof is based on a removable singularity property of the Laplacian Δ . *To cite this article: J. Dávila, A. C. Ponce, C. R. Acad. Sci. Paris, Ser. I XXX (XXXX).*

Résumé

Étant donnés $\alpha > 0$ et un domaine borné $\Omega \subset \mathbb{R}^N$, nous prouvons que pour toute solution d'énergie finie $u \geq 0$ de l'équation (1), l'ensemble $[u = 0]$ a une dimension de Hausdorff inférieure ou égale à $N - 2 + \frac{2}{\alpha+1}$. La démonstration de ce résultat repose sur une propriété de singularité éliminable du laplacien Δ . *Pour citer cet article : J. Dávila, A. C. Ponce, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Soit $\Omega \subset \mathbb{R}^N$, $N \geq 2$, un domaine borné. Étant donné $\alpha > 0$, on considère l'équation des films minces (1), où $f \in L^1(\Omega)$. On s'intéresse à déterminer une borne supérieure de la dimension de Hausdorff de l'ensemble de rupture $[u = 0]$. Dans cette direction, Dupaigne-Ponce-Porretta [6] ont montré le

Théorème 0.1 *Si $u \in L^1(\Omega)$, $u \geq 0$ p.p., vérifie (1) au sens des distributions, alors*

$$\mathcal{H}^{N-2+\frac{2}{\alpha+1}}([u = 0]) = 0. \quad (2)$$

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Nous prouvons le résultat suivant :

Théorème 0.2 *Soit $u \in H^1(\Omega) \cap C^0$ avec $u \geq 0$ p.p. et tel que l'ensemble $[u = 0]$ a mesure de Lebesgue nulle. On suppose que u satisfait (5) au sens des distributions, avec $\alpha > 0$ et $f \in L^1(\Omega)$. Alors, $u^{-\alpha} \in L^1_{\text{loc}}(\Omega)$, $-\Delta u + u^{-\alpha} \leq f$ dans $\mathcal{D}'(\Omega)$ et l'ensemble $[u = 0]$ vérifie (2).*

Le théorème ci-dessus est une amélioration des résultats récemment obtenus par Jiang-Lin [9] et Guo-Wei [8]. En vu des exemples présentés dans [6], l'Éq. (2) est optimale.

Le Théorème 0.2 est une conséquence de [6, Theorem 12] (voir Theorem 2.1 ci-dessous) et d'une propriété de singularité éliminable satisfaite par le laplacien :

Théorème 0.3 *Soient $u \in H^1(\Omega)$ et $\Sigma \subset \Omega$ un ensemble relativement fermé. Supposons que $u \geq 0$ p.p. dans Ω et $\Delta u \geq \nu$ dans $\mathcal{D}'(\Omega \setminus \Sigma)$ pour une mesure finie ν dans Ω . Si $u = 0$ q.p. (= quasi-partout) sur Σ , alors $\Delta u \geq \nu|_{\Omega \setminus \Sigma}$ dans $\mathcal{D}'(\Omega)$.*

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain. A simplified model for the thickness $u \geq 0$ of a thin film in Ω is given by the equation (see [11])

$$-\Delta u + \frac{1}{u^\alpha} = f(x) \quad \text{in } \Omega, \quad (3)$$

where $\alpha > 0$. In this paper, we are motivated by the following question: what is the Hausdorff dimension of the rupture set $[u = 0]$?

An answer has been recently provided by Dupaigne-Ponce-Porretta [6]; see Theorem 2.1 below. As a corollary of their result, one immediately deduces the following

Theorem 1.1 *Given $\alpha > 0$, let $u \in L^1(\Omega)$, $u \geq 0$ a.e., be such that $u^{-\alpha} \in L^1(\Omega)$. Assume that u satisfies (3) in the sense of distributions, where $f \in L^1(\Omega)$. Then,*

$$\mathcal{H}^{N-2+\frac{2}{\alpha+1}}([u = 0]) = 0. \quad (4)$$

We denote by \mathcal{H}^β the Hausdorff measure of dimension $\beta \geq 0$. Recall (see [1]) that every function $u \in L^1(\Omega)$ such that Δu is a finite measure is well-defined outside some set of zero Newtonian (H^1) capacity, denoted “cap”. Assertion (4) then makes sense since for any Borel set $E \subset \Omega$ with zero capacity we have $\mathcal{H}^{N-2+\theta}(E) = 0$, $\forall \theta > 0$.

Under the assumptions of Theorem 1.1, Jiang-Lin [9] proved that the dimension of the rupture set $[u = 0]$ is at most $N - 2 + \frac{4}{\alpha+2}$, which is strictly larger than $N - 2 + \frac{2}{\alpha+1}$. The dimension provided by Theorem 1.1 cannot be improved. Indeed, in [6, Lemma 10] the authors show that for any $0 < \theta < \frac{2}{\alpha+1}$ there exists $u_\theta \in H^1(\Omega) \cap C^0$ such that $u_\theta^{-\alpha} \in L^1(\Omega)$, u_θ solves (3) for some $f_\theta \in L^1(\Omega)$, and $0 < \mathcal{H}^{N-2+\theta}([u_\theta = 0]) < \infty$.

The assumption “ $u^{-\alpha} \in L^1(\Omega)$ ” in Theorem 1.1 is needed in order to give a meaning to (3) in the sense of distributions. Jiang-Lin [9] and Guo-Wei [8] also considered a different notion of solution of (3). Following [9], we then say that u is a *finite energy solution* of (3) if $u \in H^1(\Omega) \cap C^0$, $u \geq 0$ in Ω , $u^{1-\alpha} \in L^1(\Omega)$ and

$$-\Delta u + \frac{1}{u^\alpha} = f(x) \quad \text{in } [u > 0], \quad (5)$$

in the sense of distributions.

Remark 1 Finite energy solutions can be seen as critical points of the energy functional associated to (3). Actually, it follows from Theorem 1.2 below that finite energy solutions satisfy (3) in the sense of

distributions, with right-hand side $f(x) + \mu$ for some nonpositive measure μ concentrated on the set $[u = 0]$.

In [8,9], it is proved that if u is a finite energy solution of (3) and $\alpha > 1$, then $\mathcal{H}^{\mu_1}([u = 0]) = 0$, where $\mu_1 = N - 2 + \frac{4}{\alpha+1}$. This dimension is also strictly larger than the one provided by Theorem 1.1. This raises the question of whether (4) still holds for finite energy solutions. We show that this is indeed the case. In fact, one of our main results is the

Theorem 1.2 *Let $u \in H^1(\Omega) \cap C^0$ be such that $u \geq 0$ a.e. and $[u = 0]$ has zero Lebesgue measure. Assume that u satisfies (5), where $\alpha > 0$ and $f \in L^1(\Omega)$. Then, (4) holds. Moreover, $u^{-\alpha} \in L^1_{\text{loc}}(\Omega)$ and*

$$-\Delta u + \frac{1}{u^\alpha} \leq f \quad \text{in } \mathcal{D}'(\Omega). \quad (6)$$

One major difference with respect to the results contained in [8,9] is that we do not assume that $u^{1-\alpha} \in L^1_{\text{loc}}(\Omega)$ but only that $[u = 0]$ has zero Lebesgue measure (see Remark 2 below); we conclude a posteriori that $u^{-\alpha} \in L^1_{\text{loc}}(\Omega)$.

Theorem 1.2 will be derived from the Hausdorff dimension estimates provided in [6] combined with the following “removable singularity” result:

Theorem 1.3 *Let $u \in H^1(\Omega)$ and $\Sigma \subset \Omega$ be a relatively closed set. Assume that $u \geq 0$ a.e. in Ω and*

$$\Delta u \geq \nu \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma) \quad (7)$$

for some $\nu \in \mathcal{M}_{\text{loc}}(\Omega)$. If $u = 0$ q.e. in Σ , then

$$\Delta u \geq \nu|_{\Omega \setminus \Sigma} \quad \text{in } \mathcal{D}'(\Omega). \quad (8)$$

In other words, $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega)$ and $\Delta u \geq 0$ in Σ .

Throughout the paper, for every open set $A \subset \mathbb{R}^N$ we denote by $\mathcal{M}_{\text{loc}}(A)$ the space of locally finite measures in A . More precisely, $\mu \in \mathcal{M}_{\text{loc}}(A)$ if and only if for every open set $\omega \subset\subset A$ there exists $C_\omega > 0$ such that $|\mu|(\omega) \leq C_\omega$; $\mu \in \mathcal{M}(A)$ if the constant C_ω can be chosen independently of ω . We say that $u = 0$ q.e. (=quasi-everywhere) in Σ if there exists a Borel set $E \subset \Sigma$ of zero capacity such that $u(x) = 0$, $\forall x \in \Sigma \setminus E$. This property makes sense for every function $u \in H^1(\Omega)$; see Section 2 below.

Note that from (7) one can only infer that $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega \setminus \Sigma)$; under the assumptions of Theorem 1.3 we are able to prove that $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega)$. Similar properties had been investigated by the authors (see [4,5]).

Combining results in [6] and [10], one obtains the following theorem related to problem (3):

Theorem 1.4 *Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $|u|^{-\alpha} \in L^1(\Omega)$ for some $\alpha \geq 1$, then either $u \geq 0$ a.e. or $u \leq 0$ a.e.*

Simple examples show that the conclusion of Theorem 1.4 is no longer true if one only assumes $|u|^{-\alpha} \in L^1(\Omega)$ for some $0 < \alpha < 1$.

2. Proofs of the main results

Let us first recall the following result established in [6, Theorem 12]:

Theorem 2.1 *Let $u \in L^1(\Omega)$, $u \geq 0$ a.e., be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u^{-\alpha} \in L^1(\Omega)$ for some $\alpha > 0$, then*

$$\mathcal{H}^{N-2+\frac{2}{\alpha+1}}([u = 0]) = 0. \quad (9)$$

Theorem 1.1 trivially follows from Theorem 2.1 as a special case.

It is well-known (see e.g. [7]) that for every $u \in H^1(\Omega)$, its precise representative \tilde{u} is quasicontinuous. More precisely, for every $\varepsilon > 0$ there exists an open set $\omega \subset \subset \Omega$ such that $\text{cap}(\omega) < \varepsilon$ and \tilde{u} is continuous on $\Omega \setminus \omega$. We shall systematically identify u and \tilde{u} and say that u is quasicontinuous, meaning \tilde{u} . Since \tilde{u} is well-defined outside some set of zero capacity, the value of $u(x)$ (i.e. $\tilde{u}(x)$) makes sense q.e.

Proof of Theorem 1.3. Replacing Ω by an open set $\Omega' \subset \subset \Omega$ if necessary we may assume that $\nu \in \mathcal{M}(\Omega)$. Moreover, we can always suppose that u is defined in \mathbb{R}^N and $u \in H^1(\mathbb{R}^N)$. Fix $\delta > 0$ and an open set $D \subset \subset \Omega$. Given $\rho \in C_0^\infty(B_1)$, consider $\rho_n(x) = n^N \rho(nx)$, $\forall x \in \mathbb{R}^N$, and $u_n = \rho_n * u$. In particular, since $u_n(x) \rightarrow u(x)$ q.e. in Ω and $u = 0$ q.e. on Σ , we have $\text{cap}(K_n) \rightarrow 0$ as $n \rightarrow \infty$, where $K_n = [u_n \geq \frac{\delta}{2}] \cap \overline{D} \cap \Sigma$. Let $\zeta_n \in C_0^\infty(\Omega)$ be such that

$$0 \leq \zeta_n \leq 1 \text{ in } \Omega, \quad \zeta_n = 1 \text{ on a neighborhood of } K_n, \quad \int_{\Omega} |\nabla \zeta_n|^2 \leq 2 \text{cap}(K_n).$$

In particular, $\zeta_n \rightarrow 0$ in $H_0^1(\Omega)$. Clearly, $D \cap \Sigma \subset [u_n < \delta] \cup K_n$; hence,

$$D \cap \Sigma \subset \text{int}([u_n < \delta] \cup [\zeta_n = 1]).$$

We thus have

$$\text{supp}\{S_\delta(u_n)(1 - \zeta_n)\} \cap D \cap \Sigma = \emptyset, \quad (10)$$

where $S_\delta : \mathbb{R} \rightarrow \mathbb{R}$ is the function given by

$$S_\delta(t) = \begin{cases} 0 & \text{if } t \leq \delta, \\ \frac{t - \delta}{\delta} & \text{if } \delta < t < 2\delta, \\ 1 & \text{if } t \geq 2\delta. \end{cases}$$

Note that, by (7), we have $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega \setminus \Sigma)$. Moreover, $u \in H^1(\Omega)$ implies that Δu does not charge sets of zero capacity. In other words, $(\Delta u)_d = \Delta u$ in Ω (the subscript “d” denotes the diffuse part of the measure with respect to capacity; see [2] for details). It then follows from (7) that

$$\Delta u = (\Delta u)_d \geq \nu_d \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma). \quad (11)$$

Let $\varphi \in C_0^\infty(D)$ be such that $\varphi \geq 0$ in Ω . Write

$$\begin{aligned} \int_{\Omega} u \Delta \varphi &= \int_{\Omega} u \Delta [(1 - S_\delta(u))\varphi] + \int_{\Omega} u \Delta [(S_\delta(u) - S_\delta(u_n))\varphi] + \\ &\quad + \int_{\Omega} u \Delta [S_\delta(u_n)\zeta_n \varphi] + \int_{\Omega} u \Delta [S_\delta(u_n)(1 - \zeta_n)\varphi] =: I + II + III + IV. \end{aligned} \quad (12)$$

We now estimate I – IV separately. Note that

$$\begin{aligned} I &= \int_{\Omega} \nabla u \cdot \nabla S_\delta(u) \varphi - \int_{\Omega} \nabla u \cdot \nabla \varphi (1 - S_\delta(u)) \\ &= \frac{1}{\delta} \int_{[\delta < u < 2\delta]} |\nabla u|^2 \varphi - \int_{\Omega} \nabla u \cdot \nabla \varphi (1 - S_\delta(u)) \geq - \int_{\Omega} \nabla u \cdot \nabla \varphi (1 - S_\delta(u)). \end{aligned} \quad (13)$$

Since $S_\delta(u_n) \rightarrow S_\delta(u)$ in $H^1(\Omega)$, we have

$$II = - \int_{\Omega} \nabla u \cdot \nabla [(S_\delta(u_n) - S_\delta(u))\varphi] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (14)$$

We now observe that $\zeta_n \rightarrow 0$ in $H_0^1(\Omega)$ and $(S_\delta(u_n))_{n \geq 1}$ is bounded in $H^1(\Omega)$; thus,

$$III = - \int_{\Omega} \nabla u \cdot \nabla [S_\delta(u_n)\zeta_n \varphi] \rightarrow 0. \quad (15)$$

By (10), we have $S_\delta(u_n)(1 - \zeta_n)\varphi \in C_0^\infty(\Omega \setminus \Sigma)$. Using (11),

$$IV \geq \int_{\Omega} S_\delta(u_n)(1 - \zeta_n)\varphi d\nu_d.$$

Since $S_\delta(u_n)(1 - \zeta_n)\varphi \rightarrow S_\delta(u)\varphi$ in $H^1(\Omega)$ and ν_d is a diffuse measure, we then get

$$IV \geq \int_{\Omega} S_\delta(u)\varphi d\nu_d + o(1). \quad (16)$$

Combining (12)–(16), we conclude that

$$\int_{\Omega} u\Delta\varphi \geq - \int_{\Omega} \nabla u \cdot \nabla \varphi (1 - S_\delta(u)) + \int_{\Omega} S_\delta(u)\varphi d\nu_d + o(1) \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\int_{\Omega} u\Delta\varphi \geq - \int_{\Omega} \nabla u \cdot \nabla \varphi (1 - S_\delta(u)) + \int_{\Omega} S_\delta(u)\varphi d\nu_d \quad \forall \delta > 0. \quad (17)$$

Note that $0 \leq S_\delta(u) \leq 1$ in Ω and $S_\delta(u) \rightarrow \chi_{[u>0]}$ q.e.; moreover, $\nabla u = 0$ a.e. on the set $[u = 0]$. As we let $\delta \rightarrow 0$ in (17), it follows from dominated convergence that

$$\int_{\Omega} u\Delta\varphi \geq - \int_{[u=0]} \nabla u \cdot \nabla \varphi + \int_{[u>0]} \varphi d\nu_d = \int_{[u>0]} \varphi d\nu_d.$$

This inequality holds for every $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ in Ω , and every open set $D \subset\subset \Omega$. Hence, $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega)$ and $\Delta u \geq \chi_{[u>0]}\nu_d$ in Ω . In particular, since $u = 0$ q.e. on Σ and ν_d is a diffuse measure,

$$(\Delta u)|_{\Sigma} \geq (\chi_{[u>0]}\nu_d)|_{\Sigma} = 0. \quad (18)$$

On the other hand, by (7) we also have

$$(\Delta u)|_{\Omega \setminus \Sigma} \geq \nu|_{\Omega \setminus \Sigma}. \quad (19)$$

Combining (18)–(19), we deduce that

$$\Delta u = (\Delta u)|_{\Sigma} + (\Delta u)|_{\Omega \setminus \Sigma} \geq \nu|_{\Omega \setminus \Sigma} \quad \text{in } \Omega.$$

The proof of Theorem 1.3 is complete. \square

Proof of Theorem 1.2. Let $\Sigma = [u = 0]$. Since $u^{-\alpha} \geq 0$ a.e. in Ω and u satisfies (5), we have $\Delta u \geq -f$ in $\mathcal{D}'(\Omega \setminus \Sigma)$. By Theorem 1.3, we conclude that $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega)$ and

$$\Delta u \geq 0 \quad \text{in } \Sigma. \quad (20)$$

Let us denote by $(\Delta u)_a$ and $(\Delta u)_s$ the absolutely continuous and the singular parts of Δu with respect to the Lebesgue measure, respectively. Since Σ has zero Lebesgue measure, by (5)

$$\frac{1}{u^\alpha} = (\Delta u)_a + f \quad \text{a.e. in } \Omega. \quad (21)$$

On the other hand, by (20) we also have

$$(\Delta u)_s \geq 0 \quad \text{in } \Omega. \quad (22)$$

Since $(\Delta u)_a \in L_{\text{loc}}^1(\Omega)$ and $f \in L^1(\Omega)$, we deduce from (21) that $u^{-\alpha} \in L_{\text{loc}}^1(\Omega)$. Combining (21)–(22), we then get

$$\Delta u = (\Delta u)_a + (\Delta u)_s \geq \frac{1}{u^\alpha} - f \quad \text{in } \Omega,$$

from which (6) follows. It remains to show that (4) holds. For every open set $D \subset\subset \Omega$, we have $\Delta u \in \mathcal{M}(D)$ and $u^{-\alpha} \in L^1(D)$. Thus, by Theorem 2.1, $\mathcal{H}^{N-2+\frac{2}{\alpha+1}}([u = 0] \cap D) = 0$. Since D is arbitrary, (4) follows. \square

Remark 2 Without the assumption “ $[u = 0]$ has zero Lebesgue measure”, the conclusion of Theorem 1.2 becomes $u^{-\alpha}\chi_{[u>0]} \in L^1_{\text{loc}}(\Omega)$ and

$$-\Delta u + \frac{1}{u^\alpha}\chi_{[u>0]} \leq f \quad \text{in } \mathcal{D}'(\Omega). \quad (23)$$

One may wonder whether (4) still holds for any $\alpha > 0$. The answer is no if $0 < \alpha < 1$. In fact, there are examples of functions $u \in H^1(\Omega) \cap C^0$ satisfying (5) for which $[u = 0]$ has positive measure; see Dávila-Montenegro [3]. However, in the case $\alpha \geq 1$ we do not know if (23) implies that $[u = 0]$ has zero Lebesgue measure, in which case (4) would be true by Theorem 1.2.

In order to establish Theorem 1.4, we need the following version of the Intermediate Value Theorem recently established by Van Schaftingen-Willem [10, Proposition 2.11]:

Theorem 2.2 *Let $u \in W^{1,1}(\Omega)$. If $\mathcal{H}^{N-1}([u = 0]) = 0$, then either $u \geq 0$ a.e. or $u \leq 0$ a.e.*

Proof of Theorem 1.4. Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. By [2], we have $\Delta|u| \in \mathcal{M}_{\text{loc}}(\Omega)$. On the other hand, note that if $|u|^{-\alpha} \in L^1(\Omega)$ for some $\alpha \geq 1$, then $|u|^{-1} \in L^1(\Omega)$. Thus, by Theorem 2.1,

$$\mathcal{H}^{N-1}([u = 0]) = \mathcal{H}^{N-1}([|u| = 0]) = 0.$$

Moreover, by standard elliptic regularity theory, $u \in W^{1,1}_{\text{loc}}(\Omega)$. Applying Theorem 2.2, we conclude that $u \geq 0$ a.e. or $u \leq 0$ a.e. \square

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