KATO'S INEQUALITY UP TO THE BOUNDARY

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Abstract. We show that if Δu is a finite measure in Ω then, under suitable assumptions on u near $\partial\Omega$, Δu^+ is also a finite measure in Ω. We also study properties of the normal derivatives $\frac{\partial u}{\partial n}$ and $\frac{\partial u^+}{\partial n}$ on $\partial\Omega$.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Given $u \in L^1(\Omega)$ with $\Delta u \in L^1(\Omega)$, Kato's inequality (see [9]; see also [4]) asserts that

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \quad \text{in } \mathcal{D}'(\Omega). \tag{1.1}$$

In particular, (1.1) implies that Δu^+ is a locally finite measure in Ω. Our goal in this paper is to address the question whether Δu^+ is a *finite* measure up to the boundary of Ω, i.e., whether

$$\int_\Omega |\Delta u^+| < \infty.$$

In general, the answer is negative: one can even construct harmonic functions $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ such that Δu^+ is not a finite measure in Ω; see Proposition A.1 below. With further assumptions on u (for instance if $u \in W^{2,1}(\Omega)$ or if u vanishes on the boundary) we will see that the answer is positive.

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The following class of functions will play a central role. We say that $u \in \mathbb{X}$ if $u \in W^{1,1}(\Omega)$ and if there exists a constant $C > 0$ such that

$$\left| \int_\Omega \nabla u \cdot \nabla \psi \right| \le C \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}), \tag{1.2}$$

in which case we set

$$[u]_{\mathbb{X}} = \sup_{\substack{\psi \in C^1(\overline{\Omega}) \\ \|\psi\|_{L^\infty} \le 1}} \int_\Omega \nabla u \cdot \nabla \psi.$$

Note that if $u \in \mathbb{X}$, then there exists a unique $T \in \big[C(\overline{\Omega})\big]^* = \mathcal{M}(\overline{\Omega})$ such that

$$\langle T, \psi \rangle = \int_\Omega \nabla u \cdot \nabla \psi \quad \forall \psi \in C^1(\overline{\Omega}).$$

On the other hand, by the Riesz Representation Theorem any $T \in \mathcal{M}(\overline{\Omega})$ admits a unique decomposition

$$\langle T, \psi \rangle = \int_{\partial\Omega} \psi \, d\nu + \int_\Omega \psi \, d\mu \quad \forall \psi \in C(\overline{\Omega}),$$

where $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$. As usual, $\mathcal{M}(\Omega)$ and $\mathcal{M}(\partial\Omega)$ denote the spaces of finite measures in Ω and $\partial\Omega$, respectively, equipped with the norm $\| \cdot \|_{\mathcal{M}}$; measures in $\mathcal{M}(\Omega)$ are identified with measures in $\overline{\Omega}$ which do not charge $\partial\Omega$. When $u \in \mathbb{X}$, we will denote

$$\mu = -\Delta u \quad \text{and} \quad \nu = \frac{\partial u}{\partial n}.$$

Throughout the paper, whenever $u \in \mathbb{X}$ we use the notation Δu and $\frac{\partial u}{\partial n}$ in the above sense. If $u \in \mathbb{X}$, then

$$\int_\Omega \nabla u \cdot \nabla \psi = \int_{\partial\Omega} \psi \, \frac{\partial u}{\partial n} - \int_\Omega \psi \, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}),$$

and consequently,

$$\int_{\partial\Omega} u \, \frac{\partial \psi}{\partial n} - \int_\Omega u \, \Delta \psi = \int_{\partial\Omega} \psi \, \frac{\partial u}{\partial n} - \int_\Omega \psi \, \Delta u \quad \forall \psi \in C^2(\overline{\Omega}).$$

Also, note that if $u \in \mathbb{X}$, then

$$[u]_{\mathbb{X}} = \int_\Omega |\Delta u| + \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|.$$

In particular, $[\,\cdot\,]_{\mathbb{X}}$ defines a seminorm in $\mathbb{X}$ and $[u]_{\mathbb{X}} = 0$ if, and only if, u is constant in Ω. In order to verify this last assertion, one may use the fact that for every $h \in C^\infty(\overline{\Omega})$ with $\int_\Omega h = 0$, there exists $\psi \in C^\infty(\overline{\Omega})$ such that $-\Delta\psi = h$ in Ω with $\frac{\partial \psi}{\partial n} = 0$ on $\partial\Omega$.

Clearly, any function $u \in W^{2,1}(\Omega)$ belongs to $\mathbb{X}$ and our notation is consistent with the usual meaning of Δu and $\frac{\partial u}{\partial n}$. Recall that, for any function $u \in L^1(\Omega)$, Δu is well-defined as a distribution. When $u \in \mathbb{X}$, the distribution Δu belongs to $\mathcal{M}(\Omega)$, but the converse is not true; see, e.g., Proposition A.1 below.

We now present our main results.

Theorem 1.1. *If $u \in \mathbb{X}$, then $u^+ \in \mathbb{X}$ and*

$$[u^+]_{\mathbb{X}} \le [u]_{\mathbb{X}}. \tag{1.3}$$

In other words,

$$\int_\Omega |\Delta u^+| + \int_{\partial\Omega} \left|\frac{\partial u^+}{\partial n}\right| \le \int_\Omega |\Delta u| + \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|. \tag{1.4}$$

Our next result gives additional properties when u vanishes on the boundary:

Theorem 1.2. *If $u \in W_0^{1,1}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions), then $u \in \mathbb{X}$ (hence $u^+ \in \mathbb{X}$). Moreover,*

$$\int_\Omega |\Delta u^+| \le \int_\Omega |\Delta u|. \tag{1.5}$$

In addition, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ with

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right| \le \int_\Omega |\Delta u|. \tag{1.6}$$

Note that assertions (1.5)–(1.6) fail if u does not vanish on $\partial\Omega$; simply take $\Omega = B_1$, the unit ball in $\mathbb{R}^N$, and $u(x) = x_1$.

We now state our extension of Kato's inequality up to the boundary:

Theorem 1.3. *Let $u \in \mathbb{X}$ be such that $\Delta u \in L^1(\Omega)$ and $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Then,*

$$\int_{\partial\Omega} \nabla u^+ \cdot \nabla\psi \le \int_{\partial\Omega} H\psi - \int_\Omega G\psi \quad \forall \psi \in C^1(\overline{\Omega}),\ \psi \ge 0 \text{ in } \Omega, \tag{1.7}$$

where $G \in L^1(\Omega)$ and $H \in L^1(\partial\Omega)$ are given by

$$G = \begin{cases} \Delta u & \text{on } [u > 0], \\ 0 & \text{on } [u \le 0], \end{cases} \quad \text{and} \quad H = \begin{cases} \frac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\frac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{1.8}$$

Thus,

$$\begin{cases} \Delta u^+ \ge G & \text{in } \Omega, \\ \dfrac{\partial u^+}{\partial n} \le H & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

We conclude this introduction with the following problems:

Open Problem 1. *Let $u \in \mathbb{X}$. Is it true that*

$$\left|\frac{\partial u^+}{\partial n}\right| \le \left|\frac{\partial u}{\partial n}\right| \quad \text{on } \partial\Omega\ ? \tag{1.10}$$

This problem is open even under the additional assumption that $u \in W_0^{1,1}(\Omega)$.

Open Problem 2. *Assume that $u \in \mathbb{X}$ and $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Is it true that $\frac{\partial u^+}{\partial n} \in L^1(\partial\Omega)$? More precisely, does one have*

$$\frac{\partial u^+}{\partial n} = H, \tag{1.11}$$

where H is the function given by (1.8)?

The answer to both Open Problems 1 and 2 is positive if $u \in W^{2,1}(\Omega)$; see Theorem 7.1 below.

Addendum. *Recently, A. Ancona informed us that he gave a positive answer to Open Problems 1 and 2 in full generality. His argument strongly relies on tools from Potential Theory; see* [2].

2. Properties of functions in $\mathbb{X}$

In this section, we investigate properties satisfied by elements in $\mathbb{X}$. We first show that condition (1.2) required for a function to belong to $\mathbb{X}$ can be replaced by

$$\left|\int_\Omega u\,\Delta\zeta\right| \le C\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}), \tag{2.1}$$

where

$$C^2_{\mathrm{N}}(\overline{\Omega}) = \left\{\zeta \in C^2(\overline{\Omega});\ \frac{\partial \zeta}{\partial n} = 0 \text{ on } \partial\Omega\right\}. \tag{2.2}$$

Proposition 2.1. *Let $u \in L^1(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if,*

$$\sup_{\substack{\zeta \in C^2_{\mathrm{N}}(\overline{\Omega})\\ \|\zeta\|_{L^\infty} \le 1}} \left|\int_\Omega u\,\Delta\zeta\right| < \infty. \tag{2.3}$$

Moreover,

(*i*) *the quantity in* (2.3) *equals* $[u]_{\mathbb{X}}$;
(*ii*) $u \in W^{1,p}(\Omega)$ *for every* $1 \le p < \frac{N}{N-1}$; *moreover,* $\|\nabla u\|_{L^p(\Omega)} \le C[u]_{\mathbb{X}}$.

In the proof of Proposition 2.1, we need the following variant of the classical De Giorgi-Stampacchia estimate (see [7,8]) for the Neumann problem:

Lemma 2.1. *Given $F \in C_0^\infty(\Omega;\mathbb{R}^N)$, let w be the unique solution of*

$$\begin{cases} -\Delta w = \operatorname{div} F & \text{in } \Omega,\\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

such that $\int_\Omega w = 0$. Then, for every $q > N$ we have

$$\|w\|_{L^\infty} \le C\|F\|_{L^q}. \tag{2.5}$$

We present a sketch of the proof of Lemma 2.1 in Appendix C.

Proof of Proposition 2.1. Note that if $u \in \mathbb{X}$, then

$$\left|\int_\Omega u\,\Delta\zeta\right| = \left|\int_\Omega \nabla u \cdot \nabla\zeta\right| \le [u]_{\mathbb{X}}\,\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}). \tag{2.6}$$

This gives the implication "$\Rightarrow$". We now assume that (2.3) holds. We split the proof of the converse into two steps:

Step 1. $u \in W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$ and

$$\|\nabla u\|_{L^p(\Omega)} \le CK,$$

where K denotes the quantity in (2.3).

Clearly, we may assume that $1 < p < \frac{N}{N-1}$. Given $F \in C_0^\infty(\Omega;\mathbb{R}^N)$, let w be the unique solution of (2.4) such that $\int_\Omega w = 0$. By (2.3) and (2.5), we have

$$\left|\int_\Omega u \operatorname{div} F\right| = \left|\int_\Omega u\,\Delta w\right| \le K\|w\|_{L^\infty} \le KC\|F\|_{L^{p'}} \quad \forall F \in C_0^\infty(\Omega;\mathbb{R}^N).$$

The conclusion follows by duality.

Step 2. $u \in \mathbb{X}$ and $[u]_{\mathbb{X}} = K$.

It suffices to show that

$$\left|\int_\Omega \nabla u \cdot \nabla \psi\right| \le K\|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{2.7}$$

Indeed, this implies $u \in \mathbb{X}$ and $[u]_{\mathbb{X}} \le K$. Since by (2.6), $K \le [u]_{\mathbb{X}}$, equality must hold. We now turn ourselves to the proof of (2.7). Given $\psi \in C^2(\overline{\Omega})$, we first show that there exists a sequence (ζ_k) such that

$$\zeta_k \in C^2_{\mathrm{N}}(\overline{\Omega}), \quad \|\nabla\zeta_k\|_{L^\infty} \le C, \quad \zeta_k \to \psi \quad \text{uniformly in } \Omega \tag{2.8}$$

and

$$\nabla\zeta_k \to \nabla\psi \quad \text{a.e. in } \Omega. \tag{2.9}$$

Indeed, let $\Phi \in C_0^\infty(\mathbb{R})$ and $\eta \in C^2(\overline{\Omega})$ with $\eta = 0$ on $\partial\Omega$ be such that

$$\Phi(t) = t \quad \forall t \in [-1,1] \quad \text{and} \quad \frac{\partial\eta}{\partial n} = \frac{\partial\psi}{\partial n} \quad \text{on } \partial\Omega.$$

Take

$$\zeta_k = \psi - \frac{1}{k}\Phi(k\eta) \quad \text{in } \overline{\Omega}.$$

Clearly, (2.8) holds. On the other hand,

$$\nabla\left[\frac{1}{k}\Phi(k\eta)\right] = \Phi'(k\eta)\nabla\eta \to \chi_{[\eta=0]}\nabla\eta \quad \text{in } \Omega.$$

Since $\nabla\eta = 0$ a.e. on the set $[\eta = 0]$, (2.9) follows. For every $k \ge 1$, we thus have

$$\left|\int_\Omega \nabla u \cdot \nabla\zeta_k\right| = \left|\int_\Omega u\,\Delta\zeta_k\right| \le K\|\zeta_k\|_{L^\infty}.$$

As $k \to \infty$, we obtain (2.7) with test functions $\psi \in C^2(\overline{\Omega})$. Using a density argument, one then gets (2.7). The proof is complete. □

Remark 2.1. Using Proposition 2.1, one deduces that given measures $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$, the Neumann problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \nu & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

has a solution $u \in \mathbb{X}$ if, and only if,

$$\mu(\Omega) + \nu(\partial\Omega) = 0. \tag{2.11}$$

The solution is unique up to an additive constant and belongs to $u \in W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$. In particular, if $\int_\Omega u = 0$, then

$$\|u\|_{W^{1,p}(\Omega)} \le C[u]_{\mathbb{X}}.$$

The following result complements Proposition 2.1:

Proposition 2.2. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} \zeta\,d\nu + \int_\Omega \zeta\,d\mu \quad \forall\zeta \in C^2_{\mathrm{N}}(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \overline{\Omega} \tag{2.12}$$

for some $\mu \in \mathcal{M}(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$. Then, $u \in \mathbb{X}$,

$$[u]_{\mathbb{X}} \le 2\Big(\|\mu^+\|_{\mathcal{M}(\Omega)} + \|\nu^+\|_{\mathcal{M}(\partial\Omega)}\Big) \tag{2.13}$$

and

$$\begin{cases} -\Delta u \le \mu & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} \le \nu & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Proof. By (2.12), we have

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} \zeta\,d\nu^+ + \int_\Omega \zeta\,d\mu^+ \quad \forall\zeta \in C^2_{\mathrm{N}}(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \overline{\Omega}. \tag{2.15}$$

For every $\zeta \in C^2_{\mathrm{N}}(\overline{\Omega})$, we apply (2.15) with test functions $\|\zeta\|_{L^\infty} \pm \zeta$ to get

$$\left|\int_\Omega u\,\Delta\zeta\right| \le 2\Big(\|\mu^+\|_{\mathcal{M}(\Omega)} + \|\nu^+\|_{\mathcal{M}(\partial\Omega)}\Big)\|\zeta\|_{L^\infty}. \tag{2.16}$$

By Proposition 2.1, it follows that $u \in \mathbb{X}$ and (2.13) holds. Proceeding as in Step 2 of the proof of Proposition 2.1 (more precisely, using (2.8)–(2.9)), one deduces from (2.12) that

$$\int_\Omega \nabla u \cdot \nabla\psi \le \int_{\partial\Omega} \psi\,d\nu + \int_\Omega \psi\,d\mu \quad \forall\psi \in C^2(\overline{\Omega}),\ \psi \ge 0 \text{ in } \overline{\Omega}.$$

Therefore,

$$\int_{\partial\Omega} \psi\,\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Delta u \le \int_{\partial\Omega} \psi\,d\nu + \int_\Omega \psi\,d\mu \quad \forall\psi \in C^2(\overline{\Omega}),\ \psi \ge 0 \text{ in } \overline{\Omega}.$$

This gives (2.14). □

3. Proof of Theorem 1.1

We begin by establishing the following lemma:

Lemma 3.1. *If $u \in C^2(\overline{\Omega})$, then*

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} \psi\,\frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\ge 0]}} \psi\,\Delta u \quad \forall\psi \in C^1(\overline{\Omega}),\ \psi \ge 0 \text{ in } \overline{\Omega}. \tag{3.1}$$

Proof. We first prove the

Claim. If $u \in C^2(\overline{\Omega})$ and $\Phi \in C^2(\mathbb{R})$ is convex, then

$$\int_\Omega \nabla\Phi(u) \cdot \nabla\psi \le \int_{\partial\Omega} \psi\,\Phi'(u)\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Phi'(u)\Delta u \quad \forall\psi \in C^1(\overline{\Omega}),\ \psi \ge 0 \text{ in } \overline{\Omega}. \tag{3.2}$$

Note that

$$\frac{\partial\Phi(u)}{\partial n} = \Phi'(u)\frac{\partial u}{\partial n} \quad \text{on } \partial\Omega$$

and, by the convexity of Φ,

$$\Delta\Phi(u) \ge \Phi'(u)\Delta u \quad \text{in } \Omega.$$

Thus, for every $\psi \in C^1(\overline{\Omega})$, $\psi \geq 0$ in $\overline{\Omega}$,

$$\int_\Omega \nabla\Phi(u)\cdot\nabla\psi = \int_{\partial\Omega} \psi\,\frac{\partial\Phi(u)}{\partial n} - \int_\Omega \psi\,\Delta\Phi(u) \leq \int_{\partial\Omega} \psi\,\Phi'(u)\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Phi'(u)\Delta u.$$

This establishes the claim.

We now apply (3.2) with $\Phi = \Phi_k$, where (Φ_k) is a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi_k'\|_{L^\infty} \leq 1$ and satisfying

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \geq 0,\\ 0 & \text{if } t < 0.\end{cases}$$

As $k \to \infty$, we obtain (3.1). □

We now prove a special case of Theorem 1.1 for functions in $C^2(\overline{\Omega})$:

Lemma 3.2. *Let $u \in C^2(\overline{\Omega})$. Then, $u^+ \in \mathbb{X}$ and*

$$[u^+]_{\mathbb{X}} \leq [u]_{\mathbb{X}}. \tag{3.3}$$

Proof. Note that $u^+ \in W^{1,1}(\Omega)$. In order to establish the lemma, it thus suffices to show that

$$\left|\int_\Omega \nabla u^+\cdot\nabla\psi\right| \leq [u]_{\mathbb{X}}\,\|\psi\|_{L^\infty} \quad \forall\psi \in C^1(\overline{\Omega}). \tag{3.4}$$

For this purpose, given $\tilde\psi \in C^1(\overline{\Omega})$ we apply (3.1) with $\psi = \|\tilde\psi\|_{L^\infty} + \tilde\psi$. We then get

$$\int_\Omega \nabla u^+\cdot\nabla\tilde\psi \leq \Bigg(\int_{\substack{\partial\Omega\\ [u\geq 0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\geq 0]}} \Delta u\Bigg)\|\tilde\psi\|_{L^\infty} + \int_{\substack{\partial\Omega\\ [u\geq 0]}} \tilde\psi\,\frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\geq 0]}} \tilde\psi\,\Delta u. \tag{3.5}$$

Since

$$\int_{\substack{\partial\Omega\\ [u\geq 0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\geq 0]}} \Delta u = -\int_{\substack{\partial\Omega\\ [u<0]}} \frac{\partial u}{\partial n} + \int_{\substack{\Omega\\ [u<0]}} \Delta u,$$

estimate (3.5) becomes

$$\begin{aligned}\int_\Omega \nabla u^+\cdot\nabla\tilde\psi &\leq -\Bigg(\int_{\substack{\partial\Omega\\ [u<0]}} \frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u<0]}} \Delta u\Bigg)\|\tilde\psi\|_{L^\infty} + \int_{\substack{\partial\Omega\\ [u\geq 0]}} \tilde\psi\,\frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\geq 0]}} \tilde\psi\,\Delta u\\ &\leq \left(\int_{\partial\Omega}\left|\frac{\partial u}{\partial n}\right| + \int_\Omega |\Delta u|\right)\|\tilde\psi\|_{L^\infty} = [u]_{\mathbb{X}}\,\|\tilde\psi\|_{L^\infty}.\end{aligned}$$

This relation holds for every $\tilde\psi \in C^1(\overline{\Omega})$. Replacing $\tilde\psi$ by $-\tilde\psi$, we obtain (3.4). This establishes the lemma. □

Proof of Theorem 1.1. Since $u \in \mathbb{X}$,

$$\int_\Omega \nabla u\cdot\nabla\psi = \int_{\partial\Omega} \psi\,\frac{\partial u}{\partial n} - \int_\Omega \psi\,\Delta u \quad \forall\psi \in C^1(\overline{\Omega}).$$

Taking $\psi = 1$ as a test function, we get

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega \Delta u. \tag{3.6}$$

Let $(\mu_k) \subset C^\infty(\overline{\Omega})$ and $(\nu_k) \subset C^\infty(\partial\Omega)$ be two sequences such that

$$\mu_k \overset{*}{\rightharpoonup} -\Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad \|\mu_k\|_{L^1(\Omega)} \to \|\Delta u\|_{\mathcal{M}(\Omega)},$$
$$\nu_k \overset{*}{\rightharpoonup} \frac{\partial u}{\partial n} \quad \text{weak}^* \text{ in } \mathcal{M}(\partial\Omega) \quad \text{and} \quad \|\nu_k\|_{L^1(\partial\Omega)} \to \left\|\frac{\partial u}{\partial n}\right\|_{\mathcal{M}(\partial\Omega)}.$$

In view of (3.6) we may also assume that

$$\int_{\partial\Omega} \nu_k = -\int_\Omega \mu_k \quad \forall k \geq 1.$$

For each $k \geq 1$, let $u_k \in C^2(\overline{\Omega})$ be the unique function such that

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega,\\ \dfrac{\partial u_k}{\partial n} = \nu_k & \text{on } \partial\Omega, \end{cases}$$

and

$$\int_\Omega u_k = \int_\Omega u.$$

Then, by Remark 2.1 applied to $u_k - \int_\Omega u$, the sequence (u_k) is bounded in $W^{1,p}(\Omega)$ for every $1 \leq p < \frac{N}{N-1}$. Since $u_k \to u$ a.e., one deduces that

$$\nabla u_k^+ \rightharpoonup \nabla u^+ \quad \text{weakly in } L^1(\Omega).$$

On the other hand, applying Lemma 3.2 to u_k, we get

$$\left|\int_\Omega \nabla u_k^+ \cdot \nabla\psi\right| \leq [u_k^+]_{\mathbb{X}}\, \|\psi\|_{L^\infty} \leq [u_k]_{\mathbb{X}}\, \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}).$$

As $k \to \infty$, we obtain

$$\left|\int_\Omega \nabla u^+ \cdot \nabla\psi\right| \leq [u]_{\mathbb{X}}\, \|\psi\|_{L^\infty} \quad \forall \psi \in C^1(\overline{\Omega}),$$

from which the conclusion follows. □

4. Properties of $\frac{\partial u}{\partial n}$

We start with a result which seems intuitively true, but still requires a proof:

Proposition 4.1. *Let $u \in W^{1,\infty}(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if, $\Delta u \in \mathcal{M}(\Omega)$ (in the sense of distributions). In this case, $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$ and*

$$\left\|\frac{\partial u}{\partial n}\right\|_{L^\infty(\partial\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}. \tag{4.1}$$

If $u \in C^1(\overline{\Omega}) \cap \mathbb{X}$, then $\frac{\partial u}{\partial n}$ coincides with the standard normal derivative on $\partial\Omega$.

Proof. We first assume that $u \in W^{1,\infty}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$. Given a sequence of mollifiers (ρ_k) such that $\operatorname{supp}\rho_k \subset B_{1/k}$, let

$$u_k(x) = \int_\Omega \rho_k(x-y)u(y)\,dy \quad \forall x \in \Omega.$$

Note that if $d(x,\partial\Omega) > 1/k$, then

$$\nabla u_k(x) = \int_\Omega \rho_k(x-y)\nabla u(y)\,dy \quad \text{and} \quad \Delta u_k(x) = \int_\Omega \rho_k(x-y)\Delta u(y)\,dy.$$

Denote

$$\Omega_\delta = \big\{x \in \Omega;\ d(x, \partial\Omega) > \delta\big\}; \tag{4.2}$$

for $\delta_0 > 0$ small enough, Ω_δ is smooth for every $\delta \in (0, \delta_0)$.
For every $k \geq 1$ and $\delta \in (0, \delta_0)$ such that $1/k < \delta$ we then have

$$\left\| \frac{\partial u_k}{\partial n} \right\|_{L^\infty(\partial\Omega_\delta)} \leq \|\nabla u_k\|_{L^\infty(\Omega_\delta)} \leq \|\nabla u\|_{L^\infty(\Omega)}. \tag{4.3}$$

Thus, for every $\psi \in C^1(\overline{\Omega})$,

$$\left| \int_{\Omega_\delta} \psi\, \Delta u_k + \int_{\Omega_\delta} \nabla\psi \cdot \nabla u_k \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)}. \tag{4.4}$$

Note that for a.e. $\delta \in (0, \delta_0)$

$$\int_{\partial\Omega_\delta} |\Delta u| = 0; \tag{4.5}$$

hence, for any such $\delta > 0$,

$$\int_{\Omega_\delta} \psi\, \Delta u_k \to \int_{\Omega_\delta} \psi\, \Delta u \quad \text{as } k \to \infty.$$

Indeed, this is a general fact (see, e.g., [5, Theorem 1, p.54]): if $\mu \in \mathcal{M}(\Omega)$ and $|\mu|(\partial\Omega_\delta) = 0$, then

$$\int_{\Omega_\delta} \psi\, (\rho_k * \mu) \to \int_{\Omega_\delta} \psi\, d\mu \quad \forall \psi \in C^0(\overline{\Omega}_\delta).$$

For any $\delta \in (0, \delta_0)$ verifying (4.5), as $k \to \infty$ in (4.4) we get

$$\left| \int_{\Omega_\delta} \psi\, \Delta u + \int_{\Omega_\delta} \nabla\psi \cdot \nabla u \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.6}$$

From this estimate, one deduces that for every $\psi \in C^1(\overline{\Omega})$,

$$\begin{aligned} \left| \int_{\Omega_\delta} \nabla\psi \cdot \nabla u \right| &\leq \|\Delta u\|_{\mathcal{M}(\Omega)} \|\psi\|_{L^\infty(\Omega_\delta)} + \|\nabla u\|_{L^\infty(\Omega)} \|\psi\|_{L^1(\partial\Omega_\delta)} \\ &\leq \big(\|\Delta u\|_{\mathcal{M}(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} |\partial\Omega_\delta| \big) \|\psi\|_{L^\infty(\Omega)}. \end{aligned}$$

As $\delta \to 0$, we conclude that $u \in \mathbb{X}$.

In order to prove that $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$, we return to estimate (4.6). Given $\phi \in C^1(\partial\Omega)$, we fix an extension $\psi \in C^1(\overline{\Omega})$ of ϕ; note that

$$\|\psi\|_{L^1(\partial\Omega_\delta)} \leq \|\phi\|_{L^1(\partial\Omega)} + C\delta \quad \forall \delta \in (0, \delta_0),$$

for some constant $C > 0$. Insert this test function ψ in (4.6). As $\delta \to 0$ we obtain, by dominated convergence,

$$\left| \int_\Omega \psi\, \Delta u + \int_\Omega \nabla\psi \cdot \nabla u \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)}.$$

Hence,

$$\left| \int_{\partial\Omega} \phi\, \frac{\partial u}{\partial n} \right| \leq \|\nabla u\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\partial\Omega)} \quad \forall \phi \in C^1(\partial\Omega).$$

Therefore, by duality $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega)$ and (4.1) holds.

We now assume that $u \in C^1(\overline{\Omega}) \cap \mathbb{X}$ and we denote by h the normal derivative of u in the standard sense. By Lemma B.1 and Remark B.1, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ satisfying (B.2)–(B.3) and such that

$$u_k \to u \quad \text{in } C^1(\overline{\Omega}).$$

In particular,

$$\frac{\partial u_k}{\partial n} \to h \quad \text{uniformly on } \partial\Omega.$$

Thus,

$$\int_\Omega \nabla u \cdot \nabla \psi + \int_\Omega \psi \, \Delta u = \int_{\partial\Omega} h \, \psi \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.7}$$

Hence, the normal derivative $\frac{\partial u}{\partial n}$ in the sense of the space $\mathbb{X}$ coincides with h. □

When $u \in \mathbb{X}$ the measure $\frac{\partial u}{\partial n}$ need not be an L^1-function. Surprisingly, this is always true if u vanishes on $\partial\Omega$:

Proposition 4.2. *Let $u \in W_0^{1,1}(\Omega)$. Then, $u \in \mathbb{X}$ if, and only if, $\Delta u \in \mathcal{M}(\Omega)$ in the sense of distributions. Moreover, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and*

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial\Omega)} \le \|\Delta u\|_{\mathcal{M}(\Omega)}. \tag{4.8}$$

Proof. We split the proof into two steps:

Step 1. Proof of (4.8) if u is smooth in a neighborhood of $\partial\Omega$.

Under this assumption, $\frac{\partial u}{\partial n}$ is a smooth function on $\partial\Omega$. Denote by v_1 and v_2 the solutions of

$$\begin{cases} -\Delta v_1 = \mu^+ & \text{in } \Omega, \\ \quad v_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_2 = \mu^- & \text{in } \Omega, \\ \quad v_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu = -\Delta u$. In particular,

$$u = v_1 - v_2 \quad \text{in } \Omega.$$

Since μ is smooth in a neighborhood of $\partial\Omega$, μ^+ and μ^- are Lipschitz continuous near $\partial\Omega$. Hence, v_1 and v_2 are of class C^2 near $\partial\Omega$. Moreover, $v_1 \ge 0$ in Ω and $v_1 = 0$ on $\partial\Omega$; thus,

$$\frac{\partial v_1}{\partial n} \le 0 \quad \text{on } \partial\Omega.$$

It follows that

$$\int_{\partial\Omega} \left| \frac{\partial v_1}{\partial n} \right| = -\int_{\partial\Omega} \frac{\partial v_1}{\partial n} = \int_\Omega \mu^+.$$

Similarly,

$$\int_{\partial\Omega} \left| \frac{\partial v_2}{\partial n} \right| = \int_\Omega \mu^-.$$

Therefore,

$$\int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right| \le \int_{\partial\Omega} \left| \frac{\partial v_1}{\partial n} \right| + \int_{\partial\Omega} \left| \frac{\partial v_2}{\partial n} \right| = \int_\Omega (\mu^+ + \mu^-) = \int_\Omega |\Delta u|.$$

Step 2. Proof of the proposition completed.

Let $(\varphi_k) \subset C_0^\infty(\Omega)$ be a sequence of test functions such that

$$0 \le \varphi_k \le 1 \quad \text{in } \overline{\Omega} \quad \text{and} \quad \varphi_k(x) = 1 \quad \text{if } d(x, \partial\Omega) \ge \frac{1}{k}.$$

Take $\mu_k = -\varphi_k \, \Delta u$, $\forall k \ge 1$. Then, $(\mu_k) \subset \mathcal{M}(\Omega)$ is a sequence of measures such that $\operatorname{supp} \mu_k \subset \Omega$ and, by dominated convergence,

$$\mu_k \to -\Delta u \quad \text{strongly in } \mathcal{M}(\Omega). \tag{4.9}$$

For each $k \ge 1$, let u_k be the unique solution of

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega, \\ \quad u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that u_k is harmonic in a neighborhood of $\partial\Omega$. We claim that

$$\int_{\partial\Omega} \phi \, \frac{\partial u_k}{\partial n} \to \int_{\partial\Omega} \phi \, \frac{\partial u}{\partial n} \quad \forall \phi \in C^1(\partial\Omega). \tag{4.10}$$

Indeed, since $u_k \to u$ in $L^1(\Omega)$ and (∇u_k) is bounded in $W_0^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$, (see [10]) we have

$$\int_\Omega \nabla\psi \cdot \nabla u_k \to \int_\Omega \nabla\psi \cdot \nabla u \quad \forall \psi \in C^1(\overline{\Omega}). \tag{4.11}$$

Assertion (4.10) then follows from (4.9) and (4.11).

Applying Step 1 to the function $u_i - u_j$, we have

$$\left\| \frac{\partial u_i}{\partial n} - \frac{\partial u_j}{\partial n} \right\|_{L^1(\partial\Omega)} \le \|\mu_i - \mu_j\|_{\mathcal{M}(\Omega)} \quad \forall i, j \ge 1.$$

In view of the strong convergence of (μ_k) in $\mathcal{M}(\Omega)$, $\big(\frac{\partial u_k}{\partial n}\big)$ is a Cauchy sequence in $L^1(\partial\Omega)$. Hence, this sequence converges in $L^1(\partial\Omega)$ to some function h. By (4.10), $h = \frac{\partial u}{\partial n}$; hence,

$$\frac{\partial u_k}{\partial n} \to \frac{\partial u}{\partial n} \quad \text{in } L^1(\partial\Omega).$$

Moreover, since (4.8) holds for every u_k, it also holds for u. The proof is complete. □

We now show that if $u \in W^{1,1}(\Omega)$ and $\nabla u \in BV(\Omega)$ then the normal derivative $\frac{\partial u}{\partial n}$ in the sense of the space $\mathbb{X}$ coincides with the function $n \cdot \nabla u$ on $\partial\Omega$ defined in the sense of traces:

Proposition 4.3. *Assume that $u \in W^{1,1}(\Omega)$ and $\nabla u \in BV(\Omega)$; hence,*

$$\Delta u = \operatorname{div}(\nabla u) \in \mathcal{M}(\Omega).$$

Then, $u \in \mathbb{X}$ and $\frac{\partial u}{\partial n}$ coincides with $n \cdot \nabla u|_{\partial\Omega}$ on $\partial\Omega$, where $\nabla u|_{\partial\Omega}$ is understood in the sense of traces. In particular, $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\partial\Omega)} \le C \|\nabla u\|_{BV(\Omega)}. \tag{4.12}$$

In the proof of Proposition 4.3 we use the notion of strict convergence in $BV(A)$, where $A \subset \mathbb{R}^N$ is a Lipschitz domain. We recall that a sequence $(f_n) \subset BV(A)$ converges strictly to $f \in BV(A)$ if

$$f_n \to f \quad \text{strongly in } L^1(A) \quad \text{and} \quad \int_A |Df_n| \to \int_A |Df|.$$

By [1, Theorem 3.88], the trace operator

$$f \in BV(A) \longmapsto f|_{\partial A} \in L^1(\partial A)$$

is continuous from $BV(A)$ (under strict convergence) into $L^1(\partial A)$ (under strong convergence).

Proof of Proposition 4.3. By Lemma B.1 and Remark B.1, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ satisfying (B.1)–(B.3) and (B.12). Since (∇u_k) converges strictly to ∇u in $BV(\Omega)$, we have

$$\nabla u_k|_{\partial\Omega} \to \nabla u|_{\partial\Omega} \quad \text{in } L^1(\partial\Omega). \tag{4.13}$$

Hence,

$$\int_\Omega \nabla u \cdot \nabla \psi + \int_\Omega \psi \, \Delta u = \int_{\partial\Omega} \big(n \cdot \nabla u|_{\partial\Omega}\big) \, \psi \quad \forall \psi \in C^1(\partial\Omega).$$

This implies that $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$ and equals $n \cdot \nabla u|_{\partial\Omega}$. By the BV-trace theory, (4.12) holds. □

5. Proof of Theorem 1.2

We first establish Theorem 1.2 for functions in $C^2_{\mathrm{D}}(\overline{\Omega})$, where

$$C^2_{\mathrm{D}}(\overline{\Omega}) = \Big\{\zeta \in C^2(\overline{\Omega});\ \zeta = 0 \text{ on } \partial\Omega\Big\}. \tag{5.1}$$

Lemma 5.1. *Let $u \in C^2_{\mathrm{D}}(\overline{\Omega})$. Then, $\Delta u^+ \in \mathcal{M}(\Omega)$ and*

$$\|\Delta u^+\|_{\mathcal{M}} \le \|\Delta u\|_{L^1}. \tag{5.2}$$

Proof. Apply (3.3) with $u + a$, where $a > 0$. We deduce that

$$\big[(u+a)^+\big]_{\mathbb{X}} \le [u+a]_{\mathbb{X}} = [u]_{\mathbb{X}}. \tag{5.3}$$

Since $(u+a)^+ = u + a$ in a neighborhood of $\partial\Omega$,

$$\frac{\partial}{\partial n}(u+a)^+ = \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega. \tag{5.4}$$

Note that

$$\big[(u+a)^+\big]_{\mathbb{X}} = \big\|\Delta(u+a)^+\big\|_{\mathcal{M}(\Omega)} + \left\|\frac{\partial}{\partial n}(u+a)^+\right\|_{L^1(\partial\Omega)},$$

$$[u]_{\mathbb{X}} = \|\Delta u\|_{L^1(\Omega)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^1(\partial\Omega)}.$$

By (5.3)–(5.4) we then have

$$\big\|\Delta(u+a)^+\big\|_{\mathcal{M}} \le \|\Delta u\|_{L^1} \quad \forall a > 0.$$

The result follows from the lower semicontinuity of the norm $\|\cdot\|_{\mathcal{M}}$ with respect to the weak* convergence as $a \to 0$. □

Proof of Theorem 1.2. Since $u \in \mathbb{X}$, $\Delta u \in \mathcal{M}(\Omega)$. Take a sequence $(\mu_k) \subset C^\infty(\overline{\Omega})$ such that

$$\mu_k \overset{*}{\rightharpoonup} -\Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega) \quad \text{and} \quad \|\mu_k\|_{L^1} \to \|\mu\|_{\mathcal{M}}.$$

For each $k \ge 1$, let $u_k \in C^2_{\mathrm{D}}(\overline{\Omega})$ be the solution of

$$-\Delta u_k = \mu_k \quad \text{in } \Omega.$$

Then, by standard elliptic estimates,

$$u_k \to u \quad \text{in } L^1(\Omega).$$

On the other hand, it follows from Lemma 5.1 that $\Delta u_k^+ \in \mathcal{M}(\Omega)$ and

$$\|\Delta u_k^+\|_{\mathcal{M}} \leq \|\Delta u_k\|_{L^1}.$$

Thus,

$$\left|\int_\Omega u_k^+ \,\Delta\zeta\right| \leq \|\Delta u_k\|_{L^1}\|\zeta\|_{L^\infty} = \|\mu_k\|_{L^1}\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{D}}(\overline{\Omega}).$$

As $k \to \infty$ we obtain

$$\left|\int_\Omega u^+ \Delta\zeta\right| \leq \|\Delta u\|_{\mathcal{M}}\|\zeta\|_{L^\infty} \quad \forall \zeta \in C^2_{\mathrm{D}}(\overline{\Omega}).$$

This gives (1.5). From Proposition 4.2, we know that $\frac{\partial u}{\partial n}, \frac{\partial u^+}{\partial n} \in L^1(\partial\Omega)$ and (1.6) holds. $\square$

6. Kato's inequality up to the boundary

Before proving Theorem 1.3, we first present some variants of Kato's inequality when Δu and $\frac{\partial u}{\partial n}$ are not necessarily L^1-functions but only finite measures. We prove for instance the following companion to [3, Proposition 4.B.5]:

Proposition 6.1. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \leq \int_{\partial\Omega} h\zeta + \int_\Omega g\zeta \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \overline{\Omega} \tag{6.1}$$

for some $g \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$. Then, $u \in W^{1,1}(\Omega)$ and

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \leq \int_{\substack{\partial\Omega\\ [u\geq 0]}} h\psi + \int_{\substack{\Omega\\ [u\geq 0]}} g\psi \quad \forall \psi \in C^1(\overline{\Omega}),\ \psi \geq 0 \text{ in } \Omega. \tag{6.2}$$

Proof. By Proposition 2.2, $u \in \mathbb{X}$. Moreover,

$$\begin{cases} -\Delta u \leq g & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} \leq h & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

We now split the proof into two steps:

Step 1. Let $\Phi \in C^2(\mathbb{R})$ be a nondecreasing convex function such that $\Phi' \in L^\infty(\mathbb{R})$. Then,

$$\int_\Omega \nabla\Phi(u)\cdot\nabla\psi \leq \int_{\partial\Omega}\psi\,\Phi'(u)h + \int_\Omega \psi\,\Phi'(u)g \tag{6.4}$$

for every $\psi \in C^1(\overline{\Omega})$ such that $\psi \geq 0$ in $\overline{\Omega}$.

Let $(g_k) \subset C^\infty(\overline{\Omega})$ and $(h_k) \subset C^\infty(\partial\Omega)$ be such that

$$g_k \to g \text{ in } L^1(\Omega) \text{ and a.e.} \quad \text{and} \quad h_k \to h \text{ in } L^1(\partial\Omega) \text{ and a.e.}$$

Next, take $(\mu_k) \subset C^\infty(\overline{\Omega})$ and $(\nu_k) \subset C^\infty(\partial\Omega)$ such that

$$\mu_k \overset{*}{\rightharpoonup} -\Delta u \text{ weak}^* \text{ in } \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad \nu_k \overset{*}{\rightharpoonup} \frac{\partial u}{\partial n} \text{ weak}^* \text{ in } \mathcal{M}(\partial\Omega).$$

In view of (6.3) and

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega \Delta u,$$

we may assume that

$$\mu_k \le g_k \text{ in } \Omega, \quad \nu_k \le h_k \text{ on } \partial\Omega \quad \text{and} \quad \int_{\partial\Omega} \nu_k = -\int_\Omega \mu_k \quad \forall k \ge 1.$$

Let $u_k \in C^\infty(\overline{\Omega})$ be the unique solution of

$$\begin{cases} -\Delta u_k = \mu_k & \text{in } \Omega,\\ \dfrac{\partial u_k}{\partial n} = \nu_k & \text{on } \partial\Omega, \end{cases}$$

such that $\int_\Omega u_k = \int_\Omega u$. By Remark 2.1, the sequence (u_k) is bounded in $W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$. Passing to a subsequence if necessary, we have

$$\nabla\Phi(u_k) \rightharpoonup \nabla\Phi(u) \quad \text{weakly in } L^1(\Omega).$$

Let $\psi \in C^1(\overline{\Omega})$, $\psi \ge 0$ in $\overline{\Omega}$. As in Lemma 3.1, for every $k \ge 1$ we have

$$\begin{aligned}\int_\Omega \nabla\Phi(u_k)\cdot\nabla\psi &\le \int_{\partial\Omega} \psi\,\Phi'(u_k)\frac{\partial u_k}{\partial n} - \int_\Omega \psi\,\Phi'(u_k)\Delta u_k\\ &\le \int_{\partial\Omega} \psi\,\Phi'(u_k)h_k + \int_\Omega \psi\,\Phi'(u_k)g_k.\end{aligned}$$

By dominated convergence we obtain (6.4) as $k \to \infty$.

Step 2. Proof of the proposition completed.

Apply (6.4) with $\Phi = \Phi_k$, where (Φ_k) is a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $0 \le \Phi_k' \le 1$ and

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \ge 0,\\ 0 & \text{if } t < 0. \end{cases}$$

The result follows as we let $k \to \infty$. □

The following variant of Proposition 6.1 will be needed below:

Proposition 6.2. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\zeta \le \int_{\partial\Omega} h\zeta + \int_\Omega \zeta\,d\mu \quad \forall\zeta \in C^2_{\mathrm{N}}(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \overline{\Omega} \tag{6.5}$$

for some $\mu \in \mathcal{M}(\Omega)$, $\mu \ge 0$, and $h \in L^1(\partial\Omega)$. Then, $u \in W^{1,1}(\Omega)$ and

$$\int_\Omega \nabla u^+\cdot\nabla\psi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall\psi \in C^1(\overline{\Omega}),\ \psi \ge 0 \text{ in } \overline{\Omega}. \tag{6.6}$$

Proof. One can proceed as in the proof of Proposition 6.1. In Step 1, one should replace (6.4) by

$$\int_\Omega \nabla\Phi(u)\cdot\nabla\psi \le \int_{\partial\Omega} \psi\,\Phi'(u)h + \|\Phi'\|_{L^\infty}\int_\Omega \psi\,d\mu. \tag{6.4'}$$

Inequality (6.4') is easily obtained by approximation, where the sequence $(g_k) \subset C^\infty(\overline{\Omega})$ is chosen so that

$$g_k \overset{*}{\rightharpoonup} \mu \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}).$$

The rest of the argument remains unchanged. □

We now prove the

Proposition 6.3. *Let $u \in \mathbb{X}$. If $\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$, then*

$$\frac{\partial u^+}{\partial n} \leq \begin{cases} \frac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\frac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{6.7}$$

Proof. Denoting by $\mu = (-\Delta u)^+$ and $h = \frac{\partial u}{\partial n}$, we have

$$-\int_\Omega u\,\Delta\zeta \leq \int_{\partial\Omega} h\zeta + \int_\Omega \zeta\,d\mu \quad \forall \zeta \in C^2_{\mathrm{N}}(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \overline{\Omega}.$$

Therefore, by Proposition 6.2, u^+ satisfies

$$\int_\Omega \nabla u^+ \cdot \nabla\psi \leq \int_{\substack{\partial\Omega \\ [u \geq 0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^1(\overline{\Omega}),\ \psi \geq 0 \text{ in } \overline{\Omega}. \tag{6.8}$$

By Theorem 1.1, we know that $u^+ \in \mathbb{X}$. It thus follows that

$$\frac{\partial u^+}{\partial n} \leq \chi_{[u \geq 0]} h = \chi_{[u \geq 0]} \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega. \tag{6.9}$$

Given $a > 0$, we now apply (6.8) with u replaced by $u - a$. As $a \to 0$, we obtain

$$\int_{\partial\Omega} u^+ \frac{\partial\psi}{\partial n} - \int_\Omega u^+\,\Delta\psi \leq \int_{\substack{\partial\Omega \\ [u > 0]}} h\psi + \int_\Omega \psi\,d\mu \quad \forall \psi \in C^1(\overline{\Omega}),\ \psi \geq 0 \text{ in } \Omega. \tag{6.10}$$

Hence,

$$\frac{\partial u^+}{\partial n} \leq \chi_{[u > 0]} h = \chi_{[u > 0]} \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega.$$

In particular,

$$\frac{\partial u^+}{\partial n} \leq 0 \quad \text{on } [u = 0]. \tag{6.11}$$

Assertion (6.7) follows by combining (6.9) and (6.11). □

We state the following consequence of Proposition 6.3:

Corollary 6.1. *Let $u \in \mathbb{X} \cap W_0^{1,1}(\Omega)$. If $u \geq 0$ in Ω, then*

$$\frac{\partial u}{\partial n} \leq 0 \quad \text{on } \partial\Omega.$$

Proof. Since $u = u^+$ in Ω and $u = 0$ on $\partial\Omega$, applying Proposition 6.3 above we get

$$\frac{\partial u}{\partial n} = \frac{\partial u^+}{\partial n} \leq \min\left\{\frac{\partial u}{\partial n}, 0\right\} \leq 0 \quad \text{on } \partial\Omega.$$ □

We now present the

Proof of Theorem 1.3. By Theorem 1.1, $u^+ \in \mathbb{X}$. Applying Kato's inequality to $u - a$, we have

$$\Delta(u - a)^+ \geq \chi_{[u \geq a]} \Delta u \quad \text{in } \Omega \tag{6.12}$$

for every $a \in \mathbb{R}$. As $a \downarrow 0$ in (6.12) we get

$$\Delta u^+ \geq \chi_{[u > 0]} \Delta u = G \quad \text{in } \Omega.$$

By this estimate and (6.7), for every $\psi \in C^1(\overline{\Omega})$ with $\psi \geq 0$ in Ω,

$$\int_\Omega \nabla u^+ \cdot \nabla \psi = \int_{\partial\Omega} \psi \, \frac{\partial u^+}{\partial n} - \int_\Omega \psi \, \Delta u^+ \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi.$$

The proof is complete. $\square$

7. Computing $\frac{\partial u^+}{\partial n}$ for $W^{2,1}$-functions

Our goal in this section is to give a positive answer to Open Problems 1 and 2 under the additional assumption that $u \in W^{2,1}(\Omega)$:

Theorem 7.1. *If $u \in W^{2,1}(\Omega)$, then $\nabla u^+ \in BV(\Omega)$ (so that, $u^+ \in \mathbb{X}$ by Proposition 4.3) and*

$$\frac{\partial u^+}{\partial n} = \begin{cases} \frac{\partial u}{\partial n} & \text{on } [u > 0], \\ 0 & \text{on } [u < 0], \\ \min\left\{\frac{\partial u}{\partial n}, 0\right\} & \text{on } [u = 0]. \end{cases} \tag{7.1}$$

We first prove the

Lemma 7.1. *If $v \in W^{1,1}(\Omega)$ and $\nabla v \in BV(\Omega)$, then*

$$\frac{\partial v}{\partial n}(x) = \lim_{t \downarrow 0} \frac{v(x) - v\big(x - tn(x)\big)}{t} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{7.2}$$

In (7.2), we identify v with its precise representative, which is well-defined outside a set of zero $\mathcal{H}^{N-1}$-Hausdorff measure; see [5, Section 4.8, Theorem 1 and Section 5.6, Theorem 3].

Proof. Since $v \in W^{1,1}(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ the function

$$t \in (0, \delta) \longmapsto v\big(x - tn(x)\big)$$

is well-defined for some $\delta > 0$ and belongs to $W^{1,1}(0, \delta)$. Thus,

$$\frac{v\big(x - tn(x)\big) - v(x)}{t} = -n(x) \cdot \int_0^1 \nabla v\big(x - stn(x)\big) \, ds. \tag{7.3}$$

Moreover, since $\nabla v \in BV(\Omega)$, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ the function

$$r \in (0, \delta) \longmapsto \nabla v\big(x - rn(x)\big)$$

belongs to $BV(0, \delta) \subset L^\infty(0, \delta)$ and (see [1, Theorem 3.108])

$$\lim_{r \downarrow 0} \nabla v\big(x - rn(x)\big) = \nabla v|_{\partial\Omega}(x). \tag{7.4}$$

We deduce from (7.3)–(7.4) that

$$\lim_{t \downarrow 0} \frac{v\big(x - tn(x)\big) - v(x)}{t} = -n(x) \cdot \nabla v|_{\partial\Omega}(x).$$

By Proposition 4.3 above, $\frac{\partial v}{\partial n} = n \cdot \nabla v|_{\partial\Omega}$ and the conclusion follows. $\square$

We also need the following elementary lemma whose proof is left to the reader:

Lemma 7.2. *Let* $v : [0, \delta] \to \mathbb{R}$ *be such that*

$$\lim_{t \downarrow 0} \frac{v(0) - v(t)}{t} = \alpha \in \mathbb{R}. \tag{7.5}$$

Then,

$$\lim_{t \downarrow 0} \frac{v^+(0) - v^+(t)}{t} = \begin{cases} \alpha & \text{if } v(0) > 0, \\ 0 & \text{if } v(0) < 0, \\ \min\{\alpha, 0\} & \text{if } v(0) = 0. \end{cases} \tag{7.6}$$

We now present the

Proof of Theorem 7.1. We split the proof into three steps:

Step 1. Proof of the assertion: $\nabla u^+ \in BV(\Omega)$.

Extending u to $\mathbb{R}^N$, we may assume that $u \in W^{2,1}(\mathbb{R}^N)$. We claim that

$$\frac{\partial^2 u^+}{\partial \mathrm{e}^2} \geq \chi_{[u \geq 0]} \frac{\partial^2 u}{\partial \mathrm{e}^2} \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{7.7}$$

for every $\mathrm{e} \in \mathbb{R}^N \setminus \{0\}$. Indeed, let (Φ_k) be a sequence of smooth convex functions such that $\Phi_k(0) = 0$, $\|\Phi_k'\|_{L^\infty} \leq 1$ and

$$\Phi_k'(t) \to \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{7.8}$$

Then,

$$\frac{\partial^2 \Phi_k(u)}{\partial \mathrm{e}^2} = \Phi_k'(u) \frac{\partial^2 u}{\partial \mathrm{e}^2} + \Phi_k''(u) \left(\frac{\partial u}{\partial \mathrm{e}} \right)^2 \geq \Phi_k'(u) \frac{\partial^2 u}{\partial \mathrm{e}^2} \quad \text{in } \mathbb{R}^N.$$

As $k \to \infty$, we obtain (7.7).

It follows from (7.7) that $\dfrac{\partial^2 u^+}{\partial \mathrm{e}^2} \in \mathcal{M}(\Omega)$ for every $\mathrm{e} \in \mathbb{R}^N \setminus \{0\}$. Applying the conclusion with $\mathrm{e} = \mathrm{e}_i, \mathrm{e}_j, \mathrm{e}_i + \mathrm{e}_j$ for every $i, j \in \{1, \ldots, N\}$ we deduce that $D^2 u^+$ is a finite measure in Ω. Thus, $\nabla u^+ \in BV(\Omega)$.

Step 2. Proof of (7.1).

By Lemma 7.1, for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, u satisfies

$$\lim_{t \downarrow 0} \frac{u(x) - u\big(x - tn(x)\big)}{t} = \frac{\partial u}{\partial n}(x). \tag{7.9}$$

Hence, by (7.2) applied to u^+ and by (7.6) applied to $v(t) = u(x - tn(x))$,

$$\frac{\partial u^+}{\partial n}(x) = \lim_{t \downarrow 0} \frac{u^+(x) - u^+\big(x - tn(x)\big)}{t} = \begin{cases} \frac{\partial u}{\partial n}(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) < 0, \\ \min\left\{\frac{\partial u}{\partial n}(x), 0\right\} & \text{if } u(x) = 0, \end{cases}$$

for every $x \in \partial\Omega$ for which (7.9) holds. Since this is true $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, (7.1) follows. The proof of Theorem 7.1 is complete. □

Appendix A. The measure Δu^+ need not be finite

In this appendix, we construct a harmonic function in dimension 2 such that $\int_\Omega |\Delta u^+| = \infty$:

Proposition A.1. *Let*

$$\Omega = \big\{(x,y) \in \mathbb{R}^2;\ x^2+y^2<1 \text{ and } x>0\big\}.$$

There exists a harmonic function $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $u|_{\partial\Omega} \in W^{1,1}(\partial\Omega)$ such that

(i) *$u \notin \mathbb{X}$ and $u^+ \notin \mathbb{X}$;*
(ii) *$\Delta u^+ \geq 0$ in the sense of distributions;*
(iii) *Δu^+ is **not** a finite measure in Ω.*

Proof. Let u be the function in $\overline{\Omega}$ given in polar coordinates by

$$u(r,\theta) = \sum_{k=1}^{\infty} r^{a_k} \sin(a_k\theta) \tag{A.1}$$

where $(a_k) \subset (0,1)$ is a sequence such that

$$\sum_{k=1}^{\infty} k a_k < \infty.$$

Since

$$|u(r,\theta)| \leq \sum_{k=1}^{\infty} \big|\sin(a_k\theta)\big| \leq \frac{\pi}{2}\sum_{k=1}^{\infty} a_k,$$

it follows that $u \in C(\overline{\Omega})$ and u is harmonic in Ω (u is a series of harmonic functions). Note that

$$|\nabla u|^2 = \sum_{j,k=1}^{\infty} a_j a_k r^{a_j+a_k-2}\cos\big((a_j-a_k)\theta\big).$$

Thus,

$$\int_\Omega |\nabla u|^2 \leq \pi \sum_{j,k=1}^{\infty} \frac{a_j a_k}{a_j+a_k} \leq 2\pi \sum_{\substack{j,k=1\\ j\leq k}}^{\infty} \frac{a_j a_k}{a_j+a_k} \leq 2\pi \sum_{k=1}^{\infty} k a_k < \infty;$$

in other words, $u \in H^1(\Omega)$. Denoting by τ the tangential unit vector of u on $\partial\Omega$, we have

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \tau}\right| = 4\sum_{k=1}^{\infty} \sin\Big(a_k\frac{\pi}{2}\Big) \leq 2\pi\sum_{k=1}^{\infty} a_k < \infty;$$

hence, $u \in W^{1,1}(\partial\Omega)$.
Since u is harmonic in Ω, u^+ is subharmonic. Thus, $\Delta u^+ \geq 0$ in Ω. We show that Δu^+ is not a finite measure in Ω. Note that u vanishes only on the x-axis. Denoting by dx ($= dr$) the 1-dimensional Lebesgue measure on the segment $(0,1)\times\{0\}$, we then have

$$\Delta u^+ = \frac{\partial u}{\partial y}(x,0)\,dx = \frac{1}{r}\frac{\partial u}{\partial \theta}(r,0)\,dr = \sum_{k=1}^{\infty} a_k r^{a_k-1}\,dr.$$

Therefore,

$$\int_\Omega |\Delta u^+| = \sum_{k=1}^{\infty}\int_0^1 a_k r^{a_k-1}\,dr = \sum_{k=1}^{\infty} 1 = \infty.$$

Hence, $u^+ \notin \mathbb{X}$ and, by Theorem 1.1, this means that $u \notin \mathbb{X}$. □

Remark A.1. This example also shows that given $\varphi \in W^{1,1}(\partial\Omega)$, it is in general not possible to construct a function $v \in W^{2,1}(\Omega)$ such that $v|_{\partial\Omega} = \varphi$. This is in contrast with the well-known result of Gagliardo [6] which asserts that the map

$$w \in W^{1,1}(\Omega) \longmapsto w|_{\partial\Omega} \in L^1(\partial\Omega)$$

is surjective.

Indeed, take $\varphi = u|_{\partial\Omega}$, where u is given by (A.1). Suppose by contradiction that there exists some $v \in W^{2,1}(\Omega)$ such that $v|_{\partial\Omega} = \varphi$. Applying Proposition 4.2 to $u - v \in W_0^{1,1}(\Omega)$, we would deduce that $\frac{\partial}{\partial n}(u-v) \in L^1(\partial\Omega)$. But $v \in W^{2,1}(\Omega)$ implies $\frac{\partial v}{\partial n} \in L^1(\partial\Omega)$ and therefore

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(u-v) + \frac{\partial v}{\partial n} \in L^1(\partial\Omega),$$

a contradiction.

Appendix B. Approximation by smooth functions in $\overline{\Omega}$

In this appendix, we establish the following

Lemma B.1. *Given $u \in \mathbb{X}$, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})$ such that*

$$u_k \to u \quad \text{in } W^{1,1}\big(\Omega\big), \tag{B.1}$$

$$\int_\Omega \psi\, \Delta u_k \to \int_\Omega \psi\, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}) \tag{B.2}$$

and

$$\int_{\partial\Omega} \psi\, \frac{\partial u_k}{\partial n} \to \int_{\partial\Omega} \psi\, \frac{\partial u}{\partial n} \quad \forall \psi \in C^1(\overline{\Omega}). \tag{B.3}$$

Proof. We split the proof into two steps:

Step 1. Given $x_0 \in \partial\Omega$, there exist $\delta > 0$ and a sequence $(v_k) \subset C^\infty(\overline{\Omega})$ such that

$$v_k \to u \quad \text{in } W^{1,1}\big(B_\delta(x_0) \cap \Omega\big), \tag{B.4}$$

$$\int_\Omega \psi\, \Delta v_k \to \int_\Omega \psi\, \Delta u \quad \forall \psi \in C^1(\overline{\Omega}) \text{ with } \operatorname{supp}\psi \subset B_\delta(x_0). \tag{B.5}$$

Since $\partial\Omega$ is smooth, there exist $\delta_1 > 0$ and an open cone $T \subset \mathbb{R}^N$ (with vertex at $0 \in \mathbb{R}^N$) such that

$$(x + T) \cap B_{\delta_1}(x) \subset \Omega \quad \forall x \in B_{\delta_1}(x_0) \cap \overline{\Omega}. \tag{B.6}$$

Let $\delta = \delta_1/2$ and $\rho \in C_0^\infty(B_\delta)$, $\rho \geq 0$, be such that $\int_{B_\delta} \rho = 1$ and

$$\operatorname{supp}\rho \subset -T. \tag{B.7}$$

Set

$$\rho_k(x) = k^N \rho(kx) \quad \forall x \in \mathbb{R}^N.$$

We show that the sequence $(v_k) \subset C^\infty(\overline{\Omega})$ given by

$$v_k(x) = \int_\Omega \rho_k(x-y) u(y)\, dy \quad \forall x \in \overline{\Omega} \tag{B.8}$$

satisfies (B.4)–(B.5).

Note that given any $x \in B_\delta(x_0) \cap \Omega$, by (B.7) $v_k(x)$ depends only on the values of

u on a compact subset of $(x+T)\cap B_{\delta_1}(x)$. In fact, from (B.6)–(B.7) and a change of variable, we can rewrite (B.8) as

$$v_k(x) = \int\limits_{T\cap B_{\delta_1}(0)} \rho_k(-z)u(x+z)\,dz \quad \forall x \in B_\delta(x_0)\cap\Omega. \tag{B.9}$$

Therefore,

$$\nabla v_k = \rho_k * (\nabla u) \quad \text{and} \quad \Delta v_k = \rho_k * (\Delta u) \quad \text{in } B_\delta(x_0)\cap\Omega. \tag{B.10}$$

In particular, (B.4)–(B.5) hold.

Step 2. Proof of the proposition completed.

By compactness of $\partial\Omega$, we can cover this set with finitely many balls $B_\delta(x_1), \dots, B_\delta(x_t)$ such that (B.4)–(B.5) hold on each ball $B_\delta(x_i)$ for some sequence $(v_k^i) \subset C^\infty(\overline{\Omega})$. We now take $(v_k^0) \subset C^\infty(\overline{\Omega})$ and $\omega \Subset \Omega$ such that $\Omega \setminus \bigcup_{i=1}^t B_\delta(x_i) \subset \omega$,

$$v_k^0 \to u \quad \text{in } W^{1,1}(\omega) \quad \text{and} \quad \Delta v_k^0 \overset{*}{\rightharpoonup} \Delta u \quad \text{weak}^* \text{ in } \mathcal{M}(\omega)$$

(such sequence can be obtained via convolution of u).
Let (φ_i) be a partition of unity subordinated to the covering $\omega, B_\delta(x_1), \dots, B_\delta(x_t)$ of $\overline{\Omega}$. One verifies that (B.1)–(B.2) hold for the sequence (u_k) given by

$$u_k = \sum_{i=0}^t \varphi_i v_k^i.$$

Assertion (B.3) immediately follows from (B.1)–(B.2). □

Remark B.1. An inspection of the proof of Lemma B.1 shows that

(*i*) if $u \in C^1(\overline{\Omega})$, then

$$u_k \to u \quad \text{in } C^1(\overline{\Omega}); \tag{B.11}$$

(*ii*) if $\nabla u \in BV(\Omega)$, then

$$\|D^2 u_k\|_{L^1(\Omega)} \to \|D^2 u\|_{\mathcal{M}(\Omega)}. \tag{B.12}$$

Appendix C. Proof of Lemma 2.1

The proof of Lemma 2.1 we present below follows the lines of [8, Lemma 7.3] (see also [7, Theorem 8.15]) with some minor modifications. We first need the following variant of the Gagliardo-Nirenberg inequality:

Proposition C.1. *Let*

$$\mathcal{A} = \left\{ v \in W^{1,1}(\Omega);\ \big|[v=0]\big| \geq \frac{|\Omega|}{3} \right\}. \tag{C.1}$$

Then,

$$\|v\|_{L^{\frac{N}{N-1}}} \leq C\|\nabla v\|_{L^1} \quad \forall v \in \mathcal{A}. \tag{C.2}$$

We denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^N$.

Proof. By a variant of the Poincaré inequality (easily proved by contradiction), we have

$$\|v\|_{L^1} \leq C\|\nabla v\|_{L^1} \quad \forall v \in \mathcal{A}. \tag{C.3}$$

On the other hand, by the standard Gagliardo-Nirenberg inequality and an extension argument,

$$\|v\|_{L^{\frac{N}{N-1}}} \leq C\big(\|\nabla v\|_{L^1} + \|v\|_{L^1}\big) \quad \forall v \in W^{1,1}(\Omega). \tag{C.4}$$

Combining (C.3)–(C.4), we obtain (C.2). □

Proof of Lemma 2.1. Replacing w by $w - a$ for some suitable constant $a \in \mathbb{R}$ if necessary, we may assume that

$$\big|[w \leq 0]\big| \geq \frac{|\Omega|}{3} \quad \text{and} \quad \big|[w \geq 0]\big| \geq \frac{|\Omega|}{3}. \tag{C.5}$$

Given $t > 0$, let

$$v_t(x) = \big[w(x) - t\big]^+ \quad \forall x \in \Omega. \tag{C.6}$$

Using v_t as a test function in (2.4), one shows that

$$\|\nabla v_t\|_{L^2} \leq \|F\|_{L^q}\big|[w > t]\big|^{\frac{1}{2}-\frac{1}{q}}.$$

On the other hand, by Hölder's inequality and Proposition C.1,

$$\|v_t\|_{L^1} \leq C\|\nabla v_t\|_{L^2}\big|[w > t]\big|^{\frac{1}{2}+\frac{1}{N}}.$$

Thus,

$$\|v_t\|_{L^1} \leq C\|F\|_{L^q}\big|[w > t]\big|^{\alpha} \quad \forall t > 0, \tag{C.7}$$

where $\alpha = 1 + \frac{1}{N} - \frac{1}{q}$. Recall that

$$\|v_t\|_{L^1} = \int_0^\infty \big|[v_t > r]\big|\, dr = \int_t^M \big|[w > s]\big|\, ds, \tag{C.8}$$

where $M = \|w^+\|_{L^\infty}$. Since $\alpha > 1$, one deduces using (C.7)–(C.8) that

$$\|w^+\|_{L^\infty} \leq C\|F\|_{L^q}^{\frac{1}{\alpha}}\|w^+\|_{L^1}^{1-\frac{1}{\alpha}}. \tag{C.9}$$

From (C.9) and $\|w^+\|_{L^1} \leq |\Omega|\|w^+\|_{L^\infty}$, we then have

$$\|w^+\|_{L^\infty} \leq C\|F\|_{L^q}.$$

Replacing w by $-w$, one obtains a similar estimate for w^-. Thus,

$$\|w\|_{L^\infty} \leq \|w^+\|_{L^\infty} + \|w^-\|_{L^\infty} \leq 2C\|F\|_{L^q}.$$ □

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