A case of density in $W^{2,p}(M;N)$

Pierre Bousquet [a], Augusto C. Ponce [b], Jean Van Schaftingen [c]

[a] *Unité de Mathématiques Pures et Appliquées (UMR CNRS 5669), ENS Lyon, 46 Allée d'Italie, 69007 Lyon, France*
[b] *Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 6083), Université de Tours, 37200 Tours, France*
[c] *Département de Mathématique, Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgique*

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Abstract

Given two compact Riemannian manifolds M^m, N^n without boundary and $m-1 < 2p < m$, we show that maps which are smooth except on finitely many points are dense in $W^{2,p}(M;N)$. If in addition $\pi_{m-1}(N)$ is trivial, then $C^\infty(M;N)$ is dense in $W^{2,p}(M;N)$. *To cite this article: P. Bousquet, A.C. Ponce, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I ??? (200?).*

Résumé

Un cas de densité dans $W^{2,p}(M;N)$ On considère deux variétés riemaniennes compactes sans bord M^m et N^n. Quand $m-1 < 2p < m$, on montre que les fonctions lisses sauf en un nombre fini de points sont denses dans $W^{2,p}(M;N)$. Si la variété N vérifie $\pi_{m-1}(N) = \{0\}$, alors $C^\infty(M;N)$ est dense dans $W^{2,p}(M;N)$. *Pour citer cet article : P. Bousquet, A.C. Ponce, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I ? ? ? (200 ?).*

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Soient $M, N \subset \mathbb{R}^l$ deux variétés riemanniennes compactes sans bord de dimension m et n respectivement. Étant donné $1 \le p < \infty$, on définit l'ensemble $W^{2,p}(M,N)$ par

$$W^{2,p}(M;N) = \Big\{ u \in W^{2,p}(M;\mathbb{R}^l);\ u(x) \in N \text{ p.p.} \Big\}, \tag{1}$$

qui est un espace métrique complet pour la distance induite par la norme (3).

Email addresses: Pierre.Bousquet@umpa.ens-lyon.fr (Pierre Bousquet), ponce@lmpt.univ-tours.fr (Augusto C. Ponce), Jean.VanSchaftingen@uclouvain.be (Jean Van Schaftingen).

Par convolution avec un noyau régularisant, on peut voir facilement que tout élément de $W^{2,p}(M,N)$ peut être approché par des applications de $C^\infty(M,\mathbb{R}^l)$. La densité de $C^\infty(M,N)$ dans $W^{2,p}(M,N)$ est une question plus délicate. Néanmoins, la réponse est aisée lorsque $2p > m$ en utilisant l'injection de Morrey $W^{2,p} \hookrightarrow C^0$, ou si $p = m/2$, auquel cas c'est une conséquence de la théorie des fonctions *VMO*. Quand $2p < m$, $C^\infty(M,N)$ n'est pas dense dans $W^{2,p}(M,N)$ en général (voir le Théorème 0.1 ci-dessous).

Le problème de densité des fonctions lisses dans $W^{1,p}(M,N)$ a été résolu par Bethuel [2] (voir aussi Hang–Lin [8]). La preuve pour $W^{1,p}$ nécessite des arguments de *recollement* qui ne se généralisent pas aisément au cas $W^{2,p}$. Dans un article en préparation (voir [5]), on s'intéresse au problème de densité de $C^\infty(M;N)$ dans $W^{2,p}(M;N)$ par rapport à la distance $W^{2,p}$ pour tout $p \geq 1$. Le but de cette Note est de présenter l'idée de notre preuve dans un cas particulier, à savoir lorsque $m-1 < 2p < m$:

Théorème 0.1 *Soit $m-1 < 2p < m$. Alors, $C^\infty(M,N)$ est dense dans $W^{2,p}(M,N)$ si et seulement si le groupe d'homotopie $\pi_{m-1}(N)$ est trivial.*

Si $\pi_{m-1}(N) \neq \{0\}$, alors $C^\infty(M,N)$ n'est pas dense dans $W^{2,p}(M,N)$. On démontre par contre le

Théorème 0.2 *Si $m-1 < 2p < m$, alors les fonctions lisses en dehors d'un nombre fini de points sont denses dans $W^{2,p}(M,N)$.*

Prenons par exemple $n = m-1$, $M = S^m$ et $N = S^{m-1}$. Comme $\pi_{m-1}(S^{m-1}) = \mathbb{Z}$, on déduit du Théorème 0.1 que $C^\infty(S^m; S^{m-1})$ n'est pas dense dans $W^{2,p}(S^m, S^{m-1})$ si $m-1 < 2p < m$. On peut se demander quels sont les éléments de cet espace qui peuvent être approchés par des applications lisses. La réponse est donnée en fonction du jacobien distributionnel "Jac" introduit par Brezis-Coron-Lieb [6] :

Théorème 0.3 *Soit $u \in W^{2,p}(S^m, S^{m-1})$ avec $m-1 < 2p < m$. Alors il existe une suite dans $C^\infty(S^m, S^{m-1})$ convergeant fortement vers u dans $W^{2,p}$ si et seulement si* $\operatorname{Jac}(u) = 0$ *dans $\mathcal{D}'(S^m)$.*

Les preuves détaillées des Théorèmes 0.1–0.3 seront présentées dans [5].

1. Introduction

Let $M, N \subset \mathbb{R}^l$ be two compact Riemannian manifolds without boundary, respectively m and n-dimensional. Given $1 \leq p < \infty$, consider

$$W^{2,p}(M;N) = \Big\{ u \in W^{2,p}(M;\mathbb{R}^l);\ u(x) \in N \text{ a.e.} \Big\}. \tag{2}$$

Although $W^{2,p}(M;N)$ is not a vector space, it inherits a metric from the usual norm in $W^{2,p}(M;\mathbb{R}^l)$:

$$\|u\|_{W^{2,p}} = \|u\|_{L^p} + \|Du\|_{L^p} + \|D^2u\|_{L^p} \quad \forall u \in W^{2,p}(M;\mathbb{R}^l); \tag{3}$$

$W^{2,p}(M;N)$ is a complete metric space with respect to this distance.

By standard convolution arguments, each $u \in W^{2,p}(M;N)$ can be approximated by maps in $C^\infty(M;\mathbb{R}^l)$. A deeper question concerns whether $C^\infty(M;N)$ is dense in $W^{2,p}(M;N)$. When $2p > m$, this can be easily seen to be true by using Morrey's embedding $W^{2,p} \hookrightarrow C^0$. If $p = m/2$, then functions in $W^{2,p}$ need not be continuous but smooth maps are still dense in $W^{2,p}(M,N)$. The argument in this case relies on tools from the theory of vanishing mean oscillation (VMO) functions; see [7,10]. However, density of smooth maps is no longer true in general if $2p < m$. Take for instance $M = S^m$, $N = S^{m-1}$ and

$$u = U|_{S^m}, \quad \text{where} \quad U(x) = \frac{(x_1, \ldots, x_m)}{(x_1^2 + \cdots + x_m^2)^{1/2}} \quad \forall x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}. \tag{4}$$

Then, $u \in W^{2,p}(S^m; S^{m-1})$ for $2p < m$, but u cannot be approximated by smooth maps $\varphi : S^m \to S^{m-1}$ for instance if $m - 1 < 2p < m$ (see Theorem 2.1 below).

The problem of density of smooth maps in $W^{1,p}(M; N)$ has been solved by Bethuel [2] and later completed by Hang-Lin [8]. The strategy for $W^{1,p}$ in [2,8] relies on some gluing arguments which cannot be directly applied in the $W^{2,p}$-setting. In a work in preparation (see [5]) we address the question whether $C^\infty(M, N)$ is strongly dense in $W^{2,p}(M, N)$ for every $p \geq 1$. Our goal in this Note is to present some of our ideas in a simpler case, namely $m - 1 < 2p < m$. In Section 3 below we prove the following

Theorem 1.1 *Let $m - 1 < 2p < m$. Then, $C^\infty(M; N)$ is dense in $W^{2,p}(M; N)$ if, and only if, the homotopy group $\pi_{m-1}(N)$ is trivial.*

When $\pi_{m-1}(N) \neq \{0\}$, $C^\infty(M; N)$ is not dense in $W^{2,p}(M; N)$. However,

Theorem 1.2 *Let $m - 1 < 2p < m$. Then, every element in $W^{2,p}(M; N)$ can be strongly approximated by maps in $W^{2,p}(M; N)$ which are smooth except on finitely many points.*

In the special case when $n = m - 1$ and $N = S^{m-1}$, this result can be obtained using some tools in [4].

2. Closure of smooth maps in $W^{2,p}(S^m; S^{m-1})$

We now assume that $n = m - 1$, $M = S^m$ and $N = S^{m-1}$. Since $\pi_{m-1}(S^{m-1}) = \mathbb{Z}$, it follows from Theorem 1.1 that $C^\infty(S^m; S^{m-1})$ is not dense in $W^{2,p}(S^m, S^{m-1})$ if $m - 1 < 2p < m$. The reason for this lack of density is the existence of "topological (point) singularities". The location and strength of such singularities can be detected using a simple tool introduced by Brezis-Coron-Lieb [6]: the distributional Jacobian "Jac". Indeed, for every $u \in W^{2,p}(S^m, S^{m-1})$, let

$$\operatorname{Jac}(u) = \frac{1}{m} \operatorname{div} D(u) \quad \text{in } \mathcal{D}'(S^m), \tag{5}$$

where

$$D(u) = (D_1, \ldots, D_m) \quad \text{and} \quad D_j = \det\left[\partial_1 u, \ldots, \partial_{j-1} u, u, \partial_{j+1} u, \ldots, \partial_m u\right]. \tag{6}$$

Here, the derivatives are computed in an orthogonal positively oriented frame. Note that $D(u)$ does not depend on the local system of coordinates, hence it is globally defined on S^m. Moreover, $D(u) \in L^1(S^m, \mathbb{R}^m)$ if $2p \geq m - 1$. Thus, $\operatorname{Jac}(u)$ is a well-defined distribution on S^m.

If u is smooth, then a straightforward computation gives $\operatorname{Jac}(u) = 0$ on S^m. On the other hand, the map given by (4) satisfies

$$\operatorname{Jac}(u) = \omega_m(\delta_P - \delta_N) \quad \text{in } \mathcal{D}'(S^m), \tag{7}$$

where $P, N \in S^m$ are the North and South poles of S^m and ω_m is the volume of the m-dimensional unit ball. This map u cannot be approximated by smooth ones in $W^{2,p}(S^m; S^{m-1})$ if $m - 1 < 2p < m$. Indeed,

Theorem 2.1 *Let $u \in W^{2,p}(S^m; S^{m-1})$, where $m - 1 < 2p < m$. Then, there exists a sequence $(u_k) \subset C^\infty(S^m, S^{m-1})$ such that $u_k \to u$ strongly in $W^{2,p}$ if, and only if,* $\operatorname{Jac}(u) = 0$ *in $\mathcal{D}'(S^m)$.*

The counterpart of this result for $W^{1,p}(S^m; S^{m-1})$ was established by Bethuel [1] for $p = m - 1$ and by Bethuel-Coron-Demengel-Hélein [3] for $m - 1 < p < m$.

Detailed proofs will be presented in [5].

3. Sketch of the proofs

We assume throughout that $m-1<2p<m$ and $\lambda>0$ is a small positive parameter, subject to an upper bound which depends only on M and N. In the proofs of Theorems 1.1, 1.2 and 2.1 we cover M with balls $\big(B_r(a_i)\big)_{i\in I}$ with finitely many overlappings. Inspired by [2], we say that $B_r(a_i)$ is a *bad ball* or a *good ball* for a map $u\in W^{2,p}(M;N)$ according to whether

$$\frac{1}{r^{m-2p}}\int\limits_{B_{3r}(a_i)}|Du|^{2p}\geq\lambda \quad\text{or}\quad \frac{1}{r^{m-2p}}\int\limits_{B_{3r}(a_i)}|Du|^{2p}<\lambda, \tag{8}$$

respectively. Roughly speaking, the strategy of our proofs is the following:
– on a bad ball, we replace u by a map which is continuous in $B_r(a_i)\setminus\{a_i\}$ but possibly singular at a_i;
– on a good ball, we replace u by a map which is continuous in $B_r(a_i)$.

We now present our two main tools. In what follows, we use the notation

$$\langle u\rangle_{W^{2,p}(\Omega)}=\int\limits_{\Omega}|Du|^{2p}+\int\limits_{\Omega}|D^2u|^p \tag{9}$$

for every $u\in W^{2,p}(M;\mathbb{R}^l)\cap L^\infty$ and every open set $\Omega\subset M$. By the Gagliardo-Nirenberg inequality, this quantity is always finite.

On a bad ball, we apply the

B Lemma *If $B_r(a)$ is a bad ball for $v\in W^{2,p}(M;N)$, then one can find $w\in W^{2,p}(M;N)$ such that*
(B_1) *w is continuous in $B_{2r}(a)\setminus\{a\}$;*
(B_2) *$w=v$ in $M\setminus B_{3r}(a)$;*
(B_3) $\|w-v\|_{L^p(M)}\leq Cr\|Dv\|_{L^{2p}(B_{3r}(a))}$;
(B_4) $\langle w-v\rangle_{W^{2,p}(M)}\leq C\langle v\rangle_{W^{2,p}(B_{3r}(a))}$.

The proof of the B Lemma is based on a smooth version of the standard extension method via zero-degree homogeneous maps (see e.g. [2, Lemma 3]).

Remark 1 By a Fubini-type argument we have $v|_{\partial B_t(a)}\in W^{2,p}\big(\partial B_t(a);N\big)$ for a.e. $t\in(0,3r)$. Since $2p>m-1$, it then follows from Morrey's inequality that $v|_{\partial B_t(a)}:\partial B_t(a)\to N$ is continuous. If we happen to know that this map is homotopic to a constant in $C^0\big(\partial B_t(a),N\big)$, then using an idea of Bethuel-Zheng [4, proof of Theorem 5] we can obtain a map w satisfying (B_1)–(B_4) which is continuous even at the center a of the ball. This fact is important in the proofs of Theorems 1.1 and 2.1.

For good balls, there exists $\lambda=\lambda(M,N)>0$ such that the following lemma holds true (recall that being a bad or a good ball depends on the choice of λ):

G Lemma *If $B_r(a)$ is a good ball for $v\in W^{2,p}(M;N)$, then one can find $w\in W^{2,p}(M;N)$ such that*
(G_1) *w is continuous in $B_{2r}(a)$;*
(G_2) *$w=v$ in $M\setminus B_{3r}(a)$;*
(G_3) $\|w-v\|_{L^p(M)}\leq Cr\|Dv\|_{L^{2p}(B_{3r}(a))}$;
(G_4) $\langle w-v\rangle_{W^{2,p}(M)}\leq C\langle v\rangle_{W^{2,p}(A)}$, *for some open set $A\subset B_{3r}(a)$ such that* $|A|^{1/2p}\leq Cr\|Du\|_{L^{2p}(B_{3r}(a))}$.

The proof of the G Lemma is based on the local projection of u into a small geodesic ball in N and resembles the proof of a similar result in the $W^{1,p}$-setting (see [2]).

Remark 2 In the B and G Lemmas, we can further require that for any (small) $\eta > 0$ we get a map w such that
– whenever v is continuous in some ball $B_t(b) \subset M$, then w is continuous in $B_{\tilde{t}}(b)$ with $\tilde{t} = t - \eta r$;
– whenever v is continuous in $B_t(b) \setminus \{b\} \subset M$, then w is continuous in $B_{\tilde{t}}(b) \setminus \{b\}$.
This observation is needed in order to prevent the loss of points of continuity when applying the B and G Lemmas to balls which may overlap.

Remark 3 In the counterparts of both lemmas for $W^{1,p}$-maps (in this case $m-1 < p < m$), one can take w such that $w = v$ in $M \setminus B_{\alpha r}$ for some $\alpha \in (2,3)$; in addition,
– v is continuous in $B_{\alpha r}(a) \setminus \{a\}$ if $B_r(a)$ is a bad ball;
– v is continuous in $B_{\alpha r}(a)$ if $B_r(a)$ is a good ball.
This underlines one of the main differences between the $W^{1,p}$- and the $W^{2,p}$-cases, namely the difficulty of gluing w and v in a neighborhood of $\partial B_{3r}(a)$.

The proofs of the B and G Lemmas will appear in [5]. We now show how to deduce Theorems 1.1, 1.2 and 2.1.

Proof of Theorem 1.2 Taking $r > 0$ sufficiently small, we can cover M with balls $(B_r(a_i))_{i \in I}$ in such a way that, for every $i \in I$, $B_{3r}(a_i)$ intersects at most θ balls $B_{3r}(a_j)$, where θ depends only on the dimension of M. We can thus split the set of indices I as $I = I_1 \cup \cdots \cup I_{\theta+1}$ so that for any $i = 1, \ldots, k$ and any distinct indices $j_1, j_2 \in I_i$ we have $B_{3r}(a_{j_1}) \cap B_{3r}(a_{j_2}) = \emptyset$.

Starting from $u_0 = u$, we construct maps $u_1, \ldots, u_{\theta+1} \in W^{2,p}(M;N)$ inductively as follows. Given $k \geq 0$ and u_k we apply the B Lemma or the G Lemma to the map u_k and to each ball $B_r(a_i)$ with $i \in I_{k+1}$ until we exhaust I_{k+1}; denote by u_{k+1} the map obtained by this procedure. We emphasize that $B_r(a_i)$ is considered to be bad or good with respect to u_k and not with respect to the original map u. Since the balls $\big(B_{3r}(a_i)\big)_{i \in I_{k+1}}$ are disjoint, we have

$$\|u_{k+1} - u_k\|_{L^p(M)} \leq Cr \|Du_k\|_{L^{2p}(M)} \quad \text{and} \quad \langle u_{k+1} - u_k \rangle_{W^{2,p}(M)} \leq C \langle u_k \rangle_{W^{2,p}(E_k)} \tag{10}$$

for some open set $E_k \subset M$ such that $|E_k|^{1/2p} \leq Cr \|Du_k\|_{L^{2p}(M)}$; E_k is the union of $B_{3r}(a_i)$ among the bad balls with the sets A arising from the good balls.
By induction, it follows from (10) that for every $k = 1, \ldots, \theta+1$ we have

$$\|u_k - u\|_{L^p(M)} \leq C_k\, r\, \langle u \rangle_{W^{2,p}(M)}^{1/2p} \quad \text{and} \quad \langle u_k - u \rangle_{W^{2,p}(M)} \leq C_k \langle u \rangle_{W^{2,p}(\tilde{E}_k)}, \tag{11}$$

where $\tilde{E}_k = \bigcup_{j=1}^k E_j$ is an open set with $|\tilde{E}_k| \leq C_k\, r^{2p}\, \langle u \rangle_{W^{2,p}(M)}$. Let $w_r = u_{\theta+1}$.

By (11), $w_r \to u$ strongly in $W^{2,p}$ as $r \to 0$. Moreover, applying Remark 2 with $\eta = 1/\theta$ it follows that w_r is continuous on the set $B_r(a_i) \setminus \{a_i\}$ for every $i \in I$. Indeed, taking k such that $i \in I_k$, then u_k is continuous in $B_{2r}(a_i) \setminus \{a_i\}$. Using Remark 2 inductively to $u_{k+1}, \ldots, u_{\theta+1}$, we deduce that $u_{\theta+1}$ is continuous in $B_{t_k}(a_i) \setminus \{a_i\}$ with $t_k = 2r - \frac{\theta+1-k}{\theta}\, r$. Since $t_k \geq r$, the conclusion follows.

Thus, w_r is continuous on $M \setminus \bigcup_{i \in I} \{a_i\}$. We can now strongly approximate w_r in $W^{2,p}(M;N)$ by maps which are *smooth* on $M \setminus \bigcup_{i \in I} \{a_i\}$. The proof of Theorem 1.2 is complete. □

Remark 4 Since in the proof of Theorem 1.2 we have produced at most one singularity for each bad ball used, an inspection of the proof actually shows that the number of points of discontinuity of the map w_r grows at most like $o(r^{2p-m})$ as $r \to 0$.

Proof of Theorem 1.1 ($\Rightarrow$) This implication can be established as in the $W^{1,p}$-case (see e.g. [9, Theorem 4.4]).

($\Leftarrow$) We proceed as in the proof of Theorem 1.2. Since $\pi_{m-1}(N) = \{0\}$, every continuous map $h \in C^0\big(\partial B_t(a); N\big)$ is homotopic to some constant for every $t > 0$ small. By Remark 1, we can apply the B Lemma to get a continuous map even on bad balls. Thus, the map w_r is continuous everywhere on M. Approximating w_r by smooth maps in $W^{2,p}(M; N)$ we get the result. $\square$

Proof of Theorem 2.1 ($\Rightarrow$) Assume that there exists a sequence $(u_k) \subset C^\infty(M; N)$ such that $u_k \to u$ strongly in $W^{2,p}$. For every $k \geq 1$, $\mathrm{Jac}\,(u_k) = 0$ on S^m. Hence, by continuity of the distributional Jacobian we have $\mathrm{Jac}\,(u) = 0$ in $\mathcal{D}'(S^m)$.

($\Leftarrow$) If $\mathrm{Jac}\,(u) = 0$, then it follows from the main result in [1] that for every $a \in M$ and for a.e. $t > 0$ small $u|_{\partial B_t(a)}$ is continuous and its degree $\deg\big(u|_{\partial B_t(a)}\big)$ vanishes; thus $u|_{\partial B_t(a)}$ is contractible in $C^0\big(\partial B_t(a); S^{m-1}\big)$. The conclusion then follows as in the proof of Theorem 1.1. $\square$

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