SEMILINEAR ELLIPTIC EQUATIONS AND SYSTEMS WITH DIFFUSE MEASURES

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A Lucio

"Tu se' lo mio maestro e 'l mio autore"
(Dante, Inferno, I, 85)

Abstract. We study the equation $-\Delta u + g(x, u) = \mu$, where $g(\cdot, s)$ is finite outside sets of zero H^1-capacity, $\forall s \in \mathbb{R}$, and μ is a diffuse measure. As an application, we provide a positive answer to a question of Lucio Boccardo concerning existence of solutions of an elliptic system with absorption.

Contents

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. The original motivation of this work was a question of L. Boccardo concerning the existence of solutions of the system

$$
\begin{cases}
-\Delta u + u^{p_1} v^{q_1} = f_1 & \text{in } \Omega, \\
-\Delta v + u^{p_2} v^{q_2} = f_2 & \text{in } \Omega, \\
\qquad u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where f_1, f_2 are given nonnegative functions in $L^1(\Omega)$ and $p_i, q_i > 0$ for $i = 1, 2$. We prove in this paper that (1.1) always has a solution; see Theorem 1.1 below.

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More generally, we show that (1.1) still has a solution if f_1, f_2 are not necessarily L^1-functions, but diffuse measures.

We recall that a finite measure μ in Ω is diffuse if for every Borel set $E \subset \Omega$

$$\operatorname{cap}(E) = 0 \quad \Longrightarrow \quad |\mu|(E) = 0,$$

where "cap" denotes the Newtonian H^1-capacity. According to a result of Boccardo-Gallouët-Orsina [2], μ is diffuse if, and only if, $\mu \in L^1(\Omega) + H^{-1}(\Omega)$, i.e.

$$\mu = f - \Delta u \quad \text{in } \mathcal{D}'(\Omega), \tag{1.2}$$

for some $f \in L^1(\Omega)$ and $u \in H_0^1(\Omega)$.

One of our main results is the

THEOREM 1.1. *Assume that μ_1, μ_2 are diffuse measures in Ω and $g_1, g_2 \in C(\mathbb{R} \times \mathbb{R})$ satisfy*

(a_1) *for every $t \in \mathbb{R}$, $g_1(\cdot, t)$ is nondecreasing and $g_1(0,t) = 0$;*

(a_2) *for every $s \in \mathbb{R}$, $g_2(s, \cdot)$ is nondecreasing and $g_2(s, 0) = 0$.*

Then, the system

$$\begin{cases} -\Delta u + g_1(u,v) = \mu_1 & \text{in } \Omega, \\ -\Delta v + g_2(u,v) = \mu_2 & \text{in } \Omega, \\ \qquad\quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has a solution $(u, v) \in L^1(\Omega) \times L^1(\Omega)$.

Our proof of Theorem 1.1 is based on Schauder's fixed point theorem. Some important tools are the notions of "quasi-L^1 functions" and "equidiffuse sequences" (see Sections 2 and 3 below) as well as the following result concerning the existence of solutions of the *scalar* equation

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

THEOREM 1.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

$$g(x,s)s \geq 0 \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R} \tag{1.5}$$

and

$$\sup_{|s| \leq t} |g(x,s)| \leq G_t(x) + H_t(x) \quad \text{for a.e. } x \in \Omega,\ \forall t > 0, \tag{1.6}$$

where $G_t : \Omega \to \mathbb{R}$ is quasifinite and $H_t \in L^1(\Omega)$. If μ is a diffuse measure in Ω, then there exist a smallest and a largest solution of (1.4).

We recall that a measurable function $G : \Omega \to \mathbb{R}$ is quasifinite if for every $\varepsilon > 0$ and every $K \subset \Omega$ compact there exist $M > 0$ and an open set $\omega \subset \Omega$ such that $\operatorname{cap}(\omega) < \varepsilon$ and $|G| \leq M$ a.e. on $K \setminus \omega$ (see [13]).

The study of the equation (1.4) with datum μ in $L^1(\Omega)$ was initiated by Brezis-Strauss [7]. Later, Gallouët-Morel [11] studied the existence of solutions of (1.4) when $\mu \in L^1(\Omega)$ and

$$\sup_{|s| \leq t} |g(\cdot, s)| \in L^1(\Omega) \quad \forall t > 0;$$

in other words, when (1.6) holds with $G_t \equiv 0$.

REMARK 1.1. Quasifinite functions need not belong to $L^1(\Omega)$; for example, $1/\|x\|^N$ is quasifinite in B_1 but $1/\|x\|^N \notin L^1(B_1)$. Conversely, L^1-functions need not be quasifinite; for instance, $1/|x_1|^\alpha \in L^1(B_1)$ for any $0 < \alpha < 1$ but this function is not quasifinite since the set $[x_1 = 0] \cap B_1$ has positive H^1-capacity. This explains the presence of both G_t and H_t in (1.6). The possibility of allowing the term G_t is needed in the proof of Theorem 1.1.

It follows from Theorem 1.2 that (1.4) always has a solution with

$$g(x,s) = \frac{a(s)}{\|x\|^\alpha} \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

for every $\alpha > 0$ and $a \in C(\mathbb{R})$, where $a(s)s \geq 0$, $\forall s \in \mathbb{R}$. On the other hand, the function g given by

$$g(x,s) = \frac{s}{|x_1|^\alpha} \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}$$

does not satisfy condition (1.6) if $\alpha \geq 1$. Actually, for such g we prove the following

THEOREM 1.3. *Let $1 \leq \alpha < 2$. If $u \in L^1(B_1)$ is such that $u/|x_1|^\alpha \in L^1(B_1)$ and*

$$-\int_{B_1} u\,\Delta\zeta + \int_{B_1} \frac{u}{|x_1|^\alpha}\zeta \geq 0 \quad \forall \zeta \in C_0^2(\overline{B}_1),\ \zeta \geq 0 \text{ in } B_1, \tag{1.7}$$

then

$$u = 0 \quad \text{a.e. in } B_1. \tag{1.8}$$

Hence, according to Theorem 1.3 the equation

$$\begin{cases} -\Delta u + \dfrac{u}{|x_1|^\alpha} = \mu & \text{in } B_1, \\ \qquad\qquad u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.9}$$

has *no* solution if $1 \leq \alpha < 2$ and μ is a nonnegative measure, unless $\mu = 0$. If $\alpha \geq 2$, then we show in Section 9 below that there do exist functions $u \in L^1(\Omega)$ satisfying (1.9) for every $\mu \in L^1(B_1)$, in the sense that $u/|x_1|^\alpha \in L^1(B_1)$ and

$$-\int_{B_1} u\,\Delta\zeta + \int_{B_1} \frac{u}{|x_1|^\alpha}\zeta = \int_{B_1} \zeta\,d\mu \quad \forall \zeta \in C_0^2(\overline{B}_1). \tag{1.10}$$

In [8], Dal Maso-Mosco studied the question of existence and uniqueness of solutions for problems of the form

$$\begin{cases} -\Delta u + \nu u = f & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where ν is a nonnegative diffuse Borel measure (possibly with infinite mass) and $f \in L^2(\Omega)$. Given $\alpha \geq 1$ and $\Omega = B_1$, take

$$\nu_\alpha = \frac{1}{|x_1|^\alpha}\,dx \quad \text{and} \quad X_\alpha = H_0^1(B_1) \cap L^2(B_1;\nu_\alpha). \tag{1.12}$$

It follows from [8] that for every $f \in L^2(B_1)$ there exists a unique $u \in X_\alpha$ such that

$$\int_{B_1} \nabla u \cdot \nabla v + \int_{B_1} \frac{u}{|x_1|^\alpha} v = \int_{B_1} f v \quad \forall v \in X_\alpha. \tag{1.13}$$

We point out that their result does not contradict Theorem 1.3 above. In fact, it follows from Proposition 9.1 below that (1.13) holds for some $u \in X_\alpha$ if, and only if,

$$\int_{B_1} \nabla u \cdot \nabla \varphi + \int_{B_1} \frac{u}{|x_1|^\alpha} \varphi = \int_{B_1} f\varphi \quad \forall \varphi \in C_0^\infty\big(B_1^+ \cup B_1^-\big), \tag{1.14}$$

where

$$B_1^+ = \big\{x \in B_1;\ x_1 > 0\big\} \quad \text{and} \quad B_1^- = \big\{x \in B_1;\ x_1 < 0\big\}.$$

Moreover, for every function $u \in X_\alpha$,

$$u \in H_0^1\big(B_1^+\big) \cup H_0^1\big(B_1^-\big); \tag{1.15}$$

see Lemma 9.1 below. Hence, for every $\alpha \geq 1$, to find a solution of (1.11) in B_1 in the sense of [8] amounts to solve two *independent* Dirichlet problems on B_1^+ and B_1^-, for which we know there is a solution. Indeed, it suffices to apply Theorem 1.2 in B_1^+ and B_1^- with $G_t(x) = t/|x_1|^\alpha$ and $F_t(x) = 0$. Note that in this case the parameter α plays no role whatsoever. The fact that the solution u obtained this way satisfies (1.15) when $f \in L^2(B_1)$ follows from standard elliptic estimates (see [8, 14]).

2. Characterization of quasi-L^1 functions

In this section we discuss the concept of "quasi-L^1 functions" presented below:

Definition 2.1. *A measurable function $F : \Omega \to \mathbb{R}$ is quasi-L^1 if for every $\varepsilon > 0$ and every $K \subset \Omega$ compact there exists an open set $\omega \subset \Omega$ such that* $\operatorname{cap}(\omega) < \varepsilon$ *and $F \in L^1(K \setminus \omega)$.*

The motivation of Definition 2.1 comes from the well-known notion of quasicontinuity, which we recall below:

Definition 2.2. *A measurable function $G : \Omega \to \mathbb{R}$ is quasicontinuous if for every $\varepsilon > 0$ there exists an open set $\omega \subset \Omega$ such that* $\operatorname{cap}(\omega) < \varepsilon$ *and G is continuous on $\Omega \setminus \omega$.*

For example, if $u \in L^1(\Omega)$ is such that

$$\left| \int_\Omega u\, \Delta\varphi \right| \leq C\|\varphi\|_{L^\infty} \quad \forall \varphi \in C_0^\infty(\Omega),$$

then by the Riesz Representation Theorem Δu is a finite measure in Ω; hence, there exists a quasicontinuous function $G : \Omega \to \mathbb{R}$ such that $G = u$ a.e. (see e.g. [4, Lemma 1]). In particular, u is quasi-L^1.

Clearly,

$$F \text{ quasicontinuous} \quad \Longrightarrow \quad F \text{ quasifinite} \quad \Longrightarrow \quad F \text{ quasi-}L^1. \tag{2.1}$$

Simple examples show that the reverse implications are not true. Note in addition that

- (A_1) If $F_1, F_2 : \Omega \to \mathbb{R}$ are measurable functions such that $F_1 = F_2$ a.e. and if F_1 is quasi-L^1, then F_2 is also quasi-L^1;
- (A_2) If F is quasi-L^1 in Ω and $u : \Omega \to \mathbb{R}$ is a measurable function such that $|u| \leq F$ a.e., then u is also quasi-L^1.

Similar properties also hold for quasifinite functions, but their counterparts for quasicontinuous functions are false.

We prove the following characterization of quasi-L^1 functions:

Proposition 2.1. *A measurable function F is quasi-L^1 in Ω if, and only if, there exist G quasifinite in Ω and $H \in L^1(\Omega)$ such that*

$$|F| \le G + H \quad a.e. \tag{2.2}$$

Proof. The implication "$\Leftarrow$" is clear since $G+H$ is quasi-L^1 by (2.1), and so F is quasi-L^1 by (A_2). Conversely, assume that $F : \Omega \to \mathbb{R}$ is a quasi-L^1 function. We split the proof in two steps:

Step 1. Assume in addition that F has compact support in Ω. Then, given $\varepsilon > 0$, (2.2) holds for some G quasifinite and $H \in L^1(\Omega)$ such that $\|H\|_{L^1} < \varepsilon$.

Since F has compact support, for each $k \ge 1$ one can find an open set $\omega_k \subset\subset \Omega$ with

$$\operatorname{cap}(\omega_k) < \frac{1}{2^k} \quad \text{and} \quad F \in L^1(\Omega \setminus \omega_k). \tag{2.3}$$

We can assume that the sequence (ω_k) is non-increasing. For otherwise, we could take $\tilde{\omega}_k = \bigcup_{j=k+1}^{+\infty} \omega_j$, $\forall j \ge 1$; then, $(\tilde{\omega}_k)$ is non-increasing and still satisfies (2.3). For every $k \ge 1$, choose $M_k > 0$ sufficiently large so that

$$\int\limits_{\substack{\Omega\setminus\omega_k \\ [|F|>M_k]}} |F| < \frac{\varepsilon}{2^{k+1}}.$$

For every $x \in \Omega$, let

$$G(x) = \sum_{k\ge1} M_k\, \chi_{\omega_{k-1}\setminus\omega_k}(x),$$
$$H(x) = \sum_{k\ge1} (|F(x)| - M_k)^+ \chi_{\omega_{k-1}\setminus\omega_k}(x),$$

where $\omega_0 := \Omega$. Then,

$$|F| \le G + H \text{ a.e.,} \quad H \in L^1(\Omega) \quad \text{and} \quad \|H\|_{L^1} \le \varepsilon.$$

Since G is uniformly bounded on $\Omega \setminus \omega_j$ for every $j \ge 1$, it follows that G is quasifinite. This concludes the proof of Step 1.

Step 2. Proof of Theorem 2.1 completed.

Write $\Omega = \bigcup_{n\ge1} \Omega_n$ as an increasing union of open sets $\Omega_n \subset\subset \Omega$, and define $\Omega_0 = \emptyset$. Applying the previous step to $F\chi_{\Omega_n}$, one finds a quasifinite function $G_n : \Omega \to \mathbb{R}$ and $H_n \in L^1(\Omega)$ supported in $\overline{\Omega}_n$ such that

$$|F| \le G_n + H_n \text{ a.e. in } \Omega_n \quad \text{and} \quad \|H_n\|_{L^1} \le \frac{1}{2^n}.$$

The functions

$$G(x) = \sum_{n\ge1} G_n(x)\, \chi_{\Omega_{n-1}\setminus\Omega_n}(x) \quad \text{and} \quad H(x) = \sum_{n\ge1} H_n(x) \quad \forall x \in \Omega$$

satisfy all the required properties. The proof is complete. □

We warn the reader of the following facts:

(A_3) If G is quasifinite, then G need not belong to $L^1(\Omega)$. Indeed, $G(x) = \frac{1}{\|x\|^\alpha}$ is quasifinite in the ball $B_1 \subset \mathbb{R}^N$ for every $\alpha > 0$, but $G \in L^1(B_1)$ if, and only if, $\alpha < N$.

(A_4) If G_1 and G_2 are quasifinite functions in Ω such that $G_1 = G_2$ a.e., then it need *not* be true that $\int_\Omega G_1\, d\mu$ and $\int_\Omega G_2\, d\mu$ coincide for a given diffuse measure μ, even if G_1, G_2 are bounded functions; indeed, let S be a segment in $\mathbb{R}^2$, $G_1 = 0$, $G_2 = \chi_S$, and μ be the restriction of the 1-dimensional Hausdorff measure to S.

3. Properties of equidiffuse sequences

We denote by $\mathcal{M}(\Omega)$ the space of finite measures μ in Ω equipped with the norm

$$\|\mu\|_{\mathcal{M}} := |\mu|(\Omega) = \int_\Omega |\mu|. \tag{3.1}$$

We recall the (see [6])

Definition 3.1. *Given a sequence of finite measures (μ_n) in Ω, we say that (μ_n) is equidiffuse if*

(*i*) *(μ_n) is bounded in $\mathcal{M}(\Omega)$;*

(*ii*) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for every Borel set $E \subset \Omega$*

$$\operatorname{cap}(E) < \delta \quad \Longrightarrow \quad |\mu_n|(E) < \varepsilon \quad \forall n \geq 1. \tag{3.2}$$

If a sequence (μ_n) is equidiffuse, then each measure μ_n is diffuse in view of the following

Lemma 3.1. *Let μ be a finite measure in Ω. Then, μ is diffuse if, and only if,*

$$\lim_{\operatorname{cap}(E)\to 0} \mu(E) = 0.$$

Proof. $(\Leftarrow)$ Given $\varepsilon > 0$, let $\delta > 0$ be such that

$$\operatorname{cap}(E) < \delta \quad \Longrightarrow \quad |\mu|(E) < \varepsilon.$$

If $E_0 \subset \Omega$ is a Borel set such that $\operatorname{cap}(E_0) = 0$, then $|\mu|(E_0) < \varepsilon$, $\forall \varepsilon > 0$. Thus, $|\mu|(E_0) = 0$ and μ is diffuse.

$(\Rightarrow)$ We may assume that $\mu \geq 0$; the case of signed measures then follows by applying the conclusion to $|\mu|$. Reasoning by contradiction, suppose that there exist $\varepsilon_0 > 0$ and a sequence (E_n) of Borel subsets of Ω such that $\operatorname{cap}(E_n)$ tends to zero but $\mu(E_n) > \varepsilon_0$ for every $n \geq 1$. If the sequence (E_n) is decreasing, then the set $E = \bigcap_{n=1}^{+\infty} E_n$ has zero capacity and is such that $\mu(E) \geq \varepsilon_0$, a contradiction. If the sequence (E_n) is not decreasing, consider a subsequence (E_{n_j}) such that $\operatorname{cap}(E_{n_j}) < 2^{-j}$ for every $j \geq 1$. Then, the sequence (F_k) given by $F_k = \bigcup_{j=k+1}^{+\infty} E_{n_j}$ is decreasing, with capacity smaller than 2^{-k}, and is such that $\mu(F_k) > \varepsilon_0$. The conclusion then follows as before. □

A first example of an equidiffuse sequence (μ_n) is the

(B_1) $\mu_n = \mu$, where μ is a given diffuse measure.

This follows from Lemma 3.1 above. Other examples are

(B_2) $\mu_n = \rho_n * \mu$, where μ is diffuse and (ρ_n) is a sequence of mollifiers;

(B_3) $\mu_n = f_n$, where (f_n) is an equi-integrable sequence in $L^1(\Omega)$;

(B_4) $\mu_n = \Delta u_n$, where (u_n) is a bounded sequence in $H_0^1(\Omega)$.

Clearly, sums of equidiffuse sequences are still equidiffuse. In view of (B_3)–(B_4), one deduces that if (μ_n) is a bounded sequence of measures such that

$$\mu_n = f_n - \Delta u_n \quad \text{in } \mathcal{D}'(\Omega), \tag{3.3}$$

where f_n and u_n are as above, then (μ_n) is equidiffuse. It would be interesting to have a characterization of equidiffuse sequences in the spirit of the Boccardo-Gallouët-Orsina decomposition (1.2):

Open Problem 1. *Let (μ_n) be an equidiffuse sequence converging weakly* in Ω. Can one find $(f_n) \subset L^1(\Omega)$ and $(u_n) \subset H_0^1(\Omega)$ such that*

- (b_1) *$\mu_n = f_n - \Delta u_n$ in $\mathcal{D}'(\Omega)$;*
- (b_2) *(f_n) converges strongly in $L^1(\Omega)$;*
- (b_3) *(u_n) is bounded in $H_0^1(\Omega)$?*

A connection between Definitions 2.1 and 3.1 is provided by the following

Lemma 3.2. *Let (f_n) be a sequence in $L^1(\Omega)$ such that*

- (c_1) *$f_n \to f$ a.e.;*
- (c_2) *$|f_n| \le F$ a.e., $\forall n \ge 1$, for some quasi-L^1 function F in Ω;*
- (c_3) *(f_n) is equidiffuse.*

Then,

$$f_n \to f \quad \text{in } L^1(\Omega; \rho_0\, dx), \tag{3.4}$$

where $\rho_0(x) = \operatorname{dist}(x, \partial\Omega)$, $\forall x \in \Omega$.

A variant of this result was established by Lin-Ponce-Yang [13]. If F is a (genuine) L^1-function in Ω, then Lemma 3.2 just follows from Lebesgue's dominated convergence theorem (in which case (c_3) is not needed). For general quasi-L^1 functions F, the conclusion (3.4) need not be true if (c_3) fails.

Proof. Replacing f_n by $f_n - f$ if necessary, we may assume that

$$f_n \to 0 \quad \text{a.e.} \tag{3.5}$$

For every open set $A \subset\subset \Omega$, we show that

$$f_n \to 0 \quad \text{in } L^1(A). \tag{3.6}$$

By (c_3), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{cap}(E) < \delta \quad \Longrightarrow \quad \int_E |f_n| < \varepsilon \quad \forall n \ge 1. \tag{3.7}$$

Since F is quasi-L^1, one finds an open set $\omega \subset \Omega$ such that $\operatorname{cap}(\omega) < \delta$ and $F \in L^1(\overline{A} \setminus \omega)$. Thus, by (c_1), (c_2), and dominated convergence,

$$f_n\, \chi_{A\setminus\omega} \to 0 \quad \text{in } L^1(A).$$

Moreover, since we have $\operatorname{cap}(\omega) < \delta$, it follows from (3.7) that

$$\int_\omega |f_n| < \varepsilon \quad \forall n \ge 1.$$

Thus,

$$\limsup_{n\to\infty} \int_A |f_n| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\lim_{n\to\infty} \int_A |f_n| = 0.$$

This establishes (3.6) for every $A \subset\subset \Omega$. Since (f_n) is bounded in $L^1(\Omega)$, (3.4) follows. □

4. Stability of solutions of (1.4)

We recall that a function $u \in L^1(\Omega)$ is a solution of

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

for a given finite measure μ in Ω if

$$-\int_\Omega u\Delta\zeta = \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),$$

where

$$C_0^2(\overline{\Omega}) = \big\{ \zeta \in C^2(\overline{\Omega});\ \zeta = 0 \text{ on } \partial\Omega \big\}.$$

We say that $u \in L^1(\Omega)$ satisfies

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, if $g(\cdot, u) \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(x,u)\zeta = \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

The main result of this section is the

Proposition 4.1. *Let $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ be a sequence of Carathéodory functions such that*

(i) $g_n(x,s)s \geq 0$ for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$;
(ii) $g_n(\cdot, s_n) \to g(\cdot, s)$ a.e. whenever $s_n \to s$ in $\mathbb{R}$;
(iii) for every $t \in \mathbb{R}$, there exists a quasi-L^1 function F_t in Ω such that

$$\sup_{|s|\leq t} |g_n(x,s)| \leq F_t(x) \quad \text{for a.e. } x \in \Omega, \quad \forall n \geq 1.$$

Given a diffuse measure μ in Ω, assume that

$$\begin{cases} -\Delta u_n + g_n(x,u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

has a solution u_n, $\forall n \geq 1$. Then, up to subsequences, u_n converges strongly in $L^1(\Omega)$ to a solution u of (4.2).

In order to prove Proposition 4.1, we first recall some known results. We start with the uniform estimates for (4.1) (see [17]):

Lemma 4.1. *Every solution u of (4.1) belongs to $W_0^{1,q}(\Omega)$, for $1 \leq q < \frac{N}{N-1}$, and*

$$\|u\|_{W_0^{1,q}} \leq C_q \|\mu\|_{\mathcal{M}}. \tag{4.4}$$

The analog of Lemma 4.1 for the semilinear problem (4.2) is the following (see [3, Proposition B.3])

Lemma 4.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

$$g(x,s)s \geq 0 \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R}. \tag{4.5}$$

Then, every solution u of (4.2) *satisfies*

$$\|g(\cdot,u)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}. \tag{4.6}$$

In particular, (4.4) *holds.*

The notion of (weak) sub and supersolutions of (4.2) we consider in this paper is given below:

Definition 4.1. *A subsolution of* (4.2) *is a function $\underline{u} \in L^1(\Omega)$ such that $g(\cdot,\underline{u}) \in L^1(\Omega)$ and*

$$-\int_\Omega \underline{u}\,\Delta\zeta + \int_\Omega g(x,\underline{u})\,\zeta \leq \int_\Omega \zeta\,d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{4.7}$$

Analogously, a supersolution of (4.2) *is a function $\overline{u} \in L^1(\Omega)$ such that $g(\cdot,\overline{u}) \in L^1(\Omega)$ and*

$$-\int_\Omega \overline{u}\,\Delta\zeta + \int_\Omega g(x,\overline{u})\,\zeta \geq \int_\Omega \zeta\,d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{4.8}$$

We show that if g satisfies (4.5), then sub and supersolutions of (4.2) have "bounds" from above and from below:

Proposition 4.2. *Let μ be a finite measure in Ω, and let $\underline{U}$ and $\overline{U}$ be the (unique) solutions of*

$$\begin{cases} -\Delta\underline{U} = -\mu^- & \text{in } \Omega, \\ \underline{U} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta\overline{U} = \mu^+ & \text{in } \Omega, \\ \overline{U} = 0 & \text{on } \partial\Omega. \end{cases}$$

If (4.5) *holds, then any subsolution $\underline{w}$ of* (4.2) *satisfies $\underline{w} \leq \overline{U}$ and any supersolution $\overline{w}$ of* (4.2) *satisfies $\underline{U} \leq \overline{w}$.*

We present a proof of Proposition 4.2 based on the following version of Kato's inequality (see [3, Proposition B.5]):

Lemma 4.3. *Let $w \in L^1(\Omega)$ and $f \in L^1(\Omega;\rho_0\,dx)$ be such that*

$$-\int_\Omega w\,\Delta\zeta \leq \int_\Omega f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega.$$

Then,

$$-\int_\Omega w^+\,\Delta\zeta \leq \int_{[w\geq 0]} f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega.$$

Proof of Proposition 4.2. If $\underline{w}$ is a subsolution of (4.2), then for every $\zeta \in C_0^2(\overline{\Omega})$ with $\zeta \geq 0$ in Ω we have

$$\begin{aligned} -\int_\Omega (\underline{w} - \overline{U})\,\Delta\zeta &\leq -\int_\Omega g(x,\underline{w})\,\zeta + \int_\Omega \zeta\,d\mu - \int_\Omega \zeta\,d\mu^+ \\ &= -\int_\Omega g(x,\underline{w})\,\zeta - \int_\Omega \zeta\,d\mu^- \\ &\leq -\int_\Omega g(x,\underline{w})\,\zeta. \end{aligned}$$

Therefore, by Lemma 4.3,

$$-\int_\Omega (\underline{w}-\overline{U})^+\,\Delta\zeta \le -\int\limits_{[\underline{w}\ge\overline{U}]} g(x,\underline{w})\,\zeta \quad \forall\zeta\in C_0^2(\overline{\Omega}),\ \zeta\ge 0 \text{ in } \Omega. \tag{4.9}$$

Being $\overline{U}\ge 0$, we have $\underline{w}\ge 0$ on the set $[\underline{w}\ge\overline{U}]$; hence, by (4.5),

$$g(x,\underline{w})\ge 0 \quad \text{a.e. on } [\underline{w}\ge\overline{U}]. \tag{4.10}$$

Combining (4.9)–(4.10), we deduce that

$$-\int_\Omega (\underline{w}-\overline{U})^+\,\Delta\zeta \le 0 \quad \forall\zeta\in C_0^2(\overline{\Omega}),\ \zeta\ge 0 \text{ in } \Omega.$$

Therefore, $(\underline{w}-\overline{U})^+=0$ a.e. in Ω; equivalently, $\underline{w}\le\overline{U}$ a.e. The inequality $\overline{w}\ge\underline{U}$ is proved in the same way. $\square$

The following result will be useful in the sequel

Lemma 4.4. *Let (μ_n) be a sequence of measures in Ω such that*

$$\mu_n = f_n - \Delta u_n \quad \textit{in } \mathcal{D}'(\Omega) \tag{4.11}$$

for some $f_n\in L^1(\Omega)$ and $u_n\in W_0^{1,1}(\Omega)$ such that $f_nu_n\ge 0$ a.e., $\forall n\ge 1$. If (μ_n) is equidiffuse, then (f_n) is also equidiffuse.

We refer the reader to [6] for a proof of Lemma 4.4; see also [15].

We now present the

Proof of Proposition 4.1. Using *(i)* and Lemma 4.2, we deduce that (u_n) is bounded in $W_0^{1,q}(\Omega)$ for every $1\le q<\frac{N}{N-1}$. Therefore, by the Rellich-Kondrachov theorem, there exists a subsequence of (u_n) (still denoted by (u_n)) which converges strongly in $L^1(\Omega)$, and almost everywhere, to a function u. In particular, by *(ii)*

$$g_n(\cdot,u_n)\to g(\cdot,u) \quad \text{a.e.} \tag{4.12}$$

We claim that $\big(g_n(\cdot,u_n)\big)$ also satisfies assumptions (c_2)–(c_3) of Lemma 3.2. In fact, by Lemma 4.4, this sequence is equidiffuse. By Proposition 4.2, we know that

$$\underline{U}\le u_n\le\overline{U} \quad \text{a.e.,} \quad \forall n\ge 1.$$

Let $V=\max\big\{-\underline{U},\overline{U}\big\}$; V is quasicontinuous since $\underline{U}$ and $\overline{U}$ are quasicontinuous. In particular, V is quasifinite and $|u_n|\le V$ a.e. Let

$$W(x)=\sup_{n\in\mathbb{N}}\sup_{|s|\le V(x)}|g_n(x,s)| \quad \text{for a.e. } x\in\Omega.$$

Claim 1. W is measurable.

Indeed, note that for each $n\ge 1$

$$G_n(x,t)=\sup_{|s|\le t}|g_n(x,s)| \quad \forall (x,t)\in\Omega\times\mathbb{R}$$

is a Carathéodory function (see e.g. [10]). Since V is measurable, it follows that $G_n(\cdot,V)$ is measurable as well. Hence, W is also measurable being the supremum of countably many measurable functions.

Claim 2. W is quasi-L^1.

Given $K \subset \Omega$ and $\varepsilon > 0$, let $M > 0$ and $\omega_1 \subset \Omega$ be an open set such that $\operatorname{cap}(\omega_1) < \varepsilon/2$ and

$$|V(x)| \le M \quad \text{a.e. on } K \setminus \omega_1.$$

Let $\omega_2 \subset \Omega$ be such that $\operatorname{cap}(\omega_2) < \varepsilon/2$ and

$$F_M \in L^1(K \setminus \omega_2).$$

Take $\omega_0 = \omega_1 \cup \omega_2$. Then, $\operatorname{cap}(\omega_0) < \varepsilon$ and

$$0 \le W(x) \le \sup_{n\in\mathbb{N}} \sup_{|s|\le M} |g_n(x,s)| \le F_M(x) \quad \text{a.e. on } K \setminus \omega_0.$$

Thus, the function W belongs to $L^1(K \setminus \omega_0)$, and is therefore quasi-L^1. This establishes Claim 2.

Since W is quasi-L^1,

$$|g_n(\cdot, u_n)| \le W \quad \text{a.e.,} \quad \forall n \ge 1,$$

and $\big(g_n(\cdot, u_n)\big)$ is equidiffuse, by Lemma 3.2 we have

$$g_n(\cdot, u_n) \to g(\cdot, u) \quad \text{in } L^1(\Omega; \rho_0\, dx).$$

Note that for every function ζ in $C_0^2(\overline{\Omega})$ there exists a constant $C > 0$ such that $|\zeta| \le C\,\rho_0$. Hence, the convergence of $\big(g_n(\cdot, u_n)\big)$ in $L^1(\Omega; \rho_0\, dx)$ is enough in order to pass to the limit in the weak formulation of (4.3) and we get

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(x,u)\zeta = \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

In view of the pointwise convergence (4.12), by Lemma 4.2 and Fatou's lemma we have $g(\cdot, u) \in L^1(\Omega)$. Thus, u is a solution of (4.2). □

5. A variant of the method of sub and supersolutions

Thanks to the stability result in the previous section, we can now establish the following version of the method of sub and supersolutions for problem (1.4):

Proposition 5.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (1.5)–(1.6). *Given a diffuse measure μ in Ω, assume that* (1.4) *has sub and supersolutions $\underline{w} \le \overline{w}$ a.e. Then, there exists a solution u of* (1.4) *with $\underline{w} \le u \le \overline{w}$ a.e.*

In Appendix B below, we show that the conclusion need not hold if the measure μ is not diffuse. The main difference between Proposition 5.1 and [16, Corollary 5.4] (whose conclusion is true for *every* finite measure μ) is that in the statement above it need not be true that

$$\underline{w} \le v \le \overline{w} \quad \Longrightarrow \quad g(\cdot, v) \in L^1(\Omega).$$

Proof. Let

$$h(x) = |g(x, \underline{w}(x))| + |g(x, \overline{w}(x))| + 1;$$

then h belongs to $L^1(\Omega)$. Let

$$g_n(x,s) = \begin{cases} -n\,h(x) & \text{if } g(x,s) \le -n\,h(x), \\ g(x,s) & \text{if } -n\,h(x) < g(x,s) < n\,h(x), \\ n\,h(x) & \text{if } g(x,s) \ge n\,h(x). \end{cases}$$

In particular,

$$|g_n(\cdot, s)| \le n\,h \quad \forall s \in \mathbb{R}, \quad \forall n \ge 1, \tag{5.1}$$

where $n\,h \in L^1(\Omega)$. Since

$$g_n(x,\overline{w}) = g(x,\overline{w}) \quad \text{and} \quad g_n(x,\underline{w}) = g(x,\underline{w}),$$

it follows that $\overline{w}$ and $\underline{w}$ still are sub and supersolutions of

$$\begin{cases} -\Delta u + g_n(x,u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Since, by (5.1),

$$\underline{w} \le v \le \overline{w} \quad \Longrightarrow \quad g_n(\cdot, v) \in L^1(\Omega)\,,$$

applying [16, Corollary 5.4] we deduce that equation (5.2) has a solution u_n such that

$$\underline{w} \le u_n \le \overline{w} \quad \text{a.e.}$$

Clearly, the sequence (g_n) satisfies assumptions *(i)–(ii)* of Proposition 4.1. Observe now that, by construction, $|g_n| \le |g|$ in $\Omega \times \mathbb{R}$; thus, by (1.6),

$$\sup_{|s|\le t} |g_n(x,s)| \le G_t(x) + H_t(x) \quad \text{a.e.} \quad \forall t > 0.$$

By Proposition 2.1, $G_t + H_t$ is quasi-L^1. It thus follows from Proposition 4.1 that (up to a subsequence) (u_n) strongly converges in $L^1(\Omega)$ to a solution u of

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $\underline{w} \le u \le \overline{w}$ a.e. as desired. □

6. Proof of Theorem 1.2

In order to apply the results of the previous section, we first need to show that for any given diffuse measure μ equation (1.4) does have sub and supersolutions associated to μ in the sense of Definition 4.1. This is established in our next

Lemma 6.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (1.5)–(1.6). *Given a diffuse measure μ on Ω, then equation* (1.4) *has sub and supersolutions $\underline{w} \le \overline{w}$ a.e. such that any solution u of* (1.4) *satisfies*

$$\underline{w} \le u \le \overline{w} \quad \text{a.e.}$$

Proof. We first show the existence of $\overline{w}$. For this purpose, let

$$g_n(x,s) = \begin{cases} g(x,s) & \text{if } g(x,s) \le n, \\ n & \text{if } g(x,s) > n. \end{cases}$$

Since g_n is bounded from above, by [16, Corollary 5.4] applied with sub and supersolutions $0 \le \overline{U}$, the equation

$$\begin{cases} -\Delta u + g_n(x,u) = \mu^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

has a largest solution u_n in $[0,\overline{U}]$. Since

$$g_{n-1}(x,u_n) \le g_n(x,u_n) \quad \text{a.e.},$$

u_n is a subsolution for the problem solved by u_{n-1}. By Proposition 4.2 and the maximality of u_{n-1}, this implies that $u_{n-1} \ge u_n$ a.e. In other words, the sequence (u_n) is non-increasing and bounded from below by 0. Let $\overline{w}$ be the limit of (u_n).

By Proposition 4.1, $\overline{w}$ is a solution of (1.4) with datum μ^+.
We claim that any solution u of (1.4) satisfies

$$u \le \overline{w} \quad \text{a.e.} \tag{6.2}$$

Indeed, u is a subsolution of (6.1). By maximality of u_n, we have

$$u \le u_n \quad \text{a.e.,} \quad \forall n \ge 1.$$

As $n \to \infty$, we deduce (6.2). The existence of a subsolution $\underline{w}$ is established in a similar way using $-\mu^-$ as datum. □

Remark that in the proof $\overline{w}$ has been chosen as the largest solution of (1.4) with datum μ^+ and $\underline{w}$ as the smallest solution of (1.4) with datum $-\mu^-$.

Proof of Theorem 1.2. Applying Lemma 6.1 and Proposition 5.1, we deduce that (1.4) has a solution. We now show that (1.4) has a largest solution. Before proceeding, we first establish the following

Claim. Given two solutions u_1 and u_2 of (1.4), there exists a solution u such that $u \ge \max\{u_1, u_2\}$.

By [16, Corollary 3.1], for any two solutions u_1 and u_2 of (1.4) the function $\max\{u_1, u_2\}$ is a subsolution of the same problem. Applying Proposition 5.1 with sub and supersolutions $\max\{u_1, u_2\} \le \overline{w}$, one finds a solution u such that

$$\max\{u_1, u_2\} \le u \le \overline{w} \quad \text{a.e.}$$

This establishes the claim.

In order to prove the existence of the largest solution of (1.4), we follow the lines of [16]. Let

$$A = \sup \left\{ \int_\Omega u; \ u \text{ is a solution of } (1.4) \right\}.$$

By Proposition 4.2, $\underline{U} \le u \le \overline{U}$ a.e. for every solution of (1.4); thus, since $\underline{U}$ and $\overline{U}$ are in $L^1(\Omega)$, A is finite. By the claim above, we can find a nondecreasing sequence (u_n) of solutions of (1.4) such that

$$\int_\Omega u_n \to A.$$

By monotone convergence, there exists u_0 in $L^1(\Omega)$, limit of u_n, such that

$$\int_\Omega u_0 = A. \tag{6.3}$$

Applying Proposition 4.1, we deduce that u_0 is a solution of (1.4). Hence, by the claim u_0 must be the largest solution of (1.4). The existence of the smallest solution is achieved in the same way. □

7. Proof of Theorem 1.1

Let $\overline{U}$ and $\underline{U}$ be the solutions of

$$\begin{cases} -\Delta \overline{U} = \mu^+ & \text{in } \Omega, \\ \overline{U} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta \underline{U} = -\mu^- & \text{in } \Omega, \\ \underline{U} = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly, let $\overline{V}$ and $\underline{V}$ be the solutions of the same problems with data ν^+ and $-\nu^-$. Define

$$K_\mu = \{w \in L^1(\Omega);\ \underline{U} \le w \le \overline{U}\} \quad \text{and} \quad K_\nu = \{z \in L^1(\Omega);\ \underline{V} \le z \le \overline{V}\},$$

so that both K_μ and K_ν are closed convex subsets of $L^1(\Omega)$. Since $\overline{U}$, $\underline{U}$, $\overline{V}$ and $\underline{V}$ are quasifinite, then any function in K_μ and K_ν is quasifinite. Given $z \in K_\nu$, consider

$$h_1(x, s) = g_1(s, z(x)) \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

Then, h_1 is a Carathéodory function which satisfies (1.5)–(1.6) (the latter holds since g_1 is continuous and z is quasifinite). Therefore, by Theorem 1.2 there exists a solution u of

$$\begin{cases} -\Delta u + g_1(u, z(x)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

Since $h_1(x, \cdot)$ is nondecreasing, the solution of (7.1) is unique (see [3, Corollary B.1]). Given w in K_μ define

$$h_2(x, t) = g_2(w(x), t) \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

In the same way as before, there exists a unique solution v of

$$\begin{cases} -\Delta v + g_2(w(x), v) = \nu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

By Proposition 4.2, u belongs to K_μ and v belongs to K_ν. Thus, the map

$$\begin{aligned} T : K_\mu \times K_\nu &\longrightarrow K_\mu \times K_\nu \\ (w, z) &\longmapsto (u, v) \end{aligned}$$

is well-defined. By Lemma 4.2, we have

$$\|u\|_{W_0^{1,q}} \le C\|\mu\|_{\mathcal{M}} \quad \text{and} \quad \|v\|_{W_0^{1,q}} \le C\|\nu\|_{\mathcal{M}}$$

for every $1 \le q < \frac{N}{N-1}$. Hence, by the Rellich-Kondrachov theorem, $T(K_\mu \times K_\nu)$ is bounded and relatively compact in $L^1(\Omega) \times L^1(\Omega)$.

We now show that T is continuous. For this purpose, let (z_n) be a sequence of functions in K_ν such that $z_n \to z$ in $L^1(\Omega)$. Let u_n be the corresponding solutions of

$$\begin{cases} -\Delta u_n + g_1(u_n, z_n(x)) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Proposition 4.1, there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ in $L^1(\Omega)$, where u is the solution of

$$\begin{cases} -\Delta u + g_1(u, z(x)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of u, the whole sequence (u_n) converges to u. Analogously, if the sequence (w_n) in K_μ strongly converges in $L^1(\Omega)$ to w, then the sequence (v_n) of solutions of

$$\begin{cases} -\Delta v_n + g_1(w_n(x), v_n) = \nu & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

strongly converges in $L^1(\Omega)$ to the solution of

$$\begin{cases} -\Delta v + g_2(w(x), v) = \nu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, T is continuous. Therefore, by Schauder's theorem, there exists a fixed point (u, v) of T, that is, a solution of (1.3). □

Remark 7.1. We do not know whether Theorem 1.1 still holds if (a_1)–(a_2) are replaced by the weaker assumptions:

$(\tilde{a}_1)$ $g_1(\cdot, t)t \geq 0$ for every $t \in \mathbb{R}$;
$(\tilde{a}_2)$ $g_2(s, \cdot)s \geq 0$ for every $s \in \mathbb{R}$.

Note that $(\tilde{a}_1)$–$(\tilde{a}_2)$ guarantee the existence of solutions of (7.1)–(7.2) (via Theorem 1.2), but in this case u and v need not be unique. One could define for example $T : K_\mu \times K_\nu \to K_\mu \times K_\nu$ as $T(w, z) = (\overline{u}, \overline{v})$, where $\overline{u}$ and $\overline{v}$ are the maximal solutions of (7.1) and (7.2), respectively. However, we do not know whether this compact operator T is continuous.

8. Proof of Theorem 1.3

We first prove the following

Lemma 8.1. *Let $u \in L^1(\Omega)$. Assume that $u/\rho_K \in L^1(\Omega)$ for some compact set $K \subset \Omega$, where $\rho_K(x) = d(x, K)$, $\forall x \in \Omega$. Then,*

$$\lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} |u| = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } K. \tag{8.1}$$

Proof. Let $x \in K$. For every $y \in B_r(x)$ we have $\rho_K(y) \leq r$. Thus,

$$\frac{1}{r^N} \int_{B_r(x)} |u| \leq \frac{1}{r^{N-1}} \int_{B_r(x)} \frac{|u|}{\rho_K}.$$

By [9, Theorem 3, p.77], we have $\mathcal{H}^{N-1}(\Sigma) = 0$, where

$$\Sigma = \left\{ x \in K;\ \limsup_{r \to 0} \frac{1}{r^{N-1}} \int_{B_r(x)} \frac{|u|}{\rho_K} > 0 \right\}.$$

Therefore

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r(x)} |u| = 0 \quad \forall x \in K \setminus \Sigma,$$

and the result follows. □

We now present the

Proof of Theorem 1.3. Assume by contradiction that there exists some function $u \not\equiv 0$ satisfying (1.7). Applying Lemma 4.3 with $w = -u$, one deduces that $u \geq 0$ a.e. in Ω. Since $\alpha < 2$, by the strong maximum principle (see Lemma A.2 below), we must have $\int_{B_{1/2}} u > 0$. Applying Proposition A.1 to suitable smooth subdomains $\omega \subset [x_1 > 0] \cap B_1$ and $\omega \subset [x_1 < 0] \cap B_1$ such that $[x_1 = 0] \cap B_{1/2} \subset \partial\omega$, we deduce that there exists $\varepsilon > 0$ such that

$$u(x) \geq \varepsilon |x_1| \quad \forall x \in B_{1/2}. \tag{8.2}$$

By Lemma 8.1, we have $u = 0$ $\mathcal{H}^{N-1}$-a.e. on $[x_1 = 0] \cap B_1$. Since u is quasicontinuous, this implies that $u = 0$ q.e. on $[x_1 = 0] \cap B_1$. Applying [5, Corollary 1.3], it follows from (8.2) that

$$(\Delta u)_{\mathrm{d}} \geq \varepsilon\, \Delta |x_1| = 2\varepsilon\, \mathcal{H}^{N-1} \quad \text{on } [x_1 = 0] \cap B_{1/2},$$

where the subscript "d" denotes the diffuse part of the measure Δu. On the other hand, by (1.7),

$$\Delta u \leq 0 \quad \text{on } [x_1 = 0] \cap B_{1/2}.$$

In particular,

$$(\Delta u)_{\mathrm{d}} \leq 0 \quad \text{on } [x_1 = 0] \cap B_{1/2}.$$

This gives a contradiction. $\square$

9. Study of a linear problem

In this section, we study the following problem

$$\begin{cases} -\Delta u + \dfrac{u}{|x_1|^\alpha} = \mu & \text{in } B_1, \\ \hfill u = 0 & \text{on } \partial B_1, \end{cases} \tag{9.1}$$

where $\alpha > 0$ and μ is a diffuse measure in B_1. The existence (and nonexistence) of solutions of (9.1) is provided by the following

Theorem 9.1. *Let $\alpha > 0$ and μ be a diffuse measure in Ω. We have*

(*i*) *if $\alpha < 1$, then* (9.1) *has a solution;*
(*ii*) *if $1 \leq \alpha < 2$, then* (9.1) *has no solution if $\mu \geq 0$ and $\mu \neq 0$;*
(*iii*) *if $\alpha \geq 2$, then* (9.1) *has a solution if, and only if, μ does not charge the set $[x_1 = 0] \cap B_1$.*

Proof of (*i*). This case is already covered by Theorem 1.2 since $1/|x_1|^\alpha \in L^1(B_1)$ if $0 < \alpha < 1$.

Proof of (*ii*). Let $\mu \geq 0$ be a diffuse measure such that (9.1) has a solution. In particular,

$$-\int_{B_1} u\Delta\zeta + \int_{B_1} \frac{u}{|x_1|^\alpha}\zeta = \int_{B_1} \zeta \, d\mu \geq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } B_1.$$

Thus, by Theorem 1.3, $u = 0$ a.e. We conclude that $\mu = 0$.

Proof of (*iii*). ($\Rightarrow$) Assume that (9.1) has a solution u. In particular, we have

$$-\int_{B_1} u\Delta\varphi + \int_{B_1} \frac{u}{|x_1|^\alpha}\varphi = \int_{B_1 \setminus [x_1=0]} \varphi \, d\mu, \tag{9.2}$$

for every $\varphi \in C_0^2(\overline{\Omega})$ such that $\operatorname{supp}\varphi \cap [x_1 = 0] = \emptyset$. Given $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = 1$ if $|t| \leq 1/2$ and $\operatorname{supp}\psi \subset [-1, 1]$, let

$$\psi_n(x) = \psi(nx_1) \quad \forall x \in B_1, \quad \forall n \geq 1.$$

Since $u/x_1^2 \in L^1(B_1)$, we have

$$n^2 \int_{\substack{B_1 \\ [|x_1| \leq \frac{1}{n}]}} |u| \leq \int_{\substack{B_1 \\ [|x_1| \leq \frac{1}{n}]}} \frac{|u|}{x_1^2} \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\int_{B_1} |u||\nabla\psi_n| \le Cn \int_{\substack{B_1 \\ [|x_1|\le \frac{1}{n}]}} |u| \to 0, \tag{9.3}$$

and

$$\int_{B_1} |u||\Delta\psi_n| \le Cn^2 \int_{\substack{B_1 \\ [|x_1|\le \frac{1}{n}]}} |u| \to 0. \tag{9.4}$$

Apply (9.2) with test function $\varphi = \zeta(1-\psi_n)$, where $\zeta \in C_0^2(\overline{\Omega})$. As $n \to \infty$, it follows from (9.3)–(9.4) and dominated convergence that

$$-\int_{B_1} u\Delta\zeta + \int_{B_1} \frac{u}{|x_1|^\alpha}\zeta = \int_{B_1\setminus[x_1=0]} \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

In other words, u also satisfies (9.1) with datum $\mu\lfloor_{B_1\setminus[x_1=0]}$. Therefore,

$$\mu\lfloor_{[x_1=0]} = 0$$

and the result follows.

($\Leftarrow$) Let μ be a measure in B_1 which does not charge the set $[x_1 = 0] \cap B_1$. For every $n \ge 1$, let u_n be the solution of

$$\begin{cases} -\Delta u_n + \dfrac{u_n}{|x_1|^\alpha + 1/n} = \mu & \text{in } B_1, \\ \hfill u = 0 & \text{on } \partial B_1. \end{cases} \tag{9.5}$$

Passing to a subsequence if necessary, we have $u_n \to u$ in $L^1(B_1)$. Moreover, by Lemmas 3.2 and 4.4,

$$\frac{u_n}{|x_1|^\alpha + 1/n} \overset{*}{\rightharpoonup} \frac{u}{|x_1|^\alpha} + \sigma \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega)$$

for some diffuse measure σ concentrated on the set $[x_1 = 0] \cap B_1$. Thus, u is a solution of (9.1) with datum $\mu - \sigma$. By the implication "$\Rightarrow$", $\mu - \sigma$ cannot charge the set $[x_1 = 0] \cap B_1$. Therefore, $\sigma = 0$ and u is the unique solution of (9.1) associated to μ. The proof of Theorem 9.1 is complete. □

We conclude this section by showing the equivalence between (1.13) and (1.14):

Proposition 9.1. *Let $\alpha \ge 1$ and $f \in L^2(\Omega)$. Then, $u \in X_\alpha$ satisfies*

$$\int_{B_1} \nabla u \cdot \nabla v + \int_{B_1} \frac{u}{|x_1|^\alpha} v = \int_{B_1} fv \quad \forall v \in X_\alpha \tag{9.6}$$

if, and only if,

$$\int_{B_1} \nabla u \cdot \nabla \varphi + \int_{B_1} \frac{u}{|x_1|^\alpha}\varphi = \int_{B_1} f\varphi \quad \forall \varphi \in C_0^\infty\big(B_1^+ \cup B_1^-\big). \tag{9.7}$$

We first show the following

LEMMA 9.1. *For every $v \in X_1$,*

$$v = 0 \quad \text{in } [x_1 = 0] \cap B_1 \text{ in the sense of traces.} \tag{9.8}$$

Hence,

$$v \in H^1_0(B_1^+) \cup H^1_0(B_1^-). \tag{9.9}$$

Proof. By Hölder's inequality, for every $x \in B_1$ and $r > 0$ such that $B_r(x) \subset B_1$,

$$0 \le \frac{1}{|B_r|} \int_{B_r(x)} |v| \le \left(\frac{1}{|B_r|} \int_{B_r(x)} v^2 \right)^{1/2}.$$

Applying Lemma 8.1 with function v^2 and $K = [x_1 = 0] \cap \overline{B}_a$, with any $a \in (0,1)$, we deduce that

$$\lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} |v| = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } [x_1 = 0] \cap B_1.$$

Thus, (9.8) follows. Since $v \in H^1_0(B_1)$, then v belongs to $H^1_0(B_1^+) \cup H^1_0(B_1^-)$. $\square$

The main ingredient in the proof of Proposition 9.1 is the following

LEMMA 9.2. *If $\alpha \ge 1$, then $C_0^\infty\big(B_1^+ \cup B_1^-\big)$ is dense in X_α with respect to the norm*

$$\|v\|_\alpha = \|\nabla v\|_{L^2} + \|v\|_{L^2(\nu_\alpha)}. \tag{9.10}$$

Proof. Let $v \in X_\alpha$. By the previous lemma, $v = 0$ on $[x_1 = 0] \cap B_1$ in the sense of traces. Hence, for every $\varepsilon \in (0,1)$ we have

$$\int_{\substack{B_1 \\ [|x_1| < \varepsilon]}} v^2 \le \varepsilon^2 \int_{\substack{B_1 \\ [|x_1| < \varepsilon]}} |\nabla v|^2. \tag{9.11}$$

Let $(v_k) \subset X_\alpha$ be the sequence given by

$$v_k(x) = v(x)\, S(kx_1) \quad \forall x \in B_1,$$

where $S \in C^\infty(\mathbb{R})$ is such that

$$S(t) = 0 \quad \text{if } |t| \le 1 \quad \text{and} \quad S(t) = 1 \quad \text{if } |t| \ge 2.$$

Note that each v_k vanishes in a neighborhood of the set $[x_1 = 0]$. We now show that

$$v_k \to v \quad \text{in } X_\alpha.$$

Indeed, by dominated convergence,

$$v_k \to v \quad \text{in } L^2(B_1; \nu_\alpha).$$

Also notice that

$$\big|\nabla v_k - \nabla v\big| \le C|\nabla v|\, \chi_{[|x_1| < \frac{2}{k}]} + Ck|v|\, \chi_{[\frac{1}{k} < |x_1| < \frac{2}{k}]}.$$

Using (9.11), one deduces that

$$\int_{B_1} \big|\nabla v_k - \nabla v\big|^2 \le C \int_{\substack{B_1 \\ [|x_1| < \frac{2}{k}]}} |\nabla v|^2.$$

Hence,

$$\nabla v_k \to \nabla v \quad \text{in } L^2(B_1).$$

Therefore, $v_k \to v$ in X_α.
In order to conclude the proof, take $(w_n) \subset C_0^\infty(B_1)$ to be a sequence such that

$$w_n \to v \quad \text{in } H_0^1(B_1).$$

In particular, for every $k \geq 1$,

$$w_n\, S(kx_1) \to v\, S(kx_1) \quad \text{in } X_\alpha \quad \text{as } n \to \infty.$$

Thus, for each $k \geq 1$, one can take $n_k \geq 1$ sufficiently large so that

$$\big\|(w_{n_k} - v)\, S(kx_1)]\big\|_{X_\alpha} \leq \frac{1}{k}.$$

Let

$$u_k = w_{n_k}\, S(kx_1) \quad \text{in } B_1.$$

Then, (u_k) is a sequence in $C_0^\infty\big(B_1^+ \cup B_1^-\big)$ and, by the triangle inequality,

$$\|u_k - v\|_\alpha \leq \|u_k - v_k\|_\alpha + \|v_k - v\|_\alpha \leq \frac{1}{k} + \|v_k - v\|_\alpha \to 0$$

as $k \to \infty$. □

We now present the

Proof of Proposition 9.1. The implication "⇒" is trivial, while the reverse implication "⇐" can be easily deduced from the density of $C^\infty\big(B_1^+ \cup B_1^-\big)$ in X_α. □

Appendix A. A counterpart of the Hopf lemma for weak supersolutions

In this appendix we prove the following counterpart of the Hopf lemma for the linear operator $-\Delta + b$ in the case of a (possibly) unbounded coefficient b near $\partial\Omega$:

Proposition A.1. *Let $u \in L^1(\Omega)$, $u \geq 0$ a.e., and $b \in L^\infty_{\text{loc}}(\Omega)$ be such that $bu \in L^1(\Omega)$ and*

$$-\int_\Omega u\, \Delta\varphi + \int_\Omega bu\, \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega. \tag{A.1}$$

Assume that

$$b\rho_0^\alpha \in L^\infty(\Omega) \quad \text{for some } \alpha < 2. \tag{A.2}$$

Then, for every $\omega \subset\subset \Omega$ there exists $C > 0$ such that

$$\operatorname*{ess\,inf}_\Omega \frac{u}{\rho_0} \geq C \int_\omega u. \tag{A.3}$$

We first prove the

Lemma A.1. *Let $u \in L^1(\Omega)$ and $f \in L^1(\Omega; \rho_0\, dx)$ be such that*

$$-\int_\Omega u\Delta\varphi \geq \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega. \tag{A.4}$$

If $u \geq 0$ a.e., then

$$-\int_\Omega u\Delta\zeta \geq \int_\Omega f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{A.5}$$

Proof. Clearly, (A.4) still holds if $\varphi \in C^2(\overline{\Omega})$ and $\operatorname{supp} \varphi \subset \Omega$. Let $H : \mathbb{R} \to \mathbb{R}$ be a smooth convex function such that

$$H(t) = 0 \quad \forall t \le 1 \quad \text{and} \quad H'(t) = 1 \quad \forall t \ge 2.$$

Given $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \ge 0$ in Ω, then $H(n\zeta) \in C^2(\overline{\Omega})$ and $\operatorname{supp} H(n\zeta) \subset \Omega$ for every $n \ge 1$. Moreover,

$$\Delta H(n\zeta) = nH'(n\zeta)\Delta\zeta + n^2 H''(n\zeta)|\nabla\zeta|^2 \ge nH'(n\zeta)\Delta\zeta.$$

Applying (A.4) with test function $\varphi = H(n\zeta)/n$, we then obtain

$$-\int_\Omega uH'(n\zeta)\Delta\zeta \ge \int_\Omega f\,\frac{H(n\zeta)}{n}.$$

As $n \to \infty$, (A.5) follows. □

We shall also need the following version of the weak Harnack inequality:

LEMMA A.2. *Let $u \in L^1(\Omega)$, $u \ge 0$ a.e., and $b \in L^\infty_{\mathrm{loc}}(\Omega)$ be such that*

$$-\int_\Omega u\,\Delta\varphi + \int_\Omega bu\,\varphi \ge 0 \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega. \tag{A.6}$$

Then, for every $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ such that

$$\operatorname*{ess\,inf}_\omega u \ge C_\omega \int_\omega u. \tag{A.7}$$

In particular, if $u = 0$ a.e. on a set of positive measure, then $u = 0$ a.e. in Ω.

Proof. Taking Ω smaller if necessary, we may assume that $b \in L^\infty(\Omega)$. We can also suppose that ω is path connected. We proceed in two steps:

Step 1. Proof of (A.7) when u is smooth.

By the weak Harnack inequality (see [12, Theorem 8.18]), we have

$$\inf_{B_r(x)} u \ge \frac{C}{r^N} \int\limits_{B_{2r}(x)} u \tag{A.8}$$

for every $x \in \Omega$ with $B_{4r}(x) \subset \Omega$; thus,

$$\int\limits_{B_r(x)} u \ge C \int\limits_{B_{2r}(x)} u. \tag{A.9}$$

Iterating (A.9) four times, one obtains

$$\int\limits_{B_{\frac{r}{5}}(x)} u \ge C \int\limits_{B_{\frac{16r}{5}}(x)} u \tag{A.10}$$

for every $x \in \Omega$ with $B_{\frac{32r}{5}}(x) \subset \Omega$.

Given $x_0 \in \omega$ and $j \ge 1$, let $r = d(\omega, \partial\Omega)/7$, and

$$\mathcal{A}_j = \left\{ x \in \Omega \;\middle|\; \begin{array}{l} \text{there exists } y \in \omega \text{ such that} \\ d(x,y) < r \text{ and } d_\omega(x_0, y) < jr \end{array} \right\}, \tag{A.11}$$

where d_ω is the geodesic distance in ω. We shall establish the following

Claim.

$$\int_{\mathcal{A}_j} u \geq C \int_{\mathcal{A}_{j-1}} u \quad \forall j \geq 1. \tag{A.12}$$

By the Vitali covering lemma, there exists a covering $\big(B_r(x_i)\big)_{i\in I}$ of the set

$$\mathcal{E}_j = \Big\{ x \in \omega;\ d_\omega(x, x_0) < jr \Big\}$$

such that $x_i \in \mathcal{E}_j$, $\forall i \in I$, and the balls $(B_{\frac{r}{5}}(x_i))_{i\in I}$ are disjoint (all balls are defined in terms of the standard Euclidean distance in $\mathbb{R}^N$). Clearly,

$$B_{\frac{r}{5}}(x_i) \subset \mathcal{A}_j \quad \forall i \in I. \tag{A.13}$$

We now show that

$$\big(B_{3r}(x_i)\big)_{i\in I} \quad \text{is a covering of } \mathcal{A}_{j+1}. \tag{A.14}$$

Indeed, given $z \in \mathcal{A}_{j+1}$ let $y \in \omega$ be such that

$$d(z, y) < r \quad \text{and} \quad d_\omega(x_0, y) < (j+1)r.$$

Since ω is path connected, there exists $x \in \omega$ such that

$$d_\omega(x, y) < r \quad \text{and} \quad d_\omega(x, x_0) < jr.$$

In particular, $x \in \mathcal{E}_j$. Thus, there exists $i \in I$ such that $x \in B_r(x_i)$. We then have

$$\begin{aligned} d(z, x_i) &\leq d(z, y) + d(y, x) + d(x, x_i) \\ &\leq d(z, y) + d_\omega(y, x) + d(x, x_i) < r + r + r = 3r. \end{aligned}$$

Hence, $z \in B_{3r}(x_i)$ and (A.14) follows.

Applying (A.10) and (A.13)–(A.14), we obtain

$$\int_{\mathcal{A}_j} u \geq \int\limits_{\bigcup_{i\in I} B_{\frac{r}{5}}(x_i)} u = \sum_{i\in I} \int\limits_{B_{\frac{r}{5}}(x_i)} u \geq C \sum_{i\in I} \int\limits_{B_{\frac{16r}{5}}(x_i)} u \geq C \int\limits_{\bigcup_{i\in I} B_{3r}(x_i)} u \geq C \int\limits_{\mathcal{A}_{j+1}} u.$$

This proves (A.12).

Iterating (A.12), it follows that

$$\int_{\mathcal{A}_1} u \geq C \int_{\mathcal{A}_k} u, \tag{A.15}$$

for some constant C independent of $x_0 \in \omega$, where $k \geq 1$ is the smallest integer such that

$$kr \geq \sup\big\{ d_\omega(x, y);\ x, y \in \omega \big\}.$$

Since $\mathcal{A}_1 \subset B_{2r}(x_0)$ and $\mathcal{A}_k \supset \omega$, we deduce from (A.8) and (A.15) that (recall that $r = r(\omega) = d(\omega, \partial\Omega)/7$ is fixed)

$$u(x_0) \geq C_\omega \int_\omega u \quad \forall x_0 \in \omega.$$

This implies (A.7) when u is smooth.

Step 2. Proof completed.

Replacing b by $\|b\|_{L^\infty}$, we may assume that b is constant. Take any domain $\widetilde{\Omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\Omega}$. Given $\rho \in C_0^\infty(B_1)$ such that $\rho \geq 0$ in B_1 and $\int_{B_1} \rho = 1$, let

$\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon})$, $\forall x \in \mathbb{R}^N$. Then, for $\varepsilon > 0$ small, $u_\varepsilon = \rho_\varepsilon * u$ is a smooth function satisfying

$$-\Delta u_\varepsilon + b u_\varepsilon \geq 0 \quad \text{in } \widetilde{\Omega}.$$

By the previous step,

$$\inf_\omega u_\varepsilon \geq C \int_\omega u_\varepsilon. \tag{A.16}$$

Since

$$\inf_\omega u_\varepsilon \to \operatorname*{ess\,inf}_\omega u \quad \text{and} \quad \int_\omega u_\varepsilon \to \int_\omega u,$$

the result follows as $\varepsilon \to 0$ in (A.16). □

We now establish Proposition A.1:

Proof of Proposition A.1. Replacing b by b^+ if necessary, we may assume that $b \geq 0$. Take $\delta \in (0, 1/2)$ small such that ρ_0 is smooth on $\overline{A}_\delta$ and $\omega \subset \Omega_\delta$, where

$$A_\delta = \big\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < \delta\big\} \quad \text{and} \quad \Omega_\delta = \big\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \delta\big\}.$$

By Lemma A.2,

$$\operatorname*{ess\,inf}_{\Omega_\delta} u \geq C \int_{\Omega_\delta} u \geq C \int_\omega u. \tag{A.17}$$

Moreover, applying Lemma A.1 we have

$$-\int_\Omega u\,\Delta\zeta + \int_\Omega bu\,\zeta \geq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{A.18}$$

Given $\gamma > 1$, consider

$$v(x) = \rho_0(x) + \big[\rho_0(x)\big]^\gamma \quad \forall x \in A_\delta.$$

A simple computation shows that $\Delta v \in L^1(A_\delta)$ and

$$\Delta v = (1 + \gamma\rho_0^{\gamma-1})\Delta\rho_0 + \gamma(\gamma-1)\rho_0^{\gamma-2}|\nabla\rho_0|^2 \geq \frac{\gamma(\gamma-1)}{\rho_0^{2-\gamma}} - C_1, \tag{A.19}$$

since $|\nabla\rho_0| = 1$ in A_δ and $\Delta\rho_0$ is bounded. Let $M > 0$ be such that

$$b\rho^\alpha \leq M \quad \text{a.e. in } \Omega. \tag{A.20}$$

By (A.19), we have

$$-\Delta v + \frac{M}{\rho_0^\alpha} v \leq -\frac{\gamma(\gamma-1)}{\rho_0^{2-\gamma}} + \frac{2M}{\rho_0^{\alpha-1}} + C_1,$$

where we used that $v \leq 2\rho_0$ since $\delta < 1$. We now choose γ so that $1 < \gamma < 3 - \alpha$; this is possible because $\alpha < 2$. Then, $2 - \gamma > \alpha - 1 > 0$. Hence, for $\delta > 0$ sufficiently small (possibly depending on α and M) we have

$$-\Delta v + bv \leq -\Delta v + \frac{M}{\rho_0^\alpha} v \leq 0 \quad \text{in } A_\delta.$$

Let

$$w = u - \varepsilon v \quad \text{in } A_\delta,$$

where $\varepsilon = C\int_\omega u$ and C is the constant in the right-hand side of (A.17). Since $v = 0$ on $\partial\Omega$, by (A.18) we get

$$-\int_{A_\delta} w\,\Delta\psi + \int_{A_\delta} bw\,\psi \geq 0, \tag{A.21}$$

for every $\psi \in C_0^2(\overline{A}_\delta)$, $\psi \geq 0$ in A_δ, such that

$$\operatorname{supp}\psi \subset \big\{x \in \overline{\Omega};\ \operatorname{dist}(x, \partial\Omega) < \delta\big\}.$$

Since $v < 1$ in $\overline{A}_\delta$, it follows from (A.17) that $w \geq 0$ a.e. in a neighborhood of $\partial A_\delta \cap \Omega$. Thus, by Lemma A.1 and (A.21),

$$-\int_{A_\delta} w\,\Delta\zeta + \int_{A_\delta} bw\,\zeta \geq 0 \quad \forall \zeta \in C_0^2(\overline{A}_\delta),\ \zeta \geq 0 \text{ in } A_\delta.$$

Therefore, by Lemma 4.3,

$$w \geq 0 \quad \text{a.e. in } A_\delta.$$

In other words,

$$u \geq \varepsilon v \geq \varepsilon\rho_0 \quad \text{a.e. in } A_\delta$$

which combined with (A.17) gives (A.3). □

Appendix B. Failure of the method of sub and supersolutions

Throughout this appendix, we assume that $N \geq 3$. It is well-known that problem (1.4) need not have a solution if the measure μ is not diffuse. As an example, if

$$g(x,s) = (s^+)^{\frac{N}{N-2}} \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

and $0 \in \Omega$, then (1.4) has no solution with datum $\mu = \delta_0$ (see [1]). In this case the solution of the linear problem

$$\begin{cases} -\Delta U = \delta_0 & \text{in } \Omega,\\ \quad U = 0 & \text{on } \partial\Omega, \end{cases}$$

is not a supersolution of (1.4) since $g(x,U) \sim 1/\|x\|^N$ is not integrable near the origin.

One may then wonder whether (1.4) has a solution under assumptions (1.5)–(1.6) if μ is not necessarily diffuse, but $\underline{U}$ and $\overline{U}$ *are* sub and supersolutions (i.e., if both $g(x,\underline{U})$ and $g(x,\overline{U})$ belong to $L^1(\Omega)$). This would be an extension of Proposition 5.1 for general measures μ. It turns out that this is not true. In fact,

Proposition B.1. *There exists a Carathéodory function $g : B_1 \times \mathbb{R} \to \mathbb{R}$ such that* (1.5)–(1.6) *hold,* 0 *and $k/\|x\|^{N-2}$ are sub and supersolutions of*

$$\text{(B.1)} \qquad \begin{cases} -\Delta u + g(x,u) = \delta_0 & \text{in } B_1,\\ \qquad\qquad u = 0 & \text{on } \partial B_1, \end{cases}$$

but (B.1) *has no solution.*

Proof. Given $h \in C_0^\infty(\mathbb{R})$ such that $h(t)t \geq 0$, $\forall t \in \mathbb{R}$, and $h(1) = 1$, let

$$g(x,s) = h\Big(\frac{\|x\|^{N-2}}{c_N}s\Big)\frac{1}{\|x\|^N} \quad \forall (x,s) \in B_1 \times \mathbb{R},$$

where $1/c_N = (N-2)|\partial B_1|$ and $|\partial B_1|$ is the $(N-1)$-dimensional Hausdorff measure of ∂B_1; thus,

$$-\Delta\left(\frac{c_N}{\|x\|^{N-2}}\right) = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

The function g thus defined satisfies (1.5)–(1.6). Let $M \geq 1$ be such that $\operatorname{supp} g \subset [-M, M]$. Then, $v(x) = c_N M/\|x\|^{N-2}$ is a supersolution of (B.1) since $g(\cdot, v) = 0$ and

$$-\Delta v = M\delta_0 \geq \delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

Clearly, 0 is a subsolution of (1.4).
We claim that (B.1) does not have a solution. Assume by contradiction that (B.1) has a solution u. It is well-known that

$$\lim_{r\to 0} \frac{1}{r}\int_{\partial B_r} u = (N-2). \tag{B.2}$$

By Proposition 4.2,

$$0 \le u(x) \le \frac{c_N}{\|x\|^{N-2}} \quad \text{a.e.} \tag{B.3}$$

Let $\varepsilon > 0$ be such that $h(t) \ge \frac{1}{2}$ for $|t-1| < \varepsilon$. Set

$$E_r = \Big\{x \in \partial B_r;\ u(x) \ge (1-\varepsilon)\frac{c_N}{r^{N-2}}\Big\}.$$

Thus,

$$g(x, u(x)) \ge \frac{1}{2r^N} \quad \forall x \in E_r. \tag{B.4}$$

By (B.3), we have

$$\begin{aligned} \frac{1}{r}\int_{\partial B_r} u &= \frac{1}{r}\int_{E_r} u + \frac{1}{r}\int_{\partial B_r \setminus E_r} u \\ &\le c_N \frac{|E_r|}{r^{N-1}} + (1-\varepsilon)c_N \frac{|\partial B_r \setminus E_r|}{r^{N-1}} = (N-2) - \varepsilon c_N \frac{|\partial B_r \setminus E_r|}{r^{N-1}}. \end{aligned} \tag{B.5}$$

In view of (B.2) and (B.5), we must have

$$\lim_{r\to 0} \frac{|\partial B_r \setminus E_r|}{|\partial B_r|} = 0.$$

Let $r_0 > 0$ be such that

$$\frac{|E_r|}{|\partial B_r|} \ge \frac{1}{2} \quad \text{for every } 0 < r < r_0. \tag{B.6}$$

By (B.4) and (B.6), we then have

$$\begin{aligned} \int_{B_1} g(x,u)\,dx \ge \int_{B_{r_0}} g(x,u)\,dx &\ge \int_0^{r_0} dr \int_{E_r} g(r\sigma, u)\,d\sigma \\ &\ge \int_0^{r_0} \frac{|\partial B_r|}{4}\frac{dr}{r^N} = \frac{|\partial B_1|}{4}\int_0^{r_0} \frac{dr}{r} = +\infty. \end{aligned}$$

This is a contradiction. □

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