REDUCED LIMITS FOR NONLINEAR EQUATIONS WITH MEASURES

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Abstract. We consider equations (E) $-\Delta u + g(u) = \mu$ in smooth bounded domains $\Omega \subset \mathbb{R}^N$, where g is a continuous nondecreasing function and μ is a finite measure in Ω. Given a bounded sequence of measures (μ_k), assume that for each $k \geq 1$ there exists a solution u_k of (E) with datum μ_k and zero boundary data. We show that if $u_k \to u^\#$ in $L^1(\Omega)$, then $u^\#$ is a solution of (E) relative to some finite measure $\mu^\#$. We call $\mu^\#$ the *reduced limit* of (μ_k). This reduced limit has the remarkable property that it does not depend on the boundary data, but only on (μ_k) and on g. For power nonlinearities $g(t) = |t|^{q-1}t$, $\forall t \in \mathbb{R}$, we show that if (μ_k) is nonnegative and bounded in $W^{-2,q}(\Omega)$, then μ and $\mu^\#$ are absolutely continuous with respect to each other; we then produce an example where $\mu^\# \neq \mu$.

Contents

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1. Introduction

In this paper we investigate the convergence of solutions of the equation

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $g(0) = 0$, and μ is a finite measure in Ω. By a solution of (1.1) we mean a function $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $g(u) \in L^1_{\mathrm{loc}}(\Omega)$ and (1.1) holds in the sense of distributions.

In general, equation (1.1) is not solvable for every finite measure μ. We shall denote by $\mathcal{G}(g)$ the set of finite measures for which a solution exists. When there is no risk of confusion we shall simply write $\mathcal{G}$, even though this set depends on the nonlinearity g.

Questions related to the convergence and stability of solutions of

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ \hfill u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

have been addressed in various contexts. We recall that a function u is a solution of (1.2) if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = \int_\Omega \zeta \, d\mu$$

for every $\zeta \in C^2_0(\overline{\Omega})$ (= space of functions in $C^2(\overline{\Omega})$ vanishing on $\partial\Omega$).

Let us denote by $\mathcal{G}_0(g)$ the set of finite measures for which (1.2) has a solution. Clearly, $\mathcal{G}_0(g) \subset \mathcal{G}(g)$. We prove in the Appendix below that $\mathcal{G}_0(g) = \mathcal{G}(g)$.

The space of finite measures in Ω is denoted by $\mathcal{M}(\Omega)$. If (μ_k) is a sequence in this space, the notation

$$\mu_k \overset{*}{\rightharpoonup} \mu \tag{1.3}$$

means that (μ_k) converges weakly* in $\big[C_0(\overline{\Omega})\big]^*$, where $C_0(\overline{\Omega})$ denotes the space of continuous functions in $\overline{\Omega}$ vanishing on the boundary. For brevity, we shall refer to this convergence as *weak* convergence in* Ω.

It is known that if (μ_k) is a bounded sequence of measures in Ω converging strongly to μ, then the solutions u_k of (1.2) with data μ_k always converge strongly in $L^1(\Omega)$ to the solution of (1.2) (see [6, Appendix 4B]). Similarly, if $g(t) = |t|^{q-1}t$ where $1 < q < \frac{N}{N-2}$, then (1.2) has a solution for every finite measure and if (μ_k) is a sequence converging weakly* to μ, then the solutions u_k also converge strongly in $L^1(\Omega)$ to the solution u associated to μ. However, for $q \geq \frac{N}{N-2}$, this conclusion fails; see [6, Example 1]. In fact, it may even happen that $\mu_k \overset{*}{\rightharpoonup} 1$ weakly* but $u_k \to 0$ in $L^1(\Omega)$, even though the function identically equal to 0 is not the solution of (1.2) with datum $\mu = 1$!

A natural question that comes up in this connection is the following: assuming that $q \geq \frac{N}{N-2}$ and $\mu_k \overset{*}{\rightharpoonup} \mu$, what additional 'minimal' assumptions would guarantee that solutions of (1.2) with data μ_k converge to the solution of (1.2) with datum μ? When this is not the case, what can we still say about the limit of the solutions? These are the types of problems that we address in this paper.

Our first result shows that if the sequence of solutions converges strongly in L^1 then the limit is a solution of (1.2) with *some* measure $\mu^{\#}$, in general different from the weak* limit μ.

Theorem 1.1. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded sequence such that $\mu_k \overset{*}{\rightharpoonup} \mu$. For each $k \geq 1$, denote by u_k the unique solution of (1.2) with datum μ_k. If*

$$u_k \to u^{\#} \quad \text{in } L^1(\Omega), \tag{1.4}$$

then $g(u^{\#}) \in L^1(\Omega)$ and there exists a finite measure $\mu^{\#}$ in Ω such that

$$\begin{cases} -\Delta u^{\#} + g(u^{\#}) = \mu^{\#} & \text{in } \Omega, \\ u^{\#} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Surprisingly, the measure $\mu^{\#}$ does not depend on the Dirichlet boundary condition. In fact, the sequence (u_k) may be replaced by any sequence of solutions of *equation* (1.1) with $\mu = \mu_k$, which may not even possess a boundary trace. This is the content of our next result:

Theorem 1.2. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded sequence such that $\mu_k \overset{*}{\rightharpoonup} \mu$. For every $k \geq 1$, assume that $v_k \in L^1(\Omega)$ satisfies*

$$-\Delta v_k + g(v_k) = \mu_k \quad \text{in } \Omega. \tag{1.6}$$

If

$$v_k \to v^{\#} \quad \text{in } L^1(\Omega), \tag{1.7}$$

then

$$-\Delta v^{\#} + g(v^{\#}) = \mu^{\#} \quad \text{in } \Omega, \tag{1.8}$$

where $\mu^{\#}$ is the measure given by Theorem 1.1.

We say that a sequence (μ_k) in $\mathcal{G}(g)$ has a *reduced limit* if it converges weakly* in $\mathcal{M}(\Omega)$ and if there exists a sequence $(v_k) \subset L^1(\Omega)$ satisfying (1.6)–(1.7); the *reduced limit* $\mu^{\#}$ is defined by (1.8).

We use this notation because of its simplicity, but we emphasize that the reduced limit $\mu^{\#}$ depends on (μ_k) and not just on its weak* limit. Indeed it is possible that different sequences converging weakly* to the same measure μ lead to different limits with respect to the same nonlinearity g. However, $\mu^{\#}$ does not depend on the domain: for any domain $\omega \Subset \Omega$, the reduced limit of (μ_k) in ω is simply the restriction of $\mu^{\#}$ to ω.

Further we note that every bounded sequence (μ_k) in $\mathcal{G}$ possesses a subsequence which satisfies the conditions of Theorem 1.2 and consequently has a reduced limit (see Section 6).

Following these results, we investigate some properties of $\mu^{\#}$; in particular, to what extent $\mu^{\#}$ inherits properties of the sequence (μ_k). Our next result illustrates the kind of properties that we are interested in.

Theorem 1.3. *Assume that $(\mu_k) \subset \mathcal{G}$ has reduced limit $\mu^{\#}$. If*

$$\mu_k \geq 0 \quad \forall k \geq 1, \tag{1.9}$$

then

$$\mu^{\#} \geq 0. \tag{1.10}$$

Observe that (1.10) does not follow from Fatou's lemma, which only implies in this case that $\mu^\# \le \mu$, where μ is the weak* limit of the sequence (μ_k).

Remark 1.1. The notion of reduced limit is reminiscent of the notion of reduced measure introduced by Brezis-Marcus-Ponce [6]. We recall that if $g(t) = 0$, $\forall t \le 0$, the reduced measure μ^* is the largest measure less than or equal to μ for which problem (1.2) has a solution. Our main concern in [6] was to study the approximation mechanism behind (1.2), for example via truncation of the nonlinearity g for a fixed measure μ, or via some special approximations of the datum μ for a fixed g. For instance, given a sequence of mollifiers (ρ_k) we have shown that, if g is convex, then solutions u_k of (1.2) with data $\mu_k = \rho_k * \mu$ converge to the largest subsolution u^* associated to μ. Since this function satisfies (1.2) with measure μ^*, one deduces in this case that $\mu^\# = \mu^*$.

We now focus on the case of equations with power nonlinearities, namely

$$-\Delta u + |u|^{q-1}u = \mu \quad \text{in } \Omega \tag{1.11}$$

in the supercritical range $q \ge \frac{N}{N-2}$. We recall that for a finite measure μ, equation (1.11) has a solution if and only if

$$\mu \in L^1(\Omega) + W^{-2,q}(\Omega).$$

In [6], we have showed that if (μ_k) is a bounded sequence of measures converging strongly to μ in $W^{-2,q}(\Omega)$, then $\mu^\# = \mu$. One might ask what happens if (μ_k) is just bounded in $W^{-2,q}(\Omega)$. In Theorem 1.3 the reduced limit $\mu^\#$ can be identically zero even if the sequence (μ_k) has a nonzero weak* limit. However, if $g(t) = |t|^{q-1}t$ then, boundedness in $W^{-2,q}$ guarantees that this cannot happen:

Theorem 1.4. *Assume that $(\mu_k) \subset \mathcal{G}$ is a nonnegative sequence with weak* limit μ and reduced limit $\mu^\#$. If (μ_k) is bounded in $W^{-2,q}(\Omega)$, then*

$$\mu^\# = 0 \quad \textit{if and only if} \quad \mu = 0. \tag{1.12}$$

For the proof see Section 8 below. Under the assumptions of this theorem, equation (1.11) has a solution with datum μ. Therefore, in view of (1.12) one may expect that the reduced limit $\mu^\#$ coincides with μ. Surprisingly, this conclusion does not hold in general; a counterexample is provided by Theorem 9.2 below.

Following is a description of some basic concepts and tools employed in this paper.

- (i) The notion of *equidiffuse sequence* of measures (μ_k) relative to an outer measure T. This means that (μ_k) is uniformly absolutely continuous with respect to T; more precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$E \subset \Omega \text{ Borel} \quad \text{and} \quad T(E) < \delta \quad \Longrightarrow \quad |\mu_k|(E) < \varepsilon \quad \forall k \ge 1.$$
- (ii) The notion of *concentrating sequence* of measures (μ_k) relative to an outer measure T. This means that there exists a sequence of Borel sets (E_k) of Ω such that
$$T(E_k) \to 0 \quad \text{and} \quad |\mu_k|(\Omega \setminus E_k) \to 0.$$

Let us consider for example the special case where T is a measure and $\mu_1 = \mu_2 = \ldots = \mu$ for some fixed measure μ. Then the sequence (μ_k) is equidiffuse if and only if μ is absolutely continuous with respect to T (denoted $\mu \ll T$) and (μ_k) is concentrating if and only if μ is singular with respect to T (denoted $\mu \perp T$).

Two important ingredients, related to the above concepts, are:

(iii) The Biting lemma of R. Chacon and H. Rosenthal according to which every bounded sequence of measures (μ_k) can be decomposed as a sum of an equidiffuse and a concentrating sequences; see Theorem 2.1 below.

(iv) The Inverse Maximum Principle for sequences, extending a previous result of Dupaigne-Ponce [14].

Using the Biting lemma we introduce the notions of *diffuse limit* and *concentrated limit* of a bounded sequence of measures (see Definition 2.1 below) and study some of the properties of these limits. In particular we identify the diffuse limit of a sequence $(g(u_k))$ where (u_k) converges in $L^1(\Omega)$ and $(g(u_k))$ is bounded in this space. These results, together with the counterpart of the Inverse Maximum Principle for sequences, play a crucial role in the proofs of Theorems 1.2 and 1.3.

2. Diffuse and concentrated limits

We denote by T a nonnegative outer measure defined on the class of Borel subsets of Ω. The space of finite Borel measures in Ω is denoted by $\mathcal{M}(\Omega)$ and is equipped with the norm

$$\|\mu\|_{\mathcal{M}} = \int_{\Omega} |\mu|;$$

by the Riesz representation theorem, $\mathcal{M}(\Omega) = \big[C_0(\overline{\Omega})\big]^*$.

The following result, independently proved by R. Chacon and H. Rosenthal (see Brooks-Chacon [11]), plays a central role in this section.

Theorem 2.1 (Biting lemma). *For every bounded sequence $(\mu_k) \subset \mathcal{M}(\Omega)$, there exist bounded sequences $(\alpha_k), (\sigma_k) \subset \mathcal{M}(\Omega)$ such that*

(B_1) $\mu_k = \alpha_k + \sigma_k$, $\forall k \geq 1$;

(B_2) (α_k) *is equidiffuse and* (σ_k) *is concentrating with respect to* T.

It is not difficult to see that the sequences (α_k) and (σ_k) can be chosen so that

(B_3) $\alpha_k \perp \sigma_k$, $\forall k \geq 1$.

Lemma 2.1. *Using the notation of the Biting lemma, assume that $\mu_k \overset{*}{\rightharpoonup} \mu$, $\alpha_k \overset{*}{\rightharpoonup} \alpha$ and $\sigma_k \overset{*}{\rightharpoonup} \sigma$. If (α_k') and (σ_k') is another pair of sequences satisfying (B_1)–(B_2), then $\alpha_k' \overset{*}{\rightharpoonup} \alpha$ and $\sigma_k' \overset{*}{\rightharpoonup} \sigma$.*

Proof. From the definition of equidiffuse sequences, one shows that $\alpha \ll T$. Therefore, if $\mu = 0$ then $\alpha = \sigma = 0$.
Let (α_{k_j}') and (σ_{k_j}') be subsequences converging weakly* to α' and σ' respectively. The previous statement implies that $\alpha = \alpha'$ and $\sigma = \sigma'$. This further implies that $\alpha_k' \overset{*}{\rightharpoonup} \alpha$ and $\sigma_k' \overset{*}{\rightharpoonup} \sigma$. □

In order to analyze in more detail the weak* limit of (μ_k) we shall study the weak* limits of the sequences (α_k) and (σ_k).

Definition 2.1. *Let (μ_k) be a bounded sequence in $\mathcal{M}(\Omega)$ and let (α_k) and (σ_k) be sequences satisfying conditions (B_1)–(B_2) of the Biting lemma. Assume that (μ_k) converges weakly*.*

(*a*) *If* $\alpha_k \overset{*}{\rightharpoonup} \alpha$*, we call* α *the* diffuse limit *of* (μ_k).
(*b*) *If* $\sigma_k \overset{*}{\rightharpoonup} \sigma$*, we call* σ *the* concentrated limit *of* (μ_k).

If a sequence of measures (μ_k) is bounded (but not necessarily weakly* convergent) and if every weak* convergent subsequence of (μ_k) possesses a diffuse limit α independent of the subsequence, we shall still say that this common limit α is the diffuse limit of (μ_k). Note that if (μ_k) is merely bounded, then it may possess a diffuse limit in this sense, but not a concentrated limit.

In view of Lemma 2.1, if (μ_k) possesses a diffuse limit and a concentrated limit then these limits are independent of the decomposition given by (B_1)–(B_2).

The diffuse and concentrated limits of (μ_k) depend on T. For instance, if $(\rho_k) \subset C_0^\infty(-1,1)$ is a sequence of mollifiers,

$$\rho_k \overset{*}{\rightharpoonup} \delta_0 \quad \text{weakly}^* \text{ in } \mathcal{M}(-1,1)$$

and one verifies that

(*a*) if T is the Lebesgue measure in $\mathbb{R}$, then (ρ_k) has diffuse limit 0 and concentrated limit δ_0;
(*b*) if T is the Newtonian capacity cap_{H^1}, then (ρ_k) has diffuse limit δ_0 and concentrated limit 0, since every nonnempty set in $\mathbb{R}$ has positive capacity.

We recall that if $\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}(\Omega)$, then

$$\|\mu\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}}.$$

It is worth noting the following improved version of this estimate.

Corollary 2.1. *Let* $(\mu_k) \subset \mathcal{M}(\Omega)$ *be a bounded sequence possessing diffuse and concentrated limits* α *and* σ*, respectively. Then,*

$$\|\alpha\|_{\mathcal{M}} + \|\sigma\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}}. \tag{2.1}$$

Proof. Take sequences $(\alpha_k), (\sigma_k) \subset \mathcal{M}(\Omega)$ satisfying (B_1)–(B_3). Then,

$$\alpha_k \overset{*}{\rightharpoonup} \alpha \quad \text{and} \quad \sigma_k \overset{*}{\rightharpoonup} \sigma \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega).$$

Hence,

$$\|\alpha\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \|\alpha_k\|_{\mathcal{M}} \quad \text{and} \quad \|\sigma\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \|\sigma_k\|_{\mathcal{M}}. \tag{2.2}$$

On the other hand, since $\mu_k = \alpha_k + \sigma_k$ and $\alpha_k \perp \sigma_k$, we have

$$\|\mu_k\|_{\mathcal{M}} = \|\alpha_k\|_{\mathcal{M}} + \|\sigma_k\|_{\mathcal{M}} \quad \forall k \geq 1. \tag{2.3}$$

Combining (2.2)–(2.3) we obtain (2.1). □

Corollary 2.2. *Let* $(\mu_k) \subset \mathcal{M}(\Omega)$ *be a bounded sequence of nonnegative measures with weak* limit* μ*. If* (μ_k) *has diffuse and concentrated limits* α *and* σ*, respectively, then*

$$0 \leq \alpha \leq \mu \quad \textit{and} \quad 0 \leq \sigma \leq \mu. \tag{2.4}$$

Proof. Take sequences $(\alpha_k), (\sigma_k) \subset \mathcal{M}(\Omega)$ satisfying (B_1)–(B_2) and such that $\alpha_k \perp \sigma_k$, $\forall k \geq 1$. Since

$$\alpha_k + \sigma_k = \mu_k \geq 0 \quad \text{and} \quad \alpha_k \perp \sigma_k,$$

we must have $\alpha_k, \sigma_k \geq 0$, $\forall k \geq 1$; hence, $\alpha, \sigma \geq 0$. The corollary now follows from the equality $\mu = \alpha + \sigma$. □

As a final remark, we point out that if $(\mu_k) \subset \mathcal{M}(\Omega)$ has diffuse and concentrated limits equal to α and σ, respectively, then $\alpha \ll T$, but σ need not be a measure concentrated with respect to T or with respect to α. For instance, if T is the Lebesgue measure in $\mathbb{R}^N$, $f \in L^1(\Omega)$ and (λ_k) is a convex combination of Dirac masses such that

$$\lambda_k \overset{*}{\rightharpoonup} 1 \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega),$$

then the sequence (μ_k) given by

$$\mu_k = f + \lambda_k \quad \forall k \geq 1$$

has f as diffuse limit and 1 as concentrated limit.

3. The diffuse limit of $(g(u_k))$

In this section we study the diffuse limit of the nonlinear term in the equation (1.2) with data μ_k. We start with a basic result which is independent of the PDE.

Proposition 3.1. *Let $(u_k) \subset L^1(\Omega)$ be a sequence such that $\big(g(u_k)\big)$ is bounded in $L^1(\Omega)$. If*

$$u_k \to u^\# \quad \text{in } L^1(\Omega), \tag{3.1}$$

then $g(u^\#)$ is the diffuse limit of $\big(g(u_k)\big)$ with respect to Lebesgue measure in $\mathbb{R}^N$.

Given $a > 0$, we denote by $T_a : \mathbb{R} \to \mathbb{R}$ the truncation at $\pm a$, defined as

$$T_a(t) = \begin{cases} t & \text{if } |t| \leq a, \\ a & \text{if } t > a, \\ -a & \text{if } t < -a. \end{cases} \tag{3.2}$$

We first prove the following

Lemma 3.1. *Assume that $(u_k) \subset L^1(\Omega)$ satisfies the assumptions of Proposition 3.1. Then, there exists a subsequence (u_{k_j}) such that*

$$g(u_{k_j})\chi_{[|u_{k_j}| \leq j]} \to g(u^\#) \quad \text{in } L^1(\Omega). \tag{3.3}$$

Proof. For every $j \in \mathbb{N}$, we have by dominated convergence,

$$g\big(T_j(u_k)\big) \to g\big(T_j(u^\#)\big) \quad \text{in } L^1(\Omega).$$

On the other hand, if follows from Fatou's lemma that $g(u^\#) \in L^1(\Omega)$. Thus, by monotone convergence,

$$g\big(T_j(u^\#)\big) \to g(u^\#) \quad \text{in } L^1(\Omega).$$

Using a diagonalization argument, one then finds an increasing sequence of integers (k_j) such that

$$g\big(T_j(u_{k_j})\big) \to g(u^\#) \quad \text{in } L^1(\Omega).$$

Since for every $j \geq 1$,

$$0 \leq \big|g(u_{k_j})\big|\chi_{[|u_{k_j}| \leq j]} \leq \big|g\big(T_j(u_{k_j})\big)\big| \text{ a.e.},$$

the conclusion follows by dominated convergence. $\square$

Proof of Proposition 3.1. Passing to a subsequence if necessary, we may assume that $\big(g(u_k)\big)$ has diffuse and concentrated limits α and σ, respectively. Let (u_{k_j}) be the subsequence given by Lemma 3.1. Set

$$\alpha_j = g(u_{k_j})\chi_{[|u_{k_j}|\le j]} \quad \text{and} \quad \sigma_j = g(u_{k_j})\chi_{[|u_{k_j}|> j]}. \tag{3.4}$$

We claim that (α_j) and (σ_j) satisfy conditions (B_1)–(B_2). Indeed, since (α_j) strongly converges in $L^1(\Omega)$, the sequence (α_j) is equidiffuse (or, equivalently in this case, equi-integrable). On the other hand, by the Chebyshev inequality,

$$\big|\big[|u_{k_j}| > j\big]\big| \le \frac{1}{j}\|u_{k_j}\|_{L^1} \le \frac{C}{j} \quad \forall j \ge 1.$$

Thus, the sequence (σ_j) is concentrating.
Therefore, $\alpha = g(u^\#)$. Since α is independent of the subsequence, we conclude that $g(u^\#)$ is the diffuse limit of $\big(g(u_k)\big)$. $\square$

We now examine the weak* limit of the sequence $\big(g(u_k)\big)$ when u_k is a solution of (1.1) with datum μ_k. In this case, the conclusion can be improved by replacing the Lebesgue measure with the Newtonian capacity cap_{H^1} as the outer measure T.

Proposition 3.2. *Let $(\mu_k) \subset \mathcal{M}(\Omega)$ be a bounded sequence. Assume that, for each $k \ge 1$, there exists $u_k \in L^1(\Omega)$ such that*

$$-\Delta u_k + g(u_k) = \mu_k \quad \text{in } \Omega. \tag{3.5}$$

If $\big(g(u_k)\big)$ is bounded in $L^1(\Omega)$ and

$$u_k \to u^\# \quad \text{in } L^1(\Omega), \tag{3.6}$$

then $g(u^\#)$ is the diffuse limit of $\big(g(u_k)\big)$ with respect to cap_{H^1}.

For the proof of the proposition we need the following lemma.

Lemma 3.2. *Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then,*

$$T_a(u) \in H^1_{\mathrm{loc}}(\Omega) \quad \forall a > 0. \tag{3.7}$$

Moreover, for every $\omega \Subset \Omega$ there exists $C_\omega > 0$ such that for every $a > 0$,

$$\int_\omega |\nabla T_a(u)|^2 \le C_\omega a\Big(\|u\|_{L^1(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}\Big) \tag{3.8}$$

and

$$\mathrm{cap}_{H^1}\big([|u| > a] \cap \omega\big) \le \frac{C_\omega}{a}\Big(\|u\|_{L^1(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}\Big). \tag{3.9}$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ in Ω and $\varphi = 1$ on ω. Set $v = u\varphi$. For every $a > 0$, we have

$$\int_\Omega |\nabla T_a(v)|^2 = \int_\Omega \nabla T_a(v) \cdot \nabla v = -\int_\Omega T_a(v)\Delta v \le a \int_\Omega |\Delta v|. \tag{3.10}$$

Since

$$\Delta v = \varphi \Delta u + 2\nabla\varphi \cdot \nabla u + u\Delta\varphi \quad \text{in } \Omega,$$

we have

$$\int_\Omega |\Delta v| \le \|\Delta u\|_{\mathcal{M}(\Omega)} + 2C_\varphi \int\limits_{\operatorname{supp}\varphi} |\nabla u| + C_\varphi \|u\|_{L^1(\Omega)}. \tag{3.11}$$

We recall that

$$\int_{\operatorname{supp}\varphi} |\nabla u| \le C_\varphi\Big(\|u\|_{L^1(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}\Big). \tag{3.12}$$

Combining (3.10)–(3.12), we get

$$\int_\Omega |\nabla T_a(v)|^2 \le C_\varphi a\Big(\|u\|_{L^1(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}\Big).$$

This implies (3.8). Since

$$\operatorname{cap}_{H^1}\big([|u| > a] \cap \omega\big) \le \operatorname{cap}_{H^1}\big([|v| > a]\big) \le \frac{1}{a^2}\int_\Omega |\nabla T_a(v)|^2,$$

the conclusion follows. □

Proof of Proposition 3.2. Passing to a subsequence if necessary, we may assume that $\big(g(u_k)\big)$ has diffuse and concentrated limits α and σ, respectively. Take (α_j) and (σ_j) as in (3.4). Since (α_j) converges strongly in $L^1(\Omega)$, it is in particular equidiffuse with respect to cap_{H^1}.
We show that the sequence (σ_k) is concentrating with respect to cap_{H^1} in every subdomain $\omega \Subset \Omega$. For this purpose, let

$$E_j = \big[|u_{k_j}| > j\big] \cap \omega.$$

By Lemma 3.2, given $\omega \Subset \Omega$ we have

$$\operatorname{cap}_{H^1}(E_j) \le \frac{C}{j}\Big(\|u_{k_j}\|_{L^1(\Omega)} + \|\mu_{k_j}\|_{\mathcal{M}(\Omega)} + \|g(u_{k_j})\|_{L^1(\Omega)}\Big).$$

Thus, $\operatorname{cap}_{H^1}(E_j) \le \frac{C}{j}$ and so (σ_j) is concentrating in ω with respect to cap_{H^1}. Therefore, $\alpha = g(u^\#)$ in ω for every $\omega \Subset \Omega$, whence $g(u^\#)$ is the diffuse limit of $\big(g(u_k)\big)$ relative to cap_{H^1}. □

4. The Inverse Maximum Principle for sequences

An important tool in the present work is an extension to sequences of the Inverse Maximum Principle of Dupaigne-Ponce [14]. We first recall their result.

Theorem 4.1 (Inverse Maximum Principle)**.** *Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \ge 0$ a.e., then*

$$(\Delta u)_{\mathrm{c}} \le 0. \tag{4.1}$$

Here, "c" denotes the concentrated part of the measure with respect to cap_{H^1}. In fact, every finite measure μ can be uniquely decomposed in terms of a diffuse part μ_{d} and a concentrated part μ_{c} with respect to an outer measure T, so that $\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}}$, $\mu_{\mathrm{d}} \ll T$ and $\mu_{\mathrm{c}} \perp T$; see e.g. [6, Lemma 4.A.1].

We prove the following extension of this result.

Theorem 4.2. *Let $(u_k) \subset L^1(\Omega)$ be a bounded sequence such that $\Delta u_k \in \mathcal{M}(\Omega)$, $\forall k \ge 1$. Assume that (Δu_k) is bounded in $\mathcal{M}(\Omega)$ and has concentrated limit $\sigma \in \mathcal{M}(\Omega)$ with respect to* cap_{H^1}*. If $u_k \ge 0$ a.e., $\forall k \ge 1$, then*

$$\sigma \le 0. \tag{4.2}$$

For the proof we use an extension of Kato's inequality (see [8]).

Lemma 4.1. *Let $u \in L^1(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then,*

$$\Delta u^+ \geq \chi_{[u\geq 0]}(\Delta u)_\mathrm{d} - |\Delta u|_\mathrm{c} \quad \text{in } \Omega. \tag{4.3}$$

We recall that if $u \in L^1(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then u is quasicontinuous with respect to cap_{H^1}; see e.g. [1, 7]. More precisely, there exists a quasicontinuous function $\tilde{u} : \Omega \to \mathbb{R}$, unique up to sets of zero H^1-capacity, such that $u = \tilde{u}$ a.e. We shall henceforth identify u with $\tilde{u}$ *pointwise* in Ω. In particular, the term $\chi_{[u\geq 0]}(\Delta u)_\mathrm{d}$ is well-defined, meaning $\chi_{[\tilde{u}\geq 0]}(\Delta u)_\mathrm{d}$.

Proof of Theorem 4.2. For every $k \geq 1$, let

$$\mu_k := \Delta u_k.$$

We denote by $(\alpha_k), (\sigma_k) \subset \mathcal{M}(\Omega)$ two sequences satisfying (B_1)–(B_2). Passing to a subsequence if necessary, we may assume that $u_k \to u$ a.e. for some function $u \in L^1(\Omega)$ and also

$$\alpha_k \overset{*}{\rightharpoonup} \alpha \quad \text{and} \quad \sigma_k \overset{*}{\rightharpoonup} \sigma \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega).$$

In particular, σ is the concentrated limit of the original sequence (μ_k).

Given $a > 0$, let T_a be as in (3.2). Since $u_k \geq 0$ a.e., $T_a(u_k) = a - (a-u)^+$. Thus, by Lemma 4.1,

$$\Delta T_a(u_k) \leq \chi_{[u_k\leq a]}(\Delta u_k)_\mathrm{d} + |\Delta u_k|_\mathrm{c}, \tag{4.4}$$

On the other hand, since each measure α_k is diffuse, one verifies that

$$\begin{aligned}(\Delta u_k)_\mathrm{d} &= (\alpha_k)_\mathrm{d} + (\sigma_k)_\mathrm{d} = \alpha_k + (\sigma_k)_\mathrm{d},\\ |\Delta u_k|_\mathrm{c} &= |\sigma_k|_\mathrm{c}.\end{aligned}$$

Thus,

$$\Delta T_a(u_k) \leq \alpha_k \chi_{[u_k\leq a]} + |\sigma_k| = \alpha_k - \alpha_k\chi_{[u_k>a]} + |\sigma_k|. \tag{4.5}$$

Let $\varepsilon > 0$. Since (α_k) is equidiffuse with respect to cap_{H^1}, there exists $\delta > 0$ such that

$$E \subset \Omega \text{ Borel} \quad \text{and} \quad \mathrm{cap}_{H^1}(E) < \delta \quad \Longrightarrow \quad |\alpha_k|(E) < \varepsilon \quad \forall k \geq 1. \tag{4.6}$$

On the other hand, given a subdomain $\omega \Subset \Omega$, by Lemma 3.2 we have

$$\mathrm{cap}_{H^1}\big([u_k > a] \cap \omega\big) \leq \frac{C_\omega}{a} \quad \forall a > 0. \tag{4.7}$$

Keeping ω fixed, by (4.6)–(4.7) there exists $a_0 > 0$ such that if $a \geq a_0$, then

$$|\alpha_k|\big([u_k > a] \cap \omega\big) \leq \varepsilon \quad \forall k \geq 1. \tag{4.8}$$

Since (σ_k) is concentrating, there exists a sequence of Borel sets $E_k \subset \Omega$ such that

$$\mathrm{cap}_{H^1}(E_k) \to 0 \quad \text{and} \quad |\sigma_k|(\Omega \setminus E_k) \to 0.$$

By inner regularity of σ_k, one can then find compact subsets $K_k \subset E_k$ such that

$$\mathrm{cap}_{H^1}(K_k) \to 0 \quad \text{and} \quad |\sigma_k|(\Omega \setminus K_k) \to 0. \tag{4.9}$$

For each $k \geq 1$, let $\zeta_k \in C_0^\infty(\Omega)$ be such that $0 \leq \zeta_k \leq 1$ in Ω, $\zeta_k = 1$ on K_k, and

$$\int_\Omega |\nabla \zeta_k|^2 \leq \mathrm{cap}_{H^1}(K_k) + \frac{1}{k}.$$

Given $\psi \in C_0^\infty(\Omega)$ with $\psi \geq 0$ in Ω and $\operatorname{supp}\psi \subset \omega$, set $\varphi_k = \psi(1-\zeta_k)$ in Ω. Then, the sequence (φ_k) satisfies

$$\begin{aligned} 0 \leq \varphi_k \leq \psi \quad & \text{in } \Omega, \\ \varphi_k = 0 \quad & \text{on } K_k, \\ \varphi_k \to \psi \quad & \text{in } H_0^1(\Omega). \end{aligned}$$

Passing to a subsequence if necessary, we may also assume that

$$\varphi_k \to \psi \quad \text{q.e.,} \tag{4.10}$$

where q.e. (= quasi-everywhere) means: outside some set of zero H^1-capacity. By (4.5), for every $k \geq 1$ and $a > 0$, we have

$$-\int_\Omega \nabla T_a(u_k) \cdot \nabla \varphi_k \leq \int_\Omega \varphi_k \, d\alpha_k - \int\limits_{[u_k > a]} \varphi_k \, d\alpha_k + \int_\Omega \varphi_k \, d|\sigma_k|. \tag{4.11}$$

It follows from Lemma 3.2 that the sequence $\big(T_a(u_k)\big)$ is bounded in $H^1(\omega)$. Since $\operatorname{supp}\varphi_k \subset \omega$ and $\varphi_k \to \psi$ in $H_0^1(\Omega)$, we then have

$$\int_\Omega \nabla T_a(u_k) \cdot \nabla \varphi_k \to \int_\Omega \nabla T_a(u) \cdot \nabla \psi \quad \text{as } k \to \infty. \tag{4.12}$$

Since $\varphi_k \to \psi$ q.e. and (α_k) is equidiffuse, (see e.g. [9, Lemma 1])

$$\int_\Omega \varphi_k \, d\alpha_k \to \int_\Omega \psi \, d\alpha \quad \text{as } k \to \infty. \tag{4.13}$$

By (4.8),

$$\Bigg| \int\limits_{[u_k > a]} \varphi_k \, d\alpha_k \Bigg| \leq \varepsilon \|\varphi_k\|_{L^\infty} \leq \varepsilon \|\psi\|_{L^\infty} \quad \forall a \geq a_0. \tag{4.14}$$

Using (4.9), we also get

$$\int_\Omega \varphi_k \, d|\sigma_k| = \int_{\Omega \setminus K_k} \varphi_k \, d|\sigma_k| \leq \|\psi\|_{L^\infty} \, |\sigma_k|(\Omega \setminus K_k) \to 0 \quad \text{as } k \to \infty. \tag{4.15}$$

As $k \to \infty$ in (4.11), we then obtain

$$-\int_\Omega \nabla T_a(u) \cdot \nabla \psi \leq \int_\Omega \psi \, d\alpha + \varepsilon \|\psi\|_{L^\infty} \quad \forall a \geq a_0.$$

Thus,

$$\int_\Omega T_a(u) \Delta \psi \leq \int_\Omega \psi \, d\alpha + \varepsilon \|\psi\|_{L^\infty} \quad \forall a \geq a_0.$$

Letting $a \to \infty$ and $\varepsilon \to 0$, we get

$$\int_\Omega u \Delta \psi \leq \int_\Omega \psi \, d\alpha.$$

Since

$$\int_\Omega u \Delta \psi = \int_\Omega \psi \Delta u = \int_\Omega \psi \, d\alpha + \int_\Omega \psi \, d\sigma,$$

we conclude that

$$\int_\Omega \psi \, d\sigma \leq 0 \quad \forall \psi \in C_0^\infty(\Omega), \ \psi \geq 0 \text{ in } \Omega.$$

Therefore, $\sigma \leq 0$. The proof of Theorem 4.2 is complete. $\square$

5. Supersolutions always converge to supersolutions

In this section we prove a result about convergence of supersolutions of equation (1.1) which appears to be stronger than Theorem 1.3 but is, in fact, equivalent to it.

Theorem 5.1. *Let $(u_k) \subset L^1(\Omega)$ be a sequence such that*

$$-\Delta u_k + g(u_k) \geq 0 \quad \text{in } \Omega. \tag{5.1}$$

If $\big(g(u_k)\big)$ is bounded in $L^1(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$, then

$$-\Delta u + g(u) \geq 0 \quad \text{in } \Omega. \tag{5.2}$$

In the proof we need a variant of Kato's inequality up to the boundary (see [6, Proposition 4.B.5]).

Lemma 5.1. *Let $u \in L^1(\Omega)$ be such that*

$$\int_\Omega u \Delta\zeta \geq \int_\Omega f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega, \tag{5.3}$$

where $f \in L^1(\Omega)$. Then,

$$\int_\Omega u^+ \Delta\zeta \geq \int_{\substack{\Omega \\ [u\geq 0]}} f\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{5.4}$$

Here, we use the notation

$$C_0^2(\overline{\Omega}) = \big\{\zeta \in C^2(\overline{\Omega})\ ;\ \zeta = 0 \text{ on } \partial\Omega\big\}.$$

Proof of Theorem 5.1. Let

$$\mu_k = -\Delta u_k + g(u_k) \quad \text{in } \Omega.$$

Since the right-hand side is a nonnegative distribution in Ω, μ_k is a locally finite (nonnegative) measure. We first show that for every $\omega \Subset \Omega$ the sequence (μ_k) is bounded in $\mathcal{M}(\omega)$. In fact, take $\varphi_\omega \in C_0^\infty(\Omega)$ such that $0 \leq \varphi_\omega \leq 1$ in Ω and $\varphi_\omega = 1$ on ω. Then,

$$\int_\Omega \varphi_\omega\, d\mu_k = -\int_\Omega u_k \Delta\varphi_\omega + \int_\Omega g(u_k)\varphi_\omega \leq C_\omega \|u_k\|_{L^1(\Omega)} + \|g(u_k)\|_{L^1(\Omega)}.$$

Since $\mu_k \geq 0$ and the sequences (u_k) and $\big(g(u_k)\big)$ are bounded in $L^1(\Omega)$, we then have

$$\|\mu_k\|_{\mathcal{M}(\omega)} \leq C_\omega \|u_k\|_{L^1(\Omega)} + \|g(u_k)\|_{L^1(\Omega)} \leq \widetilde{C}_\omega \quad \forall k \geq 1.$$

Thus, (μ_k) is bounded in $\mathcal{M}(\omega)$.
By Fatou's lemma, $g(u) \in L^1(\Omega)$. Passing to a subsequence if necessary, we may assume that

$$\mu_k \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad g(u_k) \overset{*}{\rightharpoonup} g(u) + \tau \quad \text{weakly}^* \text{ in } \mathcal{M}(\omega)$$

for some $\mu, \tau \in \mathcal{M}(\omega)$. Thus, u satisfies

$$-\Delta u + g(u) = \mu - \tau \quad \text{in } \omega. \tag{5.5}$$

From Proposition 3.2 we know that $g(u)$ is the diffuse limit of $\big(g(u_k)\big)$ with respect to cap_{H^1} and, consequently, τ must be its concentrated limit. In view of (5.5), our goal is to show that

$$\mu - \tau \geq 0 \quad \text{in } \omega. \tag{5.6}$$

We may assume that (μ_k) has a concentrated limit in $\mathcal{M}(\omega)$, which we denote by λ. By Corollary 2.2, $\mu_k \geq 0$, $\forall k \geq 1$, implies that $\lambda \leq \mu$. Since

$$\Delta u_k = g(u_k) - \mu_k \quad \forall k \geq 1,$$

the concentrated limit of (Δu_k) in ω is then given by $\tau - \lambda$. Note that

$$\tau - \mu \leq \tau - \lambda \quad \text{in } \omega. \tag{5.7}$$

Let us assume temporarily that

$$u_k \geq 0 \ \text{ a.e.} \quad \forall k \geq 1. \tag{5.8}$$

In this case, it follows from Theorem 4.2 that the concentrated limit of (Δu_k) is nonpositive. In other words,

$$\tau - \lambda \leq 0 \quad \text{in } \omega. \tag{5.9}$$

Combining (5.7) and (5.9), we obtain (5.6) under the additional assumption (5.8).

In the general case where the functions u_k need not be nonnegative we proceed as follows. Since $u_k \in W^{1,1}_{\text{loc}}(\Omega)$, we have $u_k \in L^1(\partial\omega)$. Let v_k be the harmonic function in ω with boundary value $-|u_k|$ on $\partial\omega$. We claim that

$$u_k \geq v_k \ \text{ a.e.} \tag{5.10}$$

Indeed, for every $\zeta \in C^2_0(\overline{\omega})$, $\zeta \geq 0$ in ω, we have $\dfrac{\partial\zeta}{\partial n} \leq 0$ on $\partial\omega$; thus,

$$\int_\omega (v_k - u_k)\Delta\zeta = \int_{\partial\omega} (v_k - u_k)\frac{\partial\zeta}{\partial n} + \int_\omega \big[\mu_k - g(u_k)\big]\zeta \geq -\int_\omega g(u_k)\zeta.$$

Applying Lemma 5.1 we get

$$\int_\omega (v_k - u_k)^+\Delta\zeta \geq - \int_{\substack{\omega \\ [v_k \geq u_k]}} g(u_k)\zeta \geq 0 \quad \forall \zeta \in C^2_0(\overline{\omega}),\ \zeta \geq 0 \text{ in } \omega, \tag{5.11}$$

since $v_k \leq 0$ in ω and $g(t) \leq 0$, $\forall t \leq 0$. This gives (5.10). Because

$$\Delta(u_k - v_k) = \Delta u_k = g(u_k) - \mu_k \quad \forall k \geq 1,$$

we can apply Theorem 4.2 to the sequence $(u_k - v_k)$ and deduce (5.9). Hence, u satisfies

$$-\Delta u + g(u) \geq 0 \quad \text{in } \omega.$$

Since $\omega \Subset \Omega$ is arbitrary, (5.2) holds. □

6. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. By standard estimates (see [6, Appendix 4B]),

$$\|g(u_k)\|_{L^1} \leq \|\mu_k\|_{\mathcal{M}} \quad \forall k \geq 1.$$

In particular, the sequence $\big(g(u_k)\big)$ is bounded in $L^1(\Omega)$ and, by Fatou's lemma, $g(u^\#) \in L^1(\Omega)$, with

$$\|g(u^\#)\|_{L^1} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}}.$$

Moreover, passing to a subsequence if necessary, there exists $\lambda \in \mathcal{M}(\Omega)$ such that

$$g(u_k) \overset{*}{\rightharpoonup} \lambda \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega).$$

Hence, the function $u^{\#}$ satisfies

$$\begin{cases} -\Delta u^{\#} + g(u^{\#}) = \mu^{\#} & \text{in } \Omega, \\ u^{\#} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu^{\#} = \mu + g(u^{\#}) - \lambda$. Since $\mu, \lambda \in \mathcal{M}(\Omega)$ and $g(u^{\#}) \in L^1(\Omega)$, the conclusion follows. □

In order to prove Theorem 1.2 we need a few lemmas. We first prove a local estimate for solutions of (1.1).

Lemma 6.1. *Let $u \in L^1(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ be such that*

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega. \tag{6.1}$$

Then, $u \in W^{1,1}_{\text{loc}}(\Omega)$ and for every $\omega \Subset \Omega$,

$$\|\nabla u\|_{L^1(\omega)} + \|g(u)\|_{L^1(\omega)} \leq C_\omega \Big(\|u\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} \Big). \tag{6.2}$$

Proof. Given $\delta > 0$, let

$$\Omega_\delta = \big\{ x \in \Omega;\ d(x, \partial\Omega) > \delta \big\}. \tag{6.3}$$

Let $\delta_0 > 0$ be such that $\omega \Subset \Omega_{2\delta_0}$. By standard elliptic linear estimates (see [17]), $u \in W^{1,1}_{\text{loc}}(\Omega)$ and

$$\begin{aligned} \|\nabla u\|_{L^1(\omega)} &\leq C_{\delta_0} \Big(\|u\|_{L^1(\Omega_{\delta_0})} + \|\mu\|_{\mathcal{M}(\Omega_{\delta_0})} + \|g(u)\|_{L^1(\Omega_{\delta_0})} \Big) \\ &\leq C_{\delta_0} \Big(\|u\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \|g(u)\|_{L^1(\Omega_{\delta_0})} \Big). \end{aligned} \tag{6.4}$$

Therefore, for every smooth subdomain $\omega \Subset \Omega$, u possesses a boundary trace in $L^1(\partial\omega)$. Consequently, using a Fubini-type argument, one can find $\delta_1 \in (0, \delta_0/2)$ such that

$$\|u\|_{L^1(\partial\Omega_{\delta_1})} \leq \frac{C}{\delta_0} \|u\|_{L^1(\Omega)}.$$

On the other hand, (see [15])

$$\int_{\Omega_{\delta_1}} |g(u(x))| \rho_{\delta_1}(x)\, dx \leq C \Big(\|u\|_{L^1(\partial\Omega_{\delta_1})} + \|\mu\|_{\mathcal{M}(\Omega_{\delta_1})} \Big),$$

where

$$\rho_\delta(x) = d(x, \partial\Omega_\delta) \quad \forall x \in \Omega_\delta.$$

Therefore,

$$\begin{aligned} \|g(u)\|_{L^1(\Omega_{\delta_0})} &\leq \frac{2}{\delta_0} \int_{\Omega_{\delta_1}} \big|g(u(x))\big| \rho_{\delta_1}(x)\, dx \\ &\leq C_{\delta_0} \Big(\|u\|_{L^1(\partial\Omega_{\delta_1})} + \|\mu\|_{\mathcal{M}(\Omega_{\delta_1})} \Big) \\ &\leq C_{\delta_0} \Big(\|u\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} \Big). \end{aligned} \tag{6.5}$$

Combining (6.4)–(6.5), the conclusion follows. □

We recall a result concerning the existence of solutions of (1.2) with L^1-boundary data (see [10]).

Lemma 6.2. *Let $\mu \in \mathcal{M}(\Omega)$. If the problem*

$$\begin{cases} -\Delta u + g(u) = \mu & \textit{in } \Omega, \\ u = f & \textit{on } \partial\Omega, \end{cases} \tag{6.6}$$

has a solution for some $f \in L^1(\partial\Omega)$, *in the sense that for every* $\zeta \in C_0^2(\overline{\Omega})$, $g(u)\zeta \in L^1(\Omega)$ *and*

$$-\int_\Omega u\Delta\zeta + \int_\Omega g(u)\zeta = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial n} f + \int_\Omega \zeta \, d\mu, \tag{6.7}$$

then it has a solution for every $f \in L^1(\partial\Omega)$.

In the next lemma, given two solutions u and v of (1.1), we show the existence of a solution above the subsolution $\max\{u, v\}$.

Lemma 6.3. *Let $\mu \in \mathcal{M}(\Omega)$. Assume that $u, v \in L^1(\Omega)$ satisfy*

$$-\Delta z + g(z) = \mu \quad \textit{in } \Omega. \tag{6.8}$$

Then, for every $\omega \Subset \Omega$ there exists $w \in L^1(\omega)$ such that

$$-\Delta w + g(w) = \mu \quad \textit{in } \omega,$$
$$w \geq \max\{u, v\} \quad \textit{a.e.},$$
$$\|w\|_{L^1(\omega)} \leq C_\omega\Big(\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)}\Big).$$

Proof. Using a Fubini-type argument, one can find $\delta > 0$ such that $\omega \Subset \Omega_\delta$ and

$$\|z\|_{L^1(\partial\Omega_\delta)} \leq C_\delta \|z\|_{L^1(\Omega)} \quad \text{for } z = u, v.$$

Let

$$f = \max\{u, v\} \quad \text{on } \partial\Omega_\delta.$$

By Lemma 6.2, there exists $w \in L^1(\Omega_\delta)$ such that

$$\begin{cases} -\Delta w + g(w) = \mu & \text{in } \Omega_\delta, \\ w = f & \text{on } \partial\Omega_\delta. \end{cases}$$

By elliptic estimates,

$$\|w\|_{L^1(\Omega_\delta)} \leq C\Big(\|f\|_{L^1(\partial\Omega_\delta)} + \|\mu\|_{\mathcal{M}(\Omega_\delta)}\Big).$$

Since

$$\|f\|_{L^1(\partial\Omega_\delta)} \leq \|u\|_{L^1(\partial\Omega_\delta)} + \|v\|_{L^1(\partial\Omega_\delta)} \leq C_\delta\Big(\|u\|_{L^1(\Omega_\delta)} + \|v\|_{L^1(\Omega_\delta)}\Big),$$

we deduce that

$$\|w\|_{L^1(\omega)} \leq \|w\|_{L^1(\Omega_\delta)} \leq C\Big(\|u\|_{L^1(\Omega_\delta)} + \|v\|_{L^1(\Omega_\delta)} + \|\mu\|_{\mathcal{M}(\Omega_\delta)}\Big).$$

We now show for instance that

$$w \geq u \quad \text{a.e.} \tag{6.9}$$

For every $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \geq 0$ in Ω, we have

$$\int_\Omega (u - w)\Delta\zeta = \int_{\partial\Omega} (u - w)\frac{\partial\zeta}{\partial n} + \int_\Omega \big[g(u) - g(w)\big]\zeta \geq \int_\Omega \big[g(u) - g(w)\big]\zeta.$$

Thus, by Lemma 5.1,

$$\int_\Omega (u-w)^+\Delta\zeta \ge \int_{\substack{\Omega\\ [u\ge w]}} \big[g(u)-g(w)\big]\zeta \ge 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega.$$

Therefore, $(u-w)^+ = 0$ a.e. In other words, (6.9) holds. A similar argument shows that $w \ge v$ a.e. ☐

Proof of Theorem 1.2. For every $k \ge 1$, we denote by u_k the solution of (1.2) with datum μ_k. We split the proof in two steps:

Step 1. Conclusion holds if $u_k \le v_k$ a.e., $\forall k \ge 1$.

Let $\omega \Subset \Omega$. By Lemma 6.1, both sequences $\big(g(u_k)\big)$ and $\big(g(v_k)\big)$ are bounded in $L^1(\omega)$. Passing to a subsequence if necessary, one can find $\tau_1, \tau_2 \in \mathcal{M}(\omega)$ such that

$$g(u_k) \overset{*}{\rightharpoonup} g(u^\#) + \tau_1 \quad \text{and} \quad g(v_k) \overset{*}{\rightharpoonup} g(v^\#) + \tau_2 \quad \text{weakly}^* \text{ in } \mathcal{M}(\omega).$$

Thus,

$$-\Delta u^\# + g(u^\#) = \mu - \tau_1 \quad \text{and} \quad -\Delta v^\# + g(v^\#) = \mu - \tau_2.$$

Our goal is to show that $\tau_1 = \tau_2$.

Since $u_k \le v_k$ a.e. and g is nondecreasing,

$$g(v_k) - g(u_k) \ge 0 \quad \text{a.e.}$$

Moreover,

$$g(v_k) - g(u_k) \overset{*}{\rightharpoonup} g(v^\#) - g(u^\#) + (\tau_2 - \tau_1) \quad \text{weakly}^* \text{ in } \mathcal{M}(\omega).$$

By Proposition 3.1, $g(v^\#) - g(u^\#)$ is the diffuse limit of $\big(g(v_k) - g(u_k)\big)$ with respect to Lebesgue measure; hence, $\tau_2 - \tau_1$ is its concentrated limit. Thus, by Corollary 2.2,

$$\tau_2 - \tau_1 \ge 0. \tag{6.10}$$

On the other hand,

$$\Delta(v_k - u_k) = g(v_k) - g(u_k) \quad \text{in } \omega.$$

Since $\tau_2 - \tau_1$ is also the concentrated limit of $\big(g(v_k) - g(u_k)\big)$ with respect to cap_{H^1} (see Proposition 3.2), it follows from Theorem 4.2 that

$$\tau_2 - \tau_1 \le 0. \tag{6.11}$$

Combining (6.10)–(6.11), we deduce that $\tau_1 = \tau_2$. In other words,

$$-\Delta u^\# + g(u^\#) = -\Delta v^\# + g(v^\#) \quad \text{in } \omega.$$

Since $\omega \Subset \Omega$ is arbitrary, the conclusion follows.

Step 2. Proof of Theorem 1.2 completed.

Take $\omega \Subset \tilde{\omega} \Subset \Omega$. By Lemma 6.3, there exists a bounded sequence $(w_k) \subset L^1(\tilde{\omega})$ such that

$$-\Delta w_k + g(w_k) = \mu_k \quad \text{in } \tilde{\omega},$$
$$w_k \ge \max\{u_k, v_k\} \quad \text{a.e.}$$

By Lemma 6.1, (w_k) is bounded in $W^{1,1}_{\mathrm{loc}}(\tilde{\omega})$. Passing to a subsequence if necessary, we may assume that

$$w_k \to w^\# \quad \text{in } L^1(\omega).$$

By the previous step,

$$-\Delta u^{\#} + g(u^{\#}) = -\Delta w^{\#} + g(w^{\#}) \quad \text{in } \omega,$$
$$-\Delta v^{\#} + g(v^{\#}) = -\Delta w^{\#} + g(w^{\#}) \quad \text{in } \omega.$$

Hence,

$$-\Delta u^{\#} + g(u^{\#}) = -\Delta v^{\#} + g(v^{\#}) \quad \text{in } \omega.$$

This concludes the proof. □

7. Some properties of $\mu^{\#}$

In this section we present comparison results for reduced limits in terms of the sequences (μ_k) or in terms of the nonlinearities g with which they are associated. We prove in particular a stronger version of Theorem 1.3.

Proposition 7.1. *Let $(\mu_k), (\nu_k) \subset \mathcal{G}$ be two bounded sequences with weak* limits μ, ν and reduced limits $\mu^{\#}, \nu^{\#}$, respectively. Then,*

$$\|\mu^{\#} - \nu^{\#}\|_{\mathcal{M}} \leq \|\mu - \nu\|_{\mathcal{M}} + \liminf_{k\to\infty} \|\mu_k - \nu_k\|_{\mathcal{M}}. \tag{7.1}$$

In particular, if $\mu = \nu$, then

$$\|\mu^{\#} - \nu^{\#}\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \|\mu_k - \nu_k\|_{\mathcal{M}}. \tag{7.2}$$

Proof. Let u_k and v_k be the solutions of

$$\begin{cases} -\Delta z + g(z) = \gamma & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.3}$$

associated to the measures μ_k and ν_k, respectively. By standard estimates (see [6, Corollary 4.B.1]), we have

$$\int_\Omega \big|g(u_k) - g(v_k)\big| \leq \|\mu_k - \nu_k\|_{\mathcal{M}} \quad \forall k \geq 1.$$

On the other hand, we know from Proposition 3.1 that $(\mu - \mu^{\#}) - (\nu - \nu^{\#})$ is the concentrated limit of the sequence $\big(g(u_k) - g(v_k)\big)$ with respect to Lebesgue measure. Letting $k \to \infty$, we deduce from Corollary 2.1 that

$$\big\|(\mu - \mu^{\#}) - (\nu - \nu^{\#})\big\|_{\mathcal{M}} \leq \liminf_{k\to\infty} \int_\Omega \big|g(u_k) - g(v_k)\big| \leq \liminf_{k\to\infty} \|\mu_k - \nu_k\|_{\mathcal{M}}.$$

The conclusion follows using the triangle inequality. □

If we know in addition that $\nu_k \leq \mu_k$, $\forall k \geq 1$, then one can deduce a stronger statement which implies Theorem 1.3 by taking $\nu_k = 0$, $\forall k \geq 1$.

Theorem 7.1. *Let $(\mu_k), (\nu_k) \subset \mathcal{G}$ be two bounded sequences with weak* limits μ, ν and reduced limits $\mu^{\#}, \nu^{\#}$, respectively. If*

$$\nu_k \leq \mu_k \quad \forall k \geq 1, \tag{7.4}$$

then

$$0 \leq \mu^{\#} - \nu^{\#} \leq \mu - \nu. \tag{7.5}$$

Proof. Let $u_k, v_k \in L^1(\Omega)$ be the solutions of (1.2) with data μ_k and ν_k, respectively. Then, both sequences $(u_k), (v_k) \subset L^1(\Omega)$ are bounded in $L^1(\Omega)$ and $u_k \geq v_k$ a.e. Thus,

$$g(u_k) - g(v_k) \geq 0 \quad \text{a.e.}$$

Since $(\mu - \mu^\#) - (\nu - \nu^\#)$ is the concentrated limit of $\big(g(u_k) - g(v_k)\big)$, we deduce from Corollary 2.2 that

$$(\mu - \mu^\#) - (\nu - \nu^\#) \geq 0. \tag{7.6}$$

It remains to show that $\mu^\# \geq \nu^\#$. For this purpose, write

$$\Delta(u_k - v_k) = g(u_k) - g(v_k) - (\mu_k - \nu_k).$$

Passing to a subsequence, we may assume that $(\mu_k - \nu_k)$ has a concentrated limit with respect to cap_{H^1}, which we will denote by σ. By Corollary 2.2,

$$0 \leq \sigma \leq \mu - \nu.$$

On the other hand, it follows from Proposition 3.2 that $(\mu - \mu^\#) - (\nu - \nu^\#) - \sigma$ is the concentrated limit of $\big(g(u_k) - g(v_k) - (\mu_k - \nu_k)\big)$ with respect to cap_{H^1}. Therefore, since $u_k \geq v_k$ a.e., $\forall k \geq 1$, we deduce from Theorem 4.2 that

$$(\mu - \mu^\#) - (\nu - \nu^\#) - \sigma \leq 0.$$

Hence,

$$\mu^\# - \nu^\# \geq \mu - \nu - \sigma \geq 0. \tag{7.7}$$

This establishes the proposition. $\square$

We now compare reduced limits associated to different nonlinearities.

Proposition 7.2. *Let $(\mu_k) \subset \mathcal{G}(g_1) \cap \mathcal{G}(g_2)$ be a bounded sequence with reduced limits $\mu_1^\#$ and $\mu_2^\#$ associated to g_1 and g_2, respectively. If $g_1 \leq g_2$, then*

$$\mu_1^\# \geq \mu_2^\#. \tag{7.8}$$

Proof. Let $u_k, v_k \in L^1(\Omega)$ be the solutions associated to (1.2) with datum μ_k and nonlinearities g_1 and g_2, respectively. Since $g_1 \leq g_2$, by comparison we have

$$u_k \geq v_k \quad \text{a.e.} \quad \forall k \geq 1.$$

On the other hand,

$$\Delta(u_k - v_k) = g(u_k) - g(v_k).$$

Since the concentrated limit of $\big(g(u_k) - g(v_k)\big)$ with respect to cap_{H^1} is

$$(\mu - \mu_1^\#) - (\mu - \mu_2^\#) = \mu_2^\# - \mu_1^\#,$$

it follows from Theorem 4.2 that $\mu_2^\# - \mu_1^\# \leq 0$. $\square$

The next result gives the main tool for studying reduced limits of sequences signed measures.

Proposition 7.3. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded sequence with weak* limit μ. Assume that*

$$\mu_k^+ \overset{*}{\rightharpoonup} \mu^+ \quad \text{and} \quad \mu_k^- \overset{*}{\rightharpoonup} \mu^- \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega). \tag{7.9}$$

Then, (μ_k) has a reduced limit $\mu^\#$ if and only if (μ_k^+) and $(-\mu_k^-)$ have reduced limits $\mu_1^\#$ and $\mu_2^\#$, respectively. In this case,

$$\mu_1^\# = (\mu^\#)^+ \quad \text{and} \quad \mu_2^\# = -(\mu^\#)^-. \tag{7.10}$$

In particular,

$$\mu^{\#} = \mu_1^{\#} + \mu_2^{\#} \tag{7.11}$$

and

$$\mu^{\#} = \mu \quad \text{if and only if} \quad \mu_1^{\#} = \mu^+ \ \text{and} \ \ \mu_2^{\#} = -\mu^-. \tag{7.12}$$

Proof. Passing to a subsequence if necessary, we may assume that $\mu^{\#}$, $\mu_1^{\#}$ and $\mu_2^{\#}$ exist. From Theorem 7.1, we have

$$0 \le \mu_1^{\#} - \mu^{\#} \le \mu^+ - \mu = \mu^-. \tag{7.13}$$

Applying the Hahn decomposition with respect to μ, we can write Ω in terms of two disjoint sets $E_1, E_2 \subset \Omega$, $\Omega = E_1 \cup E_2$ such that

$$\mu \ge 0 \ \text{ in } E_1 \quad \text{and} \quad \mu \le 0 \ \text{ in } E_2.$$

On the other hand, by Theorem 1.3,

$$0 \le \mu_1^{\#} \le \mu^+ \quad \text{and} \quad -\mu^- \le \mu_2^{\#} \le 0. \tag{7.14}$$

In particular, $\mu_1^{\#}$ is concentrated on E_1. It then follows from (7.13) that

$$(\mu^{\#})\lfloor_{E_1} = (\mu_1^{\#})\lfloor_{E_1} = \mu_1^{\#}.$$

Similarly, $\mu_2^{\#}$ is concentrated on E_2 and

$$(\mu^{\#})\lfloor_{E_2} = \mu_2^{\#}.$$

In particular, $\mu_1^{\#}$ and $\mu_2^{\#}$ are singular with respect to each other. Moreover,

$$\mu^{\#} = (\mu^{\#})\lfloor_{E_1} + (\mu^{\#})\lfloor_{E_2} = \mu_1^{\#} + \mu_2^{\#}.$$

Since, by (7.14), $\mu_1^{\#} \ge 0$ and $\mu_2^{\#} \le 0$, (7.10) follows. $\square$

8. Absolute continuity between μ and $\mu^{\#}$

We showed in Theorem 7.1 that if $(\mu_k) \subset \mathcal{G}$ is a bounded nonnegative sequence, then

$$0 \le \mu^{\#} \le \mu,$$

and thus $\mu^{\#} \ll \mu$. Our next result provides a sufficient condition on the sequence (μ_k) so that $\mu \ll \mu^{\#}$. This implies in particular that $\mu^{\#} = 0$ if and only if $\mu = 0$.

Theorem 8.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) = 0$ and*

$$\lim_{a,t \to +\infty} \frac{g(at)}{a g(t)} = +\infty. \tag{8.1}$$

Let $(\mu_k) \subset \mathcal{G}$ be a bounded nonnegative sequence with weak limit μ and reduced limit $\mu^{\#}$. Suppose that there exists $(U_k) \subset L^1(\Omega)$ such that for every $k \ge 1$,*

$$-\Delta U_k = \mu_k \ \text{ in } \Omega \quad \text{and} \quad g(U_k) \in L^1(\Omega). \tag{8.2}$$

If

$$\big(g(U_k)\big) \ \text{is bounded in } L^1(\Omega), \tag{8.3}$$

then μ and $\mu^{\#}$ are absolutely continuous with respect to each other.

Remark 8.1. If g is given by $g(t) = |t|^{q-1}t$, $\forall t \in \mathbb{R}$, where $q > 1$, then (8.1) holds and assumption (8.2)–(8.3) on (μ_k) is satisfied whenever (μ_k) is bounded in $W^{-2,q}(\Omega)$. In the next section, we shall study this nonlinearity in more detail in the supercritical case $q \geq \frac{N}{N-2}$.

Proof. Replacing Ω by a smaller domain if necessary, we may assume that $(U_k|_{\partial\Omega})$ is bounded in $L^1(\partial\Omega)$. Replacing g by g^+ if necessary, we may assume that

$$g(t) = 0 \quad \forall t \leq 0.$$

Given $\alpha \in (0,1)$, we then have

$$0 \leq g(\alpha U_k) \leq g(U_k) \quad \text{a.e.}$$

Thus, there exists $C_0 > 0$, independent of α, such that

$$\|g(\alpha U_k)\|_{L^1} \leq C_0 \quad \forall k \geq 1.$$

Let $\big(g(\alpha U_{k_j})\big)$ be a subsequence having diffuse and concentrated limits with respect to Lebesgue measure; denote by σ_α its concentrated limit. The proof of the theorem is based on the following assertions:

Claim 1. For every $\alpha \in (0,1)$,

$$\alpha\mu \leq \sigma_\alpha + \mu^{\#}. \tag{8.4}$$

Indeed, let v_j be such that

$$\begin{cases} -\Delta v_j + g(v_j) = \alpha\mu_{k_j} & \text{in } \Omega, \\ \qquad\qquad\quad v_j = \alpha U_{k_j} & \text{on } \partial\Omega. \end{cases} \tag{8.5}$$

Then, (v_j) is bounded in $L^1(\Omega)$ and, by comparison, $v_j \leq \alpha U_{k_j}$ a.e. Thus,

$$g(v_j) \leq g(\alpha U_{k_j}) \quad \text{a.e.}$$

Passing to a further subsequence, we may assume that $(\alpha\mu_{k_j})$ has a reduced limit $\mu_\alpha^{\#}$. It follows from Proposition 3.1 that the sequence $\big(g(v_j)\big)$ has concentrated limit $\alpha\mu - \mu_\alpha^{\#}$. Thus,

$$g(v_j) \overset{*}{\rightharpoonup} g(v_\alpha) + \alpha\mu - \mu_\alpha^{\#} \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega),$$

where v_α is the solution of (8.5) associated to $\mu_\alpha^{\#}$. Applying Corollary 2.2 to the nonnegative sequence $\big(g(\alpha U_{k_j}) - g(v_j)\big)$, we deduce that its concentrated limit is nonnegative,

$$\sigma_\alpha - \alpha\mu + \mu_\alpha^{\#} \geq 0. \tag{8.6}$$

On the other hand, since $\alpha\mu \leq \mu$, it follows from Theorem 7.1 that

$$\mu_\alpha^{\#} \leq \mu^{\#}. \tag{8.7}$$

Combining (8.6)–(8.7), we obtain (8.4).

Claim 2.

$$\lim_{\alpha \to 0} \frac{\|\sigma_\alpha\|_{\mathcal{M}}}{\alpha} = 0. \tag{8.8}$$

Given $\varepsilon > 0$, take $a_0, t_0 > 1$ such that

$$\frac{g(at)}{ag(t)} \geq \frac{1}{\varepsilon} \quad \forall a \geq a_0, \quad \forall t \geq t_0. \tag{8.9}$$

For every $\alpha \in (0, 1/a_0)$, we write

$$g(\alpha U_{k_j}) = g(\alpha U_{k_j})\chi_{[\alpha U_{k_j} < t_0]} + g(\alpha U_{k_j})\chi_{[\alpha U_{k_j} \geq t_0]}.$$

Since the first term in the right-hand side is uniformly bounded, σ_α must be the concentrated limit of $\big(g(\alpha U_{k_j})\chi_{[\alpha U_{k_j} \geq t_0]}\big)$. Thus, by Corollary 2.1,

$$\|\sigma_\alpha\|_{\mathcal{M}} \leq \liminf_{j\to\infty} \int_{[\alpha U_{k_j} \geq t_0]} g(\alpha U_{k_j}). \tag{8.10}$$

On the other hand, applying (8.9) with $a = 1/\alpha$ and $t = \alpha U_{k_j}$, we get

$$g(\alpha U_{k_j})\chi_{[\alpha U_{k_j} \geq t_0]} \leq \varepsilon \alpha g(U_{k_j}) \quad \forall j \geq 1.$$

Therefore,

$$\|\sigma_\alpha\|_{\mathcal{M}} \leq \varepsilon\alpha \liminf_{j\to\infty} \int_\Omega g(U_{k_j}) \leq \varepsilon\alpha C_0.$$

In other words,

$$\frac{\|\sigma_\alpha\|_{\mathcal{M}}}{\alpha} \leq \varepsilon C_0 \quad \forall \alpha \in (0, 1/a_0).$$

Since $\varepsilon > 0$ is arbitrary, the claim follows.

We now complete the proof of Theorem 8.1. Since $0 \leq \mu^\# \leq \mu$, we only need to show that $\mu \ll \mu^\#$. For this purpose, take a Borel set $E \subset \Omega$ such that $\mu^\#(E) = 0$. By Claim 1,

$$\alpha\mu(E) \leq \sigma_\alpha(E) \quad \forall \alpha \in (0, 1).$$

Thus,

$$\mu(E) \leq \frac{\sigma_\alpha(E)}{\alpha} \leq \frac{\|\sigma_\alpha\|_{\mathcal{M}}}{\alpha} \quad \forall \alpha \in (0, 1).$$

Letting $\alpha \to 0$, by Claim 2 we deduce that $\mu(E) = 0$. The proof is complete. □

9. Reduced limits and $W^{-2,q}$-weak convergence

In this section we assume that $N \geq 3$ and we focus on the case of power nonlinearities

$$g(t) = |t|^{q-1}t \quad \forall t \in \mathbb{R}, \tag{9.1}$$

in the supercritical range $q \geq \frac{N}{N-2}$. Denote by $\mathcal{G}^q$ the set of finite measures in Ω for which the equation

$$-\Delta u + |u|^{q-1}u = \mu \quad \text{in } \Omega \tag{9.2}$$

has a solution and we denote by $\mathcal{G}^q_0$ the set of finite measures in Ω for which the Dirichlet problem

$$\begin{cases} -\Delta u + |u|^{q-1}u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.3}$$

has a solution. For every $\mu \in \mathcal{M}(\Omega)$,

$$\mu \in \mathcal{G}^q_0 \quad \text{if and only if} \quad \mu \in L^1(\Omega) + W^{-2,q}(\Omega)$$

and Baras-Pierre [2] proved that $\mu \in \mathcal{G}^q_0$ if and only if the measure μ is diffuse relative to the capacity $\operatorname{cap}_{W^{2,q'}}$. Since, by Theorem A.1 in the Appendix, $\mathcal{G}^q = \mathcal{G}^q_0$, we have in this way a complete characterization of measures in $\mathcal{G}^q$.

Concerning sequences, if $(\mu_k) \subset \mathcal{G}^q$ is a bounded sequence strongly converging in $W^{-2,q}(\Omega)$, then its reduced limit and its weak* limit coincide; see [6, Theorem 4.13]. The goal of this section is to investigate what happens if (μ_k) is bounded in $W^{-2,q}(\Omega)$ but does not necessarily converge strongly in this space. We start by proving a more precise version of Theorem 1.4.

Theorem 9.1. *Given $q \geq \frac{N}{N-2}$, let $(\mu_k) \subset \mathcal{G}^q$ be a bounded sequence of nonnegative measures with weak* limit μ and reduced limit $\mu^\#$. If in addition (μ_k) is bounded in $W^{-2,q}(\Omega)$, then μ and $\mu^\#$ are absolutely continuous with respect to each other. Moreover, there exists $C_q > 0$ such that for every Borel set $E \subset \Omega$,*

$$\frac{C_q}{\Gamma_0^{\frac{1}{q-1}}} \big[\mu(E)\big]^{\frac{q}{q-1}} \leq \mu^\#(E) \leq \mu(E), \tag{9.4}$$

where $\Gamma_0 = \sup\limits_{k \geq 1} \big\{ \|\mu_k\|_{\mathcal{M}} + \|\mu_k\|_{W^{-2,q}}^q \big\}$.

Proof. We use the same notation as in the proof of Theorem 8.1. This theorem applies in the present case. In addition, by Theorem 7.1, $\mu^\# \leq \mu$. Therefore we only have to prove the first inequality in (9.4).
Recall that, by (8.4),

$$\alpha\mu - \sigma_\alpha \leq \mu^\# \quad \forall \alpha \in (0,1).$$

On the other hand, by (8.10),

$$\|\sigma_\alpha\|_{\mathcal{M}} \leq \alpha^q C_0 \leq \alpha^q \Gamma_0.$$

Therefore, given a Borel set $E \subset \Omega$,,

$$\alpha\mu(E) - \alpha^q \Gamma_0 \leq \alpha\mu(E) - \sigma_\alpha(E) \leq \mu^\#(E) \quad \forall \alpha \in (0,1).$$

Since $\mu(E) \leq \Gamma_0$, the left-hand side achieves a positive maximum in the interval $(0,1)$. Computing this maximum we obtain

$$\left(\frac{q-1}{q^{q-1}}\right) \frac{\big[\mu(E)\big]^{\frac{q}{q-1}}}{\Gamma_0^{\frac{1}{q-1}}} \leq \mu^\#(E). \tag{9.5}$$

This completes the proof. □

For every bounded sequence of nonnegative measures $(\mu_k) \subset \mathcal{G}^q$ converging weakly* to μ, $0 \leq \mu^\# \leq \mu$. We have just showed that if in addition (μ_k) is bounded in $W^{-2,q}(\Omega)$, then $\mu \ll \mu^\#$. Since $\mu \in W^{-2,q}(\Omega)$ and this space is contained in $\mathcal{G}^q$, one might expect that $\mu^\# = \mu$. We now present a striking example showing that this need not be the case.

Theorem 9.2. *For every $q \geq \frac{N}{N-2}$ there exists a sequence of nonnegative functions $(f_k) \subset C^\infty(\overline{\Omega})$, bounded in $L^1(\Omega)$ and in $W^{-2,q}(\Omega)$, such that its weak* limit f and its reduced limit $f^\#$ associated to the equation*

$$-\Delta u + |u|^{q-1}u = h \quad \text{in } \Omega \tag{9.6}$$

are different. In other words, if u_k is a solution of (9.6) with datum f_k and if $u_k \to u^\#$ in $L^1(\Omega)$, then $u^\#$ is not a solution of (9.6) with datum f.

We first recall some known estimates. In what follows, we say that $A \sim B$ if there exist constants $C_1, C_2 > 0$ such that $A \leq C_1 B$ and $B \leq C_2 A$.

Lemma 9.1. *Let $a > 0$. For every $R > a$ we have*

$$\int_{B_R} \frac{dx}{(|x| + a)^p} \sim \begin{cases} a^{N-p} & \text{if } p > N, \\ 1 + \log \dfrac{R}{a} & \text{if } p = N. \end{cases} \tag{9.7}$$

The proof is straightforward and will be omitted.

Given $f \in L^1(\mathbb{R}^N)$, consider the Newtonian potential associated to f:

$$Gf(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2}}\, dy \quad \forall x \in \mathbb{R}^N. \tag{9.8}$$

It is well-known that

$$-\Delta(Gf) = \gamma_N f \quad \text{in } \mathbb{R}^N,$$

where $\gamma_N = N(N-2)|B_1|$ and $|B_1|$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^N$.

Lemma 9.2. *Given $p \geq N$ and $a > 0$, let*

$$h_p(x) = \frac{1}{(|x| + a)^p} \quad \forall x \in \mathbb{R}^N. \tag{9.9}$$

Then, for every $R > a$ and every $x \in B_R$,

$$G[h_p \chi_{B_R}](x) \sim \begin{cases} \dfrac{a^{N-p}}{(|x| + a)^{N-2}} & \text{if } p > N, \\ \dfrac{1 + \log^+(|x|/a)}{(|x| + a)^{N-2}} & \text{if } p = N. \end{cases} \tag{9.10}$$

Proof. Clearly, $G[h_p \chi_{B_R}]$ is radial and

$$G[h_p \chi_{B_R}](x) \to 0 \quad \text{as } |x| \to \infty.$$

Denote $v(r) := G[h_p \chi_{B_R}](x)$, where $r = |x|$. We then have

$$\begin{aligned} v'(r) &= \frac{1}{|\partial B_r|} \int_{\partial B_r} \frac{\partial}{\partial n} G[h_p \chi_{B_R}] \\ &= \frac{C_N}{r^{N-1}} \int_{B_r} \Delta G[h_p \chi_{B_R}] = -\frac{\widetilde{C}_N}{r^{N-1}} \int_{B_r} h_p \chi_{B_R}. \end{aligned}$$

Assume that $p > N$. In this case, a straightforward computation shows that

$$\int_{B_r} h_p \chi_{B_R} \sim \begin{cases} \dfrac{r^N}{a^p} & \text{if } r \leq a, \\ a^{N-p} & \text{if } r > a. \end{cases}$$

Thus,

$$v'(r) \sim \begin{cases} -\dfrac{r}{a^p} & \text{if } r \leq a, \\ -\dfrac{a^{N-p}}{r^{N-1}} & \text{if } r > a. \end{cases}$$

Since

$$G[h_p \chi_{B_R}](x) = v(r) = -\int_r^\infty v'(t)\, dt,$$

estimate (9.10) for $p > N$ follows.
The case $p = N$ can be deduced in a similar way using

$$\int_{B_r} h_p \chi_{B_R} \sim \begin{cases} \dfrac{r^N}{a^N} & \text{if } r \le a, \\ 1 + \log \dfrac{r}{a} & \text{if } a < r < R, \\ 1 + \log \dfrac{R}{a} & \text{if } r \ge R. \end{cases}$$

This establishes the lemma. $\square$

Given $k \ge 1$, we write the unit cube $[0,1]^N$ as a union of k^N cubes of sides $\frac{1}{k}$ such that their interiors, $Q_1, \dots, Q_{k^N}$, are disjoint. If we denote by x_i the center of the open cube Q_i, then $Q_i = Q_0 + x_i$, where

$$Q_0 = \left(-\tfrac{1}{2k}, \tfrac{1}{2k}\right)^N.$$

Lemma 9.3. *Given a radially non-increasing function $h \in C^\infty(\mathbb{R}^N)$ with $h \ge 0$, let*

$$H(x) = \sum_{i=1}^{k^N} h(x - x_i)\chi_{Q_i}(x) \quad \forall x \in (0,1)^N. \tag{9.11}$$

Then, for every $i \in \{1, \dots, k^N\}$,

$$GH(x) \sim G[h\chi_{Q_0}](x - x_i) + k^N \int_{Q_0} h \quad \text{on } Q_i. \tag{9.12}$$

Proof. Given $i \in \{1, \dots, k^N\}$, let

$$J_1 = \{j \; ; \; \overline{Q}_j \cap \overline{Q}_i \ne \emptyset\} \quad \text{and} \quad J_2 = \{j \; ; \; \overline{Q}_j \cap \overline{Q}_i = \emptyset\}.$$

Denote $h_i(x) := h(x - x_i)\chi_{Q_i}(x)$. Using this notation,

$$Gh_i(x) = G[h\chi_{Q_0}](x - x_i).$$

Since h is radially non-increasing, for every $x \in Q_i$ and $j \in \{1, \dots, k^N\}$ we have

$$Gh_i(x) = G[h\chi_{Q_0}](x - x_i) \ge G[h\chi_{Q_0}](x - x_j) = Gh_j(x).$$

In particular,

$$\sum_{j \in J_1} Gh_j(x) \sim Gh_i(x) \quad \text{on } Q_i. \tag{9.13}$$

On the other hand, for every $x \in Q_i$ and $j \in J_2$,

$$Gh_j(x) \sim \frac{1}{[d(Q_j, Q_i)]^{N-2}} \int_{Q_0} h.$$

Since the number of cubes Q_t at distance $\sim \ell/k$ from Q_i is of the order of ℓ^{N-1}, then for every $x \in Q_i$ we have

$$\begin{aligned} \sum_{j \in J_2} Gh_j(x) &\sim \left\{ \sum_{\ell=1}^{k} \sum_{d(Q_t,Q_i) \sim \frac{\ell}{k}}{}_t \frac{1}{\big[d(Q_t, Q_i)\big]^{N-2}} \right\} \int_{Q_0} h \\ &\sim \left\{ \sum_{\ell=1}^{k} \frac{\ell^{N-1}}{(\ell/k)^{N-2}} \right\} \int_{Q_0} h \sim k^N \int_{Q_0} h. \end{aligned} \tag{9.14}$$

Combining (9.13)–(9.14), we obtain (9.12). □

Proof of Theorem 9.2. Without loss of generality, we may assume that $\Omega = (0,1)^N$. We split the proof in two parts:
Case 1. $q > \frac{N}{N-2}$.

Let $\varphi \in C_0^\infty(B_1)$ be a radially non-increasing function with $\varphi \geq 0$ in Ω and $\int_{B_1} \varphi = 1$. Given $\alpha > 0$, we take $a_k > 0$ so that

$$\frac{a_k^{N-(N-2)q}}{k^{N(q-1)}} = \alpha \quad \forall k \geq 1 \tag{9.15}$$

and define

$$H_k(x) = \frac{1}{k^N a_k^N} \sum_{i=1}^{k^N} \varphi\Big(\frac{x - x_i}{a_k}\Big) \quad \forall x \in (0,1)^N, \tag{9.16}$$

where $(x_i)_{i=1}^{k^N}$ are the centers of the open cubes $(Q_i)_{i=1}^{k^N}$. Let

$$f_k = \gamma_N H_k + (GH_k)^q. \tag{9.17}$$

We show that for $\alpha > 0$ sufficiently large the weak* limit and the reduced limit of (f_k) are different. For this end, let

$$\varphi_k(x) = \frac{1}{a_k^N} \varphi\Big(\frac{x - x_i}{a_k}\Big) \quad \forall x \in \mathbb{R}^N.$$

Since

$$G\varphi(x) \sim \frac{1}{(|x| + 1)^{N-2}} \quad \forall x \in \mathbb{R}^N,$$

one obtains, by scaling,

$$G\varphi_k(x) \sim \frac{1}{(|x| + a_k)^{N-2}} \quad \forall x \in \mathbb{R}^N.$$

It thus follows from Lemma 9.3 that for every $x \in Q_i$, $i = 1, \dots, k^N$,

$$GH_k(x) \sim \frac{1}{k^N} G\varphi_k(x - x_i) + 1 \sim \frac{1}{k^N} \frac{1}{(|x - x_i| + a_k)^{N-2}} + 1. \tag{9.18}$$

Thus, by Lemma 9.1,

$$\int\limits_{(0,1)^N} (GH_k)^q \sim \frac{k^N}{k^{Nq}} \int_{Q_0} \frac{dx}{(|x| + a_k)^{(N-2)q}} + 1 \sim \frac{a_k^{N-(N-2)q}}{k^{N(q-1)}} + 1 = \alpha + 1. \tag{9.19}$$

In particular,

$$\int\limits_{(0,1)^N} f_k \sim \alpha + 1 \quad \forall k \geq 1. \tag{9.20}$$

Let $A_\delta = (0,1)^N \setminus (\delta, 1-\delta)^N$. A similar computation shows that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{A_\delta} f_k < \varepsilon \quad \forall k \geq 1. \tag{9.21}$$

By (9.18),

$$(GH_k)^q(x) \sim \frac{1}{k^{Nq}} \frac{1}{(|x - x_i| + a_k)^{(N-2)q}} + 1 \quad \text{in } Q_i.$$

Applying Lemmas 9.2–9.3, for every $x \in Q_i$ we have

$$G\big[(GH_k)^q\big](x) \sim \frac{1}{k^{Nq}} \frac{a_k^{N-(N-2)q}}{(|x-x_i| + a_k)^{N-2}} + \frac{1}{(|x-x_i|+1)^{N-2}} + \alpha + 1 \tag{9.22}$$
$$\sim \frac{1}{k^{Nq}} \frac{a_k^{N-(N-2)q}}{(|x-x_i| + a_k)^{N-2}} + \alpha + 1.$$

Thus, by Lemma 9.1,

$$\int\limits_{(0,1)^N} \Big\{G\big[(GH_k)^q\big]\Big\}^q \sim k^N \bigg(\frac{a_k^{N-(N-2)q}}{k^{Nq}}\bigg)^q a_k^{N-(N-2)q} + \alpha^q + 1$$
$$= \bigg(\frac{a_k^{N-(N-2)q}}{k^{N(q-1)}}\bigg)^{q+1} + \alpha^q + 1 = \alpha^{q+1} + \alpha^q + 1 \sim \alpha^{q+1} + 1.$$

Let v_k be such that

$$\begin{cases} -\Delta v_k = f_k & \text{in } (0,1)^N, \\ \quad v_k = 0 & \text{on } \partial(0,1)^N. \end{cases}$$

Since $0 \le v_k \le Gf_k$, we have

$$\int\limits_{(0,1)^N} v_k^q \le \int\limits_{(0,1)^N} (Gf_k)^q \lesssim \alpha^{q+1} + 1 \quad \forall k \ge 1.$$

In particular, the sequence (f_k) is bounded in $W^{-2,q}(\Omega)$ and

$$\|f_k\|_{W^{-2,q}} \lesssim \alpha^{\frac{q+1}{q}} + 1 \quad \forall k \ge 1.$$

Let

$$u_k = GH_k \quad \text{in } (0,1)^N.$$

Then, u_k satisfies the equation

$$-\Delta u_k + u_k^q = f_k \quad \text{in } (0,1)^N$$

and

$$u_k \to u \quad \text{in } L^1\big((0,1)^N\big),$$

where u satisfies

$$-\Delta u = 1 \quad \text{in } (0,1)^N.$$

In other words, $f^\# = 1 + u^q$ is the reduced limit of the sequence (f_k); hence,

$$\int\limits_{(0,1)^N} f^\# \sim 1,$$

independently of α. On the other hand, passing to a subsequence if necessary, we have

$$f_k \overset{*}{\rightharpoonup} f \quad \text{weakly}^* \text{ in } \mathcal{M}\big((0,1)^N\big).$$

In view of (9.20)–(9.21),

$$\int\limits_{(0,1)^N} f \sim \alpha + 1.$$

Thus, by taking $\alpha > 0$ sufficiently large, we must have $f^\# \ne f$. This establishes the result when $q > \frac{N}{N-2}$.

Case 2. $q = \frac{N}{N-2}$.

Let H_k and f_k be given by (9.16) and (9.17), respectively, where $a_k > 0$ is now given by

$$\frac{1}{k^{\frac{2N}{N-2}}} \log \frac{1}{ka_k} = \alpha \quad \forall k \geq 1. \tag{9.15'}$$

Note that (9.18) still holds. Hence, by Lemma 9.1,

$$\int_{(0,1)^N} (GH_k)^{\frac{N}{N-2}} \sim \frac{1}{k^{\frac{2N}{N-2}}} \left(1 + \log \frac{1}{ka_k}\right) + 1 \sim \alpha + 1, \tag{9.19'}$$

from which (9.20) follows. By Lemmas 9.2–9.3, estimate (9.22) now becomes

$$G\big[(GH_k)^{\frac{N}{N-2}}\big](x) \sim \frac{1}{k^{\frac{N2}{N-2}}} \frac{1 + \log^+\left(\frac{|x-x_i|}{a_k}\right)}{\big(|x-x_i| + a_k\big)^{N-2}} + \alpha + 1 \quad \text{in } Q_i. \tag{9.22'}$$

Therefore,

$$\begin{aligned}
\int_{(0,1)^N} \Big\{G\big[(GH_k)^{\frac{N}{N-2}}\big]\Big\}^{\frac{N}{N-2}} &\sim \frac{k^N}{k^{\frac{N^3}{(N-2)^2}}} \left[1 + \left(\log \frac{1}{ka_k}\right)^{\frac{2(N-1)}{N-2}}\right] + \alpha^{\frac{N}{N-2}} + 1 \\
&\sim \left[\frac{1}{k^{\frac{2N}{N-2}}} \log \frac{1}{ka_k}\right]^{\frac{2(N-1)}{N-2}} + \alpha^{\frac{N}{N-2}} + 1 \sim \alpha^{\frac{2(N-1)}{N-2}} + 1.
\end{aligned}$$

Proceeding as in the previous case, we deduce that the weak* limit and the reduced limit of the sequence (f_k) are different for $\alpha > 0$ sufficiently large. The proof is complete. □

10. Reduced limits for $g(t) = |t|^{q-1}t$

Given a bounded sequence $(\mu_k) \subset \mathcal{G}^q$, consider a splitting (α_k) and (σ_k) into an equidiffuse and a concentrating parts relative to $\mathrm{cap}_{W^{2,q'}}$. In this section, we show that the reduced limits of (μ_k) and (α_k) associated to the nonlinearity $g(t) = |t|^{q-1}t$ coincide.

We first study the case where the sequence (μ_k) is concentrating.

Proposition 10.1. *Given $q \geq \frac{N}{N-2}$, let $(\mu_k) \subset \mathcal{G}^q$ be a bounded sequence with reduced limit $\mu^\#$. If (μ_k) is concentrating with respect to $\mathrm{cap}_{W^{2,q'}}$, then*

$$\mu^\# = 0. \tag{10.1}$$

Proof. In view of Proposition 7.3, it suffices to prove the result when the sequence (μ_k) is nonnegative. For each $k \geq 1$, assume that u_k satisfies

$$\begin{cases} -\Delta u_k + |u_k|^{q-1}u_k = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.2}$$

Passing to a subsequence if necessary, we may assume that $u_k \to u^\#$ in $L^1(\Omega)$ and a.e. By a comparison principle, $u_k \geq 0$ a.e. Let (E_k) be a sequence of Borel subset of Ω such that

$$\mathrm{cap}_{W^{2,q'}}(E_k) \to 0 \quad \text{and} \quad |\mu_k|(\Omega \setminus E_k) \to 0. \tag{10.3}$$

From the regularity of $\operatorname{cap}_{W^{2,q'}}$ and μ_k, we may assume that each E_k is compact. Moreover, there exists a sequence $(\varphi_k) \subset C_0^\infty(\Omega)$ such that

$$(10.4) \quad 0 \le \varphi_k \le 1 \ \text{ in } \Omega, \quad \varphi_k = 1 \ \text{ on } E_k \quad \text{and} \quad \int_\Omega |D^2 \varphi_k|^p \le C \operatorname{cap}_{W^{2,q'}}(E_k).$$

Let

$$F_k = \big\{ x \in \Omega \ ; \ \varphi_k(x) \ge 1/2 \big\}.$$

Then,

$$\operatorname{cap}_{W^{2,q'}}(F_k) \le 2^{q'} \int_\Omega |D^2 \varphi_k|^{q'} \to 0,$$

We claim that the sequence (u_k^q) is concentrating with respect to $\operatorname{cap}_{W^{2,q'}}$. In order to prove this, it suffices to show that

$$(10.5) \qquad \int_{\Omega \setminus F_k} u_k^q \to 0.$$

Using φ_k as a test function in (10.2), we get

$$(10.6) \qquad \int_\Omega u_k^q \varphi_k = \int_\Omega \varphi_k \, d\mu_k + \int_\Omega u_k \Delta \varphi_k \quad \forall k \ge 1.$$

In view of (10.2), $\|u_k\|_{L^q} \le \|\mu_k\|_{\mathcal{M}}$. Therefore, by (10.6),

$$(10.7) \qquad \frac{1}{2} \int_{\Omega \setminus F_k} u_k^q \le \int_\Omega u_k^q (1 - \varphi_k) \le \int_\Omega (1 - \varphi_k) \, d\mu_k - \int_\Omega u_k \Delta \varphi_k.$$

We show that both terms in the right-hand side of this estimate converge to 0 as $k \to \infty$. By (10.3),

$$(10.8) \qquad \int_\Omega (1 - \varphi_k) \, d|\mu_k| \le |\mu_k|(\Omega \setminus E_k) \to 0,$$

Furthermore, by (10.4),

$$(10.9) \qquad \left| \int_\Omega u_k \Delta \varphi_k \right| \le \|u_k\|_{L^q} \|\Delta \varphi_k\|_{L^{q'}} \le C \|D^2 \varphi_k\|_{L^{q'}} \to 0.$$

Combining (10.7)–(10.9), we get

$$\int_{\Omega \setminus F_k} u_k^q \to 0.$$

Thus, the sequence (u_k^q) is concentrating. Since $u_k \to u^\#$ a.e., this implies that $u^\# = 0$ a.e. We deduce that $u_k \to 0$ in $L^1(\Omega)$ and $\mu^\# = 0$. □

Remark 10.1. Let $q \ge \frac{N}{N-2}$. Then, for every $\mu \in \mathcal{M}(\Omega)$ there exists a bounded sequence $(\mu_k) \subset \mathcal{G}^q$ converging weakly* to μ but having reduced limit zero with respect to $g(t) = |t|^{q-1} t$. In fact, let (τ_k) be a sequence consisting of linear combinations of Dirac masses such that

$$\tau_k \overset{*}{\rightharpoonup} \mu \quad \text{weakly* in } \mathcal{M}(\Omega),$$

and let (ρ_k) be a sequence of smooth mollifiers. For every $j \ge 1$, the reduced limit of the sequence $(\rho_k * \tau_j)_{k \ge 1}$ equals the reduced measure τ_j^*, which is zero. Hence, there exists $k_j \ge j$ such that the solution of

$$\begin{cases} -\Delta u_j + |u_j|^{q-1} u_j = \rho_{k_j} * \tau_j & \text{in } \Omega, \\ \qquad\qquad\qquad u_j = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\|u_j\|_{L^1} \leq \frac{1}{j}$. Therefore, the sequence $(\rho_{k_j} * \tau_j)$ has weak* limit μ but its reduced limit is zero.

We now present the main result of this section.

Theorem 10.1. *Given $q \geq \frac{N}{N-2}$, let $(\mu_k) \subset \mathcal{G}^q$ be a bounded sequence, and let $(\alpha_k), (\sigma_k) \subset \mathcal{M}(\Omega)$ be a decomposition of (μ_k) satisfying (B_1)–(B_2) with respect to $\mathrm{cap}_{W^{2,q'}}$. If (μ_k) has a reduced limit $\mu^\#$, then $\mu^\#$ is also the reduced limit of (α_k).*

By Theorem 9.2, $\mu^\#$ need not coincide with the diffuse limit of (μ_k) with respect to $\mathrm{cap}_{W^{2,q'}}$, which is by definition the weak* limit of the sequence (α_k). However, we show that the *reduced limits* of the two sequences coincide.

For the proof of Theorem 10.1, we need two lemmas.

Lemma 10.1. *Let $(\mu_k) \subset \mathcal{G}^q$ be a bounded sequence. For each $k \geq 1$, let u_k be the solution of*

$$\begin{cases} -\Delta u_k + |u_k|^{q-1} u_k = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.10}$$

If (μ_k) is equidiffuse with respect to $\mathrm{cap}_{W^{2,q'}}$, then so is the sequence $\big(|u_k|^q\big)$.

Proof. Assume by contradiction that $\big(|u_k|^q\big)$ is not equidiffuse. Then, passing to a subsequence if necessary, one can find $\varepsilon > 0$ and a sequence of Borel subsets (E_k) of Ω such that

$$\mathrm{cap}_{W^{2,q'}}(E_k) \to 0 \quad \text{and} \quad \int_{E_k} |u_k|^q \geq \varepsilon \quad \forall k \geq 1.$$

By regularity of $\mathrm{cap}_{W^{2,q'}}$ and of the Lebesgue measure, we may assume that each set E_k is compact. Moreover, there exists a sequence $(\varphi_k) \subset C_0^\infty(\Omega)$ satisfying (10.4). In particular, $\varphi_k \to 0$ in $W^{2,q'}(\Omega)$. Passing to a subsequence if necessary, we may assume that $\varphi_k \to 0$ q.e. with respect to $\mathrm{cap}_{W^{2,q'}}$.
Let v_k be the solution of

$$\begin{cases} -\Delta v_k + |v_k|^{q-1} v_k = |\mu_k| & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.11}$$

Since $|\mu_k| \geq 0$, we have $v_k \geq 0$ a.e. Using φ_k as a test function, we get

$$\int_\Omega v_k^q \varphi_k = \int_\Omega \varphi_k \, d|\mu_k| + \int_\Omega v_k \Delta\varphi_k \quad \forall k \geq 1. \tag{10.12}$$

Since (φ_k) is uniformly bounded, $\varphi_k \to 0$ q.e. with respect to $\mathrm{cap}_{W^{2,q'}}$, and (μ_k) is equidiffuse,

$$\int_\Omega \varphi_k \, d|\mu_k| \to 0. \tag{10.13}$$

Moreover, as in the proof of Proposition 10.1,

$$\int_\Omega v_k \Delta\varphi_k \to 0. \tag{10.14}$$

Combining (10.12)–(10.14), we deduce that

$$\int_\Omega v_k^q \varphi_k \to 0.$$

Since $|u_k| \le v_k$ a.e., this contradicts the assumption

$$\int_{E_k} |u_k|^q \varphi_k \ge \varepsilon \quad \forall k \ge 1.$$

Therefore, the sequence $\big(|u_k|^q\big)$ must be equidiffuse. □

The following estimate will be used in the proof of Theorem 10.1.

Lemma 10.2. *Given $v, w \in L^q(\Omega)$, let*

$$h = |v+w|^{q-1}(v+w) - |v|^{q-1}v - |w|^{q-1}w. \tag{10.15}$$

Then, there exists a constant $C > 0$ such that for every Borel set $F \subset \Omega$,

$$\|h\|_{L^1(\Omega)} \le C\Big(\|v\|_{L^q(\Omega)}^{q-1} + \|w\|_{L^q(\Omega)}^{q-1}\Big)\Big(\|v\|_{L^q(F)} + \|w\|_{L^q(\Omega\setminus F)}\Big). \tag{10.16}$$

Proof. We first write

$$\|h\|_{L^1(\Omega)} = \int_F |h| + \int_{\Omega\setminus F} |h|. \tag{10.17}$$

We show that

$$\int_F |h| \le C\Big(\|v\|_{L^q(\Omega)}^{q-1} + \|w\|_{L^q(\Omega)}^{q-1}\Big)\|v\|_{L^q(F)}. \tag{10.18}$$

By the triangle inequality,

$$\int_F |h| \le \int_F \Big||v+w|^{q-1}(v+w) - |w|^{q-1}w\Big| + \int_F |v|^q. \tag{10.19}$$

Denote by I the first integral in the right-hand side of this inequality. In order to estimate I we use the following elementary estimate,

$$\Big||a+b|^{q-1}(a+b) - |b|^{q-1}b\Big| \le q\big(|a+b|^{q-1} + |b|^{q-1}\big)|a| \quad \forall a, b \in \mathbb{R}.$$

In fact, applying this estimate with $a = v(x)$ and $b = w(x)$, and integrating it over F, one gets

$$I \le q\bigg(\int_F |v+w|^{q-1}|v| + \int_F |w|^{q-1}|v|\bigg).$$

Thus, by Hölder's inequality,

$$I \le q\Big(\|v+w\|_{L^q(F)}^{q-1} + \|w\|_{L^q(F)}^{q-1}\Big)\|v\|_{L^q(F)} \le C\Big(\|v\|_{L^q(\Omega)}^{q-1} + \|w\|_{L^q(\Omega)}^{q-1}\Big)\|v\|_{L^q(F)}.$$

Inserting this estimate into (10.19), we get

$$\int_F |h| \le C\Big(\|v\|_{L^q(\Omega)}^{q-1} + \|w\|_{L^q(\Omega)}^{q-1}\Big)\|v\|_{L^q(F)} + \|v\|_{L^q(\Omega)}^{q-1}\|v\|_{L^q(F)}.$$

This gives (10.18). Interchanging the roles of v and w, and replacing F by $\Omega\setminus F$, one gets a similar estimate for the last integral in (10.17). Combining these estimates, one deduces (10.16). □

Proof of Theorem 10.1. For every $k \geq 1$, let v_k and w_k be the solutions of

$$\begin{cases} -\Delta z + |z|^{q-1}z = \gamma & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.20}$$

with data α_k and σ_k, respectively. Adding both equations, we observe that $v_k + w_k$ also satisfies problem (9.4) with datum

$$\lambda_k = \mu_k + h_k, \tag{10.21}$$

where $h_k \in L^1(\Omega)$ is given by

$$h_k = |v_k + w_k|^{q-1}(v_k + w_k) - |v_k|^{q-1}v_k - |w_k|^{q-1}w_k.$$

We claim that

$$h_k \to 0 \quad \text{in } L^1(\Omega). \tag{10.22}$$

Since the sequence (σ_k) is concentrating, it follows from the proof of Proposition 10.1 that the sequence $\big(|w_k|^q\big)$ is concentrating with respect to the capacity $\mathrm{cap}_{W^{2,q'}}$. Let (F_k) be a sequence of Borel subsets of Ω such that

$$\mathrm{cap}_{W^{2,q'}}(F_k) \to 0 \quad \text{and} \quad \int_{\Omega\setminus F_k} |w_k|^q \to 0.$$

Applying Lemma 10.2 with functions v_k and w_k, and Borel set F_k, we have

$$\|h_k\|_{L^1(\Omega)} \leq C\Big(\|v_k\|_{L^q(\Omega)}^{q-1} + \|w_k\|_{L^q(\Omega)}^{q-1}\Big)\Big(\|v_k\|_{L^q(F_k)} + \|w_k\|_{L^q(\Omega\setminus F_k)}\Big).$$

Since (α_k) and (σ_k) are bounded in $\mathcal{M}(\Omega)$, the sequences (v_k) and (w_k) are bounded in $L^q(\Omega)$. Thus,

$$\|h_k\|_{L^1(\Omega)} \leq \widetilde{C}\Big(\|v_k\|_{L^q(F_k)} + \|w_k\|_{L^q(\Omega\setminus F_k)}\Big) \quad \forall k \geq 1.$$

By the choice of the sets F_k, $\|w_k\|_{L^q(\Omega\setminus F_k)} \to 0$. On the other hand, since the sequence (α_k) is equidiffuse with respect to $\mathrm{cap}_{W^{2,q'}}$, $(|v_k|^q)$ is also equidiffuse by Lemma 10.1. Thus, $\|v_k\|_{L^q(F_k)} \to 0$. This implies (10.22).

We have thus showed that

$$\|\lambda_k - \mu_k\|_{\mathcal{M}} = \|h_k\|_{L^1} \to 0.$$

In particular, the sequences (λ_k) and (μ_k) have the same weak* limit μ. In order to identify their reduced limit, we note that if

$$v_k \to v^{\#} \quad \text{in } L^1(\Omega),$$

then, since $w_k \to 0$ in $L^1(\Omega)$,

$$u_k + v_k \to v^{\#} \quad \text{in } L^1(\Omega).$$

Thus, the reduced limit of (λ_k) coincides with the reduced limit of (α_k), namely $\alpha^{\#}$. But since by Proposition 7.1 the sequences (μ_k) and (λ_k) have the same reduced limits, we conclude that $\mu^{\#} = \alpha^{\#}$. This concludes the proof of the theorem. $\square$

11. Sufficient conditions for the equality $\mu^\# = \mu$

We present in this section some cases where the weak* limit and the reduced limit $\mu^\#$ of a given sequence (μ_k) are equal. The first result should be compared with Theorems 9.1 and 9.2.

Proposition 11.1. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded sequence with weak* limit μ and reduced limit $\mu^\#$. If (μ_k) is bounded in $H^{-1}(\Omega)$, then $\mu^\# = \mu$.*

Proof. For each $k \geq 1$, let u_k be such that

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

Passing to a subsequence if necessary, we may assume that $u_k \to u^\#$ in $L^1(\Omega)$ and a.e. Since $\mu_k \in H^{-1}(\Omega)$, $u_k \in H^1(\Omega)$ and (see [4, 6])

$$\int_\Omega |\nabla u_k|^2 + \int_\Omega g(u_k) u_k = \int_\Omega u_k \, d\mu_k. \tag{11.2}$$

In particular, from the boundedness of (μ_k) in $H^{-1}(\Omega)$, we deduce that the sequence (u_k) is bounded in $H^1(\Omega)$. Thus,

$$\int_\Omega g(u_k) u_k \leq \int_\Omega u_k \, d\mu_k \leq \|u_k\|_{H^1} \|\mu_k\|_{\mathcal{M}} \leq C \quad \forall k \geq 1.$$

Since $g(t)t \geq 0$, $\forall t \in \mathbb{R}$, this implies that $\big(g(u_k)\big)$ is an equi-integrable sequence in $L^1(\Omega)$. As $g(u_k) \to g(u^\#)$ a.e., it follows from Egorov's lemma that $g(u_k) \to g(u^\#)$ in $L^1(\Omega)$. Therefore, $\mu^\# = \mu$. $\square$

Proposition 11.2. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded sequence with weak* limit μ and reduced limit $\mu^\#$. Assume that there exists $\nu \in \mathcal{M}(\Omega)$ such that*

$$|\mu_k| \leq \nu \quad \forall k \geq 1. \tag{11.3}$$

Then,

$$\mu^\# = \mu. \tag{11.4}$$

Proof. We split the proof in two steps:

Step 1. (11.4) holds if, in addition,

$$\lambda_1 \leq \mu_k \leq \lambda_2 \quad \forall k \geq 1. \tag{11.5}$$

where $\lambda_1, \lambda_2 \in \mathcal{G}$.

For each $k \geq 1$, let u_k be such that

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.6}$$

Denote by v_1 and v_2 the solutions of (11.6) with data λ_1 and λ_2, respectively. By the comparison principle, we have

$$v_1 \leq u_k \leq v_2 \quad \text{a.e.} \quad \forall k \geq 1.$$

Hence, since g is nondecreasing,

$$g(v_1) \leq g(u_k) \leq g(v_2) \quad \text{a.e.} \quad \forall k \geq 1.$$

On the other hand, passing to a subsequence if necessary, we may assume that $u_k \to u$ in $L^1(\Omega)$ and a.e. Since $g(v_1), g(v_2) \in L^1(\Omega)$, we conclude that

$$g(u_k) \to g(u) \quad \text{in } L^1(\Omega).$$

Therefore, u satisfies (11.6) with right-hand side μ, whence μ is the reduced limit of the (μ_k).

Step 2. Proof completed.

In view of the previous step, it suffices to find $\lambda_1, \lambda_2 \in \mathcal{G}$ satisfying (11.5). For this purpose, note that by (11.3) we have

$$-\nu^- \leq \mu_k \leq \nu^+ \quad \forall k \geq 1.$$

We recall (see [6, Section 6]) that the reduced measure $(\nu^+)^*$ is the largest measure in $\mathcal{G}$ which is dominated by ν^+. Since $\mu_k^+ \in \mathcal{G}$ and $\mu_k^+ \leq \nu^+$,

$$\mu_k^+ \leq (\nu^+)^* \quad \forall k \geq 1.$$

Similarly, $(-\nu^-)^*$ is the smallest measure in $\mathcal{G}$ which dominates $-\nu^-$. Since $-\mu_k^- \in \mathcal{G}$ and $-\nu^- \leq -\mu_k^-$,

$$(-\nu^-)^* \leq (-\mu_k)^- \quad \forall k \geq 1.$$

Thus, (11.5) holds with $\lambda_1 = (-\nu^-)^*$ and $\lambda_2 = (\nu^+)^*$. By the previous step, (11.4) follows. ☐

We now show that the reduced limit and the weak* limit always coincide under weak-L^1 convergence.

Proposition 11.3. *Given $\nu \in \mathcal{M}(\Omega)$, let $(h_k) \subset \mathcal{G} \cap L^1(\Omega;\nu)$. If*

$$h_k \rightharpoonup h \quad \text{weakly in } L^1(\Omega;\nu), \tag{11.7}$$

then $h\nu$ is the reduced limit of the sequence $(h_k\nu)$.

Proof. By a diagonalization procedure, one can find an increasing sequence of integers (k_j) such that, for every integer $n \geq 1$, the sequence $(T_n(h_{k_j}))_{j\geq 1}$ converges weakly in $L^1(\Omega;\nu)$ to some function $\tilde{h}_n$, where T_n is given by (3.2). We may also assume that the reduced limit $\mu^\#$ of $(h_{k_j}\nu)$ exists. Since

$$\big|T_n(h_{k_j})\nu\big| \leq n\nu \quad \forall j \geq 1,$$

it follows from Proposition 11.2 that $\tilde{h}_n\nu$ is the reduced limit of the sequence $\big(T_n(h_{k_j})\nu\big)$.

On the other hand, by the Dunford-Pettis theorem (see [13]), the sequence (h_k) converges weakly in $L^1(\Omega;\nu)$ if and only if (h_k) is bounded in $L^1(\Omega;\nu)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$E \subset \Omega \text{ Borel} \quad \text{and} \quad \nu(E) < \delta \quad \Longrightarrow \quad \int_E |h_k|\, d\nu < \varepsilon \quad \forall k \geq 1. \tag{11.8}$$

Let $C_0 > 0$ be such that

$$\int_\Omega |h_k|\, d\nu \leq C_0 \quad \forall k \geq 1. \tag{11.9}$$

Let $A_{j,n} = [|h_{k_j}| > n]$; by the Chebyshev inequality,

$$\nu(A_{j,n}) \leq \frac{1}{n}\int_\Omega |h_{k_j}|\, d\nu \leq \frac{C_0}{n} \quad \forall j, n \geq 1.$$

Take $n \geq 1$ sufficiently large so that $C_0/n < \delta$. Then, by (11.8) we have

$$\big\|h_{k_j}\nu - T_n(h_{k_j})\nu\big\|_{\mathcal{M}} = \int_\Omega \big|h_{k_j} - T_n(h_{k_j})\big|\,d\nu \leq \int_{A_{j,n}} |h_{k_j}|\,d\nu < \varepsilon. \tag{11.10}$$

By lower semicontinuity of the norm in $\mathcal{M}(\Omega)$, as we let $j \to \infty$ we get

$$\|h\nu - \tilde{h}_n\nu\|_{\mathcal{M}} \leq \varepsilon. \tag{11.11}$$

Denote by $\mu^\#$ the reduced limit of the sequence $(h_{k_j}\nu)$. By Proposition 7.1 applied to $(h_{k_j}\nu)$ and $\big(T_n(h_{k_j})\nu\big)$,

$$\|\mu^\# - \tilde{h}_n\nu\|_{\mathcal{M}} \leq \|h\nu - \tilde{h}_n\nu\|_{\mathcal{M}} + \liminf_{j\to\infty} \big\|h_{k_j}\nu - T_n(h_{k_j})\nu\big\|_{\mathcal{M}} \leq 2\varepsilon. \tag{11.12}$$

Combining (11.11)–(11.12) we deduce that

$$\|\mu^\# - h\nu\|_{\mathcal{M}} \leq 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we must have $\mu^\# = h\nu$. In particular, the reduced limit $\mu^\#$ does not depend on the sequence (k_j). Therefore, the reduced limit of the whole sequence $(h_k\nu)$ exists and equals $h\nu$. □

12. Characterization of sequences for which $\mu^\# = \mu$

In the previous section, we presented some *sufficient* conditions in order that the weak* limit and the reduced limit of a given sequence (μ_k) coincide. Our goal in this section is to provide *necessary and sufficient* conditions for this property to hold. Before we present our next result, we observe that every $\mu \in \mathcal{G}$ has a decomposition of the form

$$\mu = f - \Delta v \quad \text{in } \Omega, \tag{12.1}$$

where $f \in L^1(\Omega)$, $v \in L^1(\Omega)$ and $g(v) \in L^1(\Omega)$. For instance, we can take $f = g(u)$ and $v = u$, where u is the solution of problem (1.2). But the decomposition (12.1) of μ is not unique.

Theorem 12.1. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded nonnegative sequence with weak* limit μ and reduced limit $\mu^\#$. Then,*

$$\mu^\# = \mu \tag{12.2}$$

if and only if for every $k \geq 1$ there exist $f_k \in L^1_{\mathrm{loc}}(\Omega)$ and $v_k \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\mu_k = f_k - \Delta v_k \quad \text{in } \Omega, \quad g(v_k) \in L^1_{\mathrm{loc}}(\Omega), \tag{12.3}$$

where both sequences (f_k) and $\big(g(v_k)\big)$ converge strongly in $L^1(\omega)$ for every subdomain $\omega \Subset \Omega$.

For the proof of Theorem 12.1 we need the following auxiliary results.

Lemma 12.1. *Let $(\mu_k) \subset \mathcal{G}$ be a bounded nonnegative sequence with weak* limit μ and reduced limit $\mu^\#$. Let $u_k \in L^1(\Omega)$ be the solution of*

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega,\\ u_k = 0 & \text{on } \partial\Omega \end{cases} \tag{12.4}$$

and assume that (u_k) converges in $L^1(\Omega)$. Then, the following assertions are equivalent:

(*i*) $\mu = \mu^\#$;

(ii) $\big(g(u_k)\big)$ *converges in* $L^1(\omega)$ *for every subdomain* $\omega \Subset \Omega$*;*
(iii) $\big(g(u_k)\big)$ *is equidiffuse with respect to* cap_{H^1} *in every subdomain* $\omega \Subset \Omega$*.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Since $\mu_k \geq 0$, we have $u_k \geq 0$ a.e., $\forall k \geq 1$. Let $u^\# \in L^1(\Omega)$ be such that

$$u_k \to u^\# \quad \text{in } L^1(\Omega).$$

Passing to a subsequence if necessary, we may also assume that $u_k \to u^\#$ a.e. By assumption, $\mu = \mu^\#$. Thus,

$$\int_\Omega g(u_k)\zeta \to \int_\Omega g(u^\#)\zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

By a density argument, we get

$$\int_\Omega g(u_k)\rho_0 \to \int_\Omega g(u^\#)\rho_0,$$

where

$$\rho_0(x) = d(x, \partial\Omega) \quad \forall x \in \Omega. \tag{12.5}$$

Since $g(u_k) \geq 0$ a.e., $\forall k \geq 1$, and $g(u_k)\rho_0 \to g(u^\#)\rho_0$ a.e., it follows from the Brezis-Lieb lemma (see [5]) that

$$g(u_k)\rho_0 \to g(u^\#)\rho_0 \quad \text{in } L^1(\Omega).$$

(ii) $\Rightarrow$ *(iii)*. By the Poincaré inequality,

$$|K|^{1/2} \leq C \operatorname{cap}_{H^1}(K),$$

for every compact set $K \subset \Omega$. By regularity, this inequality holds for every Borel subset of Ω. Thus, if $\big(g(u_k)\big)$ converges strongly in $L^1(\omega)$, then it is equidiffuse with respect to cap_{H^1} in ω.

(iii) $\Rightarrow$ *(i)*. By Proposition 3.2, $\mu - \mu^\#$ is the concentrated limit of $\big(g(u_k)\big)$ with respect to cap_{H^1}. In particular, if $\big(g(u_k)\big)$ is equidiffuse in ω for every $\omega \Subset \Omega$, then we must have $\mu - \mu^\# = 0$. □

Lemma 12.2. *Let* $(\mu_k) \subset \mathcal{G}$ *be a bounded nonnegative sequence with weak* limit* μ *and reduced limit* $\mu^\#$*. If* $\mu^\# = \mu$*, then for every sequence* $(h_k) \subset L^1(\Omega)$ *such that* $h_k \to h$ *strongly in* $L^1(\Omega)$*, the sequence* (λ_k) *given by*

$$\lambda_k = \mu_k + h_k \quad \forall k \geq 1 \tag{12.6}$$

has reduced limit $\lambda^\# = \mu + h$*.*

Proof. For every $k \geq 1$, let u_k be the solution of the problem

$$\begin{cases} -\Delta z + g(z) = \gamma & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.7}$$

with datum $\gamma = \mu_k$. Given $a \in (0, 1)$, let v_k be the solution of the linear problem

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.8}$$

with datum $f = T_{1/a}(h_k)$. Since $v_k \in L^\infty(\Omega)$ and $a \in (0, 1)$, it follows that $g(au_k + v_k) \in L^1(\Omega)$ and, consequently,

$$\nu_k := a\mu_k + T_{1/a}(h_k) + g(au_k + v_k) - ag(u_k) \in \mathcal{M}(\Omega).$$

We observe that $au_k + v_k$ is the solution of (12.7) with datum $\gamma = \nu_k$.
If $u_k \to u$ in $L^1(\Omega)$ then, by Lemma 12.1, $g(u_k) \to g(u)$ in $L^1(\omega)$ for every $\omega \Subset \Omega$. By dominated convergence, it follows that

$$g(au_k + v_k) \to g(au + v) \quad \text{in } L^1(\omega),$$

where v is the solution of (12.8) with $f = T_{1/a}(h)$.
Let w_k and $\tilde{w}_k$ denote the solutions of (12.7) with data

$$\beta_k = g(au_k) - ag(u_k) \quad \text{and} \quad \tau_k = a\mu_k + T_{1/a}(h_k) - ag(u_k) + g(au_k),$$

respectively. Passing to a subsequence if necessary we may assume that $w_k \to w$ and $\tilde{w}_k \to \tilde{w}$ in $L^1(\Omega)$ and a.e. For every $\omega \Subset \Omega$,

$$g(au_k) - ag(u_k) \to g(au) - ag(u) \quad \text{in } L^1(\omega).$$

Therefore, by Lemma 12.1,

$$g(w_k) \to g(w), \qquad \text{in } L^1(\omega).$$

Since

$$\beta_k \le \tau_k \le \nu_k$$

we have

$$w_k \le \tilde{w}_k \le au_k + v_k \ \text{ a.e.},$$

which implies that

$$g(w_k) \le g(\tilde{w}_k) \le g(au_k + v_k) \ \text{ a.e.}$$

Since $\big(\tilde{g}(w_k)\big)$ converges a.e. to $g(\tilde{w})$, by dominated convergence,

$$g(\tilde{w}_k) \to g(\tilde{w}) \quad \text{in } L^1(\omega)$$

for every subdomain $\omega \Subset \Omega$. This implies that $\tilde{w}$ is the solution of (12.7) with datum τ_a where τ_a is the weak* limit of (τ_k),

$$\tau_a = a\mu + T_{1/a}(h) - ag(u) + g(au).$$

Thus, $\tilde{w}$ does not depend on the subsequence and τ_a is the reduced limit of the whole sequence (τ_k). By Proposition 7.1,

$$\begin{aligned}
\|\lambda^{\#} - \tau_a\|_{\mathcal{M}(\omega)} &\le \big\|(\mu + h) - \tau_a\big\|_{\mathcal{M}(\omega)} + \liminf_{k\to\infty} \big\|\lambda_k - \tau_k\big\|_{\mathcal{M}(\omega)} \\
&\le (1-a)\|\mu\|_{\mathcal{M}(\omega)} + 2\big\|h - T_{1/a}(h)\big\|_{L^1(\omega)} + \\
&\qquad + 2\big\|ag(u) - g(au)\big\|_{L^1(\omega)} + (1-a)\limsup_{k\to\infty} \|\mu_k\|_{\mathcal{M}(\omega)}.
\end{aligned}$$

As $a \to 1$, the right-hand side of this inequality tends to 0, while

$$\tau_a \to \mu + h \quad \text{strongly in } \mathcal{M}(\omega).$$

Therefore, $\lambda^{\#} = \mu + h$ in every subdomain $\omega \Subset \Omega$, whence in Ω. □

Proof of Theorem 12.1. ($\Rightarrow$). Assume that $\mu^{\#} = \mu$. For each $k \ge 1$, let u_k be such that

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.9}$$

Then, $u_k \to u$ in $L^1(\Omega)$, where u is the solution of (12.9) with datum μ. Since by Lemma 12.1, $g(u_k) \to g(u)$ in $L^1(\omega)$ for every $\omega \Subset \Omega$, we have the conclusion with $f_k = g(u_k)$ and $v_k = u_k$.

($\Leftarrow$). We fix a subdomain $\tilde{\omega} \Subset \Omega$. By Lemma 6.1, the sequence (v_k) is relatively compact in $L^1(\Omega)$. Thus, passing to a subsequence if necessary, $v_k \to v$ in $L^1(\Omega)$. By assumption, for every $k \geq 1$,

$$-\Delta v_k + g(v_k) = \mu_k - f_k + g(v_k) \quad \text{in } \tilde{\omega}.$$

Since $g(v_k) \to g(v)$ strongly in $L^1(\tilde{\omega})$, the reduced limit $\nu^{\#}$ of $\big(\mu_k - f_k + g(v_k)\big)$ coincides with its weak* limit. Thus,

$$\nu^{\#} = \mu - f + g(v) \quad \text{in } \tilde{\omega}.$$

Since $f_k - g(v_k) \to f - g(v)$ in $L^1(\tilde{\omega})$, it follows from the previous lemma applied to the sequences $\big(\mu_k - f_k + g(v_k)\big)$ and $\big(f_k - g(v_k)\big)$ that

$$\mu^{\#} = (\mu - f + g(v)) + (f - g(v)) = \mu \quad \text{in } \tilde{\omega}.$$

Since $\mu^{\#} = \mu$ in every subdomain $\tilde{\omega} \Subset \Omega$, the conclusion follows. □

In [6, Theorem 4.5], we prove that $\mu \in \mathcal{G}(g)$ for every nonlinearity g if and only if the measure μ is diffuse with respect to cap_{H^1}. Using this result we characterize the sequences of measures (μ_k) for which the weak* limit and the reduced limit coincide for *every* g.

Theorem 12.2. *Let $(\mu_k) \subset \mathcal{M}(\Omega)$ be a bounded sequence of nonnegative measures with weak* limit μ. Assume that every measure μ_k is diffuse with respect to cap_{H^1}. Then,*

$$\mu^{\#} = \mu \quad \textit{for every nonlinearity } g \tag{12.10}$$

if and only if (μ_k) is equidiffuse with respect to cap_{H^1} in every subdomain $\omega \Subset \Omega$.

Proof. First we observe that, since μ_k is diffuse, $\mu_k \in \mathcal{G}(g)$ for every nonlinearity g.

($\Leftarrow$) Without loss of generality, we may assume that the sequence (μ_k) is equidiffuse in Ω. Let u_k be such that

$$\begin{cases} -\Delta u_k + g(u_k) = \mu_k & \text{in } \Omega, \\ \qquad\qquad\quad u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.11}$$

Passing to a subsequence if necessary, we may assume that

$$u_k \to u^{\#} \quad \text{in } L^1(\Omega).$$

Since (μ_k) is equidiffuse, it follows from [9, Lemma 3] that $\big(g(u_k)\big)$ is also equidiffuse. By Lemma 12.1, μ is the reduced limit of (μ_k) with respect to g.

($\Rightarrow$) Assume that $\mu = \mu^{\#}$. We closely follow the proof of [6, Theorem 4.5]. Suppose by contradiction that (μ_k) is not equidiffuse in some subdomain $\omega \Subset \Omega$. Passing to a subsequence if necessary, one finds $\varepsilon > 0$ and a sequence of compact sets (K_k) in ω such that

$$\mu_k(K_k) \geq \varepsilon \quad \text{and} \quad \operatorname{cap}_{H^1}(K_k) \to 0.$$

By [6, Lemma 4E.1], for every $k \geq 1$ there exists $\varphi_k \in C_0^\infty(\Omega)$ such that $0 \leq \varphi_k \leq 1$ in Ω, $\varphi_k = 1$ on K_k and

$$\int_\Omega |\Delta\varphi_k| \leq 2\operatorname{cap}_{H^1}(K_k) + \frac{1}{k} \to 0. \tag{12.12}$$

We may assume that $\operatorname{supp}\varphi_k \subset \tilde{\omega}$, $\forall k \geq 1$, where $\omega \Subset \tilde{\omega} \Subset \Omega$. Up to a subsequence we also have $\varphi_k \to 0$ a.e., $\Delta\varphi_k \to 0$ a.e. and there exists $F_1 \in L^1(\Omega)$ such that

$$|\Delta\varphi_k| \leq F_1 \ \text{ a.e.} \quad \forall k \geq 1.$$

According to a result of de La Vallée Poussin [12, Remarque 23], there exists a convex function $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, $h(s) > 0$ for $s > 0$,

$$\lim_{t\to\infty} \frac{h(t)}{t} = +\infty, \quad \text{and} \quad h(F) \in L^1(\Omega).$$

Let

$$g(t) = \begin{cases} h^*(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

where h^* is the convex conjugate (or Fenchel transform) of h. For each $k \geq 1$, let u_k be the solution of (12.11) for this nonlinearity g. Since μ coincides with the reduced limit of (μ_k), by Lemma 12.1 above we have

$$g(u_k) \to g(u) \quad \text{in } L^1(\tilde{\omega}).$$

Passing to a subsequence if necessary, one finds $F_2 \in L^1(\tilde{\omega})$, with

$$0 \leq g(u_k) \leq F_2 \ \text{ a.e.} \quad \forall k \geq 1.$$

On the other hand, for every $k \geq 1$,

$$\varepsilon \leq \mu_k(K_k) \leq \int_\Omega \varphi_k \, d\mu_k = \int_\Omega \big[g(u_k)\varphi_k - u_k \Delta\varphi_k\big]. \tag{12.13}$$

Note that

$$\big|g(u_k)\varphi_k - u_k\Delta\varphi_k\big| \to 0 \quad \text{a.e.}$$

and

$$\big|g(u_k)\varphi_k - u_k\Delta\varphi_k\big| \leq 2g(u_k)\chi_{\tilde{\omega}} + h(|\Delta\varphi_k|) \leq 2F_2\chi_{\tilde{\omega}} + F_1 \quad \forall k \geq 1.$$

By dominated convergence, the right-hand side of (12.13) converges to 0 as $k \to \infty$. This is a contradiction. Therefore, the sequence (μ_k) is equidiffuse in ω with respect to cap_{H^1}. □

13. Absolute continuity between $\mu^\#$ and $\nu^\#$

In addition to our standard assumptions on the nonlinearity g (continuity and monotonicity), throughout this section we assume that

$$g \text{ is convex.} \tag{13.1}$$

The goal of this section is to prove that if a sequence (ν_k) is uniformly absolutely continuous with respect to another sequence (μ_k), then the reduced limit $\nu^\#$ is absolutely continuous with respect to $\mu^\#$. More precisely,

Theorem 13.1. *Let $(\mu_k), (\nu_k) \subset \mathcal{G}$ be bounded sequences of nonnegative measures with reduced limits $\mu^\#$ and $\nu^\#$, respectively. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$E \subset \Omega \text{ Borel} \quad \text{and} \quad \nu_k(E) < \delta \quad \Longrightarrow \quad \mu_k(E) < \varepsilon \quad \forall k \geq 1, \tag{13.2}$$

then

$$\mu^\# \ll \nu^\#. \tag{13.3}$$

We first establish the following

Lemma 13.1. *Given nonnegative measures $\mu, \nu \in \mathcal{G}$, let u and v be the solutions of*

$$\begin{cases} -\Delta z + g(z) = \gamma & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{13.4}$$

with data μ and ν, respectively. If $\mu \leq a\nu$ for some $a \geq 1$, then

$$u \leq av \quad a.e. \tag{13.5}$$

Proof. Since $\mu \leq a\nu$, subtracting the equations satisfied by u and v we get

$$\int_\Omega (u - av)\Delta\zeta = \int_\Omega \big[g(u) - \mu - ag(v) + a\nu\big]\zeta \geq \int_\Omega \big[g(u) - ag(v)\big]\zeta,$$

for every $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \geq 0$ in Ω. Thus, by Lemma 5.1,

$$\int_\Omega (u - av)^+ \Delta\zeta \geq \int_{\substack{\Omega \\ [u \geq av]}} \big[g(u) - ag(v)\big]\zeta. \tag{13.6}$$

On the other hand, since g is convex and $g(0) = 0$, the function $g(t)/t$ is nondecreasing on $(0, \infty)$. Hence, for $a \geq 1$ we have

$$g(at) \geq ag(t) \quad \forall t \geq 0.$$

In particular,

$$g(u) - ag(v) \geq 0 \quad \text{a.e. on } [u \geq av]. \tag{13.7}$$

It follows from (13.6)–(13.7) that

$$\int_\Omega (u - av)^+ \Delta\zeta \geq 0 \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{13.8}$$

This immediately gives (13.5). □

Proposition 13.1. *Let $(\mu_k), (\nu_k) \subset \mathcal{G}$ be bounded sequences of nonnegative measures with reduced limits $\mu^\#$ and $\nu^\#$, respectively. Assume that there exists $a \geq 1$ such that*

$$\mu_k \leq a\nu_k \quad \forall k \geq 1. \tag{13.9}$$

Then,

$$\mu^\# \leq a\nu^\#. \tag{13.10}$$

Proof. Denote by $u_k, v_k \in L^1(\Omega)$ the solutions of (13.4) with data μ_k and ν_k, respectively. In particular, for every $k \geq 1$ we have

$$\Delta(av_k - u_k) = ag(v_k) - g(u_k) - a\nu_k + \mu_k \quad \text{in } \Omega.$$

Passing to a subsequence if necessary, we may assume that (μ_k) and (ν_k) have concentrated limits σ and τ, respectively. On the other hand, the sequences $\big(g(u_k)\big)$ and $\big(g(v_k)\big)$ have concentrated limits $\mu - \mu^\#$ and $\nu - \nu^\#$. Since $av_k - u_k \geq 0$ a.e. for every $k \geq 1$, it follows from Theorem 4.2 that

$$a(\nu - \nu^\#) - (\mu - \mu^\#) - a\tau + \sigma \leq 0. \tag{13.11}$$

Note that $(a\nu_k - \mu_k)$ is a sequence of nonnegative measures with weak* limit $a\nu - \mu$ and concentrated limit $a\tau - \sigma$. Hence, by Corollary 2.2,

$$a\tau - \sigma \leq a\nu - \mu. \tag{13.12}$$

Combining (13.11)–(13.12), we deduce that

$$-a\nu^\# + \mu^\# \le 0,$$

which is precisely (13.10). □

Proof of Theorem 13.1. Given $a \ge 1$, we apply the Hahn decomposition to $\mu_k - a\nu_k$. We may thus write $\Omega = E_k \cup F_k$ as a disjoint union of measurable sets such that

$$\mu_k \ge a\nu_k \quad \text{on } E_k \quad \text{and} \quad \mu_k \le a\nu_k \quad \text{on } F_k$$

(for simplicity of notation we omit the dependence of E_k and F_k on a). In particular,

$$\nu_k(E_k) \le \frac{1}{a}\mu_k(E_k) \le \frac{1}{a}\|\mu_k\|_{\mathcal{M}} \le \frac{C_0}{a} \quad \forall k \ge 1,$$

since the sequence (μ_k) is bounded in $\mathcal{M}(\Omega)$. Thus, for $a \ge 1$ sufficiently large, we have $C_0/a < \delta$. By (13.2) we deduce that

$$\mu_k(E_k) < \varepsilon \quad \forall k \ge 1. \tag{13.13}$$

Consider the sequences

$$\lambda_k = \mu_k\lfloor_{F_k} \quad \text{and} \quad \tau_k = \nu_k\lfloor_{F_k} \quad \forall k \ge 1.$$

Then,

$$\lambda_k \le a\tau_k \quad \forall k \ge 1.$$

Passing to a subsequence if necessary, we may assume that (λ_k) and (τ_k) have reduced limits $\lambda^\#$ and $\tau^\#$, respectively. Thus, by Proposition 13.1,

$$\lambda^\# \le a\tau^\#. \tag{13.14}$$

Let $E \subset \Omega$ be a Borel set such that $\nu^\#(E) = 0$. Since $0 \le \tau_k \le \nu_k$, $\forall k \ge 1$, by Theorem 7.1 we have

$$\tau^\#(E) = \nu^\#(E) = 0.$$

It follows from (13.14) and $\lambda^\# \ge 0$ that

$$\lambda^\#(E) = 0. \tag{13.15}$$

On the other hand, applying Proposition 7.1 to the sequences (μ_k) and (λ_k), we get

$$\|\mu^\# - \lambda^\#\|_{\mathcal{M}} \le \|\mu - \lambda\|_{\mathcal{M}} \le \liminf_{k\to\infty} \|\mu_k - \lambda_k\|_{\mathcal{M}} = \liminf_{k\to\infty} \mu_k(E_k) \le \varepsilon.$$

Thus, in view of (13.15),

$$\mu^\#(E) = \left|\mu^\#(E) - \lambda^\#(E)\right| \le \|\mu^\# - \lambda^\#\|_{\mathcal{M}} \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude that $\mu^\#(E) = 0$. Therefore, $\mu^\# \ll \nu^\#$. □

14. Reduced limit of $\max\{\mu_k, \nu_k\}$

Throughout this section, we assume in addition to our usual assumptions on g that

$$g \text{ is convex.}$$

Given bounded sequences $(\mu_k), (\nu_k) \subset \mathcal{M}(\Omega)$ converging weakly* to μ and ν, if $\mu \perp \nu$, then the measures $\lambda_k = \max\{\mu_k, \nu_k\}$ converge weakly* to $\max\{\mu, \nu\}$. In this section we prove the counterpart of this statement for reduced limits. In order to do so we need the following result proved in [6, Corollary 4.4]: if $\mu, \nu \in \mathcal{G}$, then $\max\{\mu, \nu\} \in \mathcal{G}$.

Theorem 14.1. *Let $(\mu_k), (\nu_k) \subset \mathcal{G}$ be bounded sequences of nonnegative measures with reduced limits $\mu^\#$ and $\nu^\#$, respectively. If $\mu^\# \perp \nu^\#$, then the sequence (λ_k) given by*

$$\lambda_k = \max\{\mu_k, \nu_k\} \quad \forall k \geq 1 \tag{14.1}$$

has reduced limit $\lambda^\# = \max\{\mu^\#, \nu^\#\}$.

We first prove a variant of Lemma 13.1.

Lemma 14.1. *Given nonnegative measures $\lambda, \mu, \nu \in \mathcal{G}$, let $w, u, v \in L^1(\Omega)$ be the solutions of*

$$\begin{cases} -\Delta z + g(z) = \gamma & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{14.2}$$

with data λ, μ and ν, respectively. If $\lambda \leq \mu + \nu$, then

$$w \leq u + v \quad \text{a.e.} \tag{14.3}$$

Proof. Since $\lambda \leq \mu + \nu$, we have

$$\int_\Omega (w-u-v)\Delta\zeta = \int_\Omega \big[g(w)-\lambda-g(u)+\mu-g(v)+\nu\big]\zeta \geq \int_\Omega \big[g(w)-g(u)-g(v)\big]\zeta,$$

for every $\zeta \in C_0^2(\overline{\Omega})$, $\zeta \geq 0$ in Ω. Thus, by Lemma 5.1,

$$\int_\Omega (w-u-v)^+\Delta\zeta \geq \int_{\substack{\Omega \\ [w \geq u+v]}} \big[g(w)-g(u)-g(v)\big]\zeta \geq 0, \tag{14.4}$$

where we used the property

$$g(s+t) \geq g(s) + g(t) \quad \forall s, t \geq 0.$$

From estimate (14.4) we deduce (14.3). □

Proof of Theorem 14.1. Since $\mu_k, \nu_k \in \mathcal{G}$, we have $\lambda_k \in \mathcal{G}$. We observe that by Proposition 7.1, $\mu^\# \leq \lambda^\#$. Thus,

$$\max\{\mu^\#, \nu^\#\} \leq \lambda^\#. \tag{14.5}$$

We now prove that

$$\lambda^\# \leq \mu^\# + \nu^\#. \tag{14.6}$$

For this purpose, let $w_k, u_k, v_k \in L^1(\Omega)$ be the solutions of (14.2) with data μ_k, ν_k and $\tilde{\lambda}_k$, respectively, where

$$\tilde{\lambda}_k = (\mu_k + \nu_k)^*.$$

In particular, since $\lambda_k \in \mathcal{G}$ and $\lambda_k \leq \mu_k + \nu_k$, $\lambda_k \leq \tilde{\lambda}_k$. Passing to a subsequence if necessary, we may assume that $(\tilde{\lambda}_k)$ has reduced limit $\tilde{\lambda}^\#$. By Lemma 14.1, we have

$$w_k \leq u_k + v_k \quad \text{a.e.} \quad \forall k \geq 1. \tag{14.7}$$

On the other hand,

$$\Delta(u_k + v_k - w_k) = g(u_k) + g(v_k) - g(w_k) - \mu_k - \nu_k + \tilde{\lambda}_k \quad \forall k \geq 1.$$

Proceeding as in the proof of Proposition 13.1, one deduces that

$$\tilde{\lambda}^\# \leq \mu^\# + \nu^\#. \tag{14.8}$$

On the other hand, since $\lambda_k \leq \tilde{\lambda}_k$, $\forall k \geq 1$, by Theorem 7.1 we also have

$$\lambda^{\#} \leq \tilde{\lambda}^{\#}. \tag{14.9}$$

Combining (14.8)–(14.9) we deduce (14.6). Since $\mu^{\#}$ and $\nu^{\#}$ are nonnegative and, by assumption, $\mu^{\#} \perp \nu^{\#}$,

$$\mu^{\#} + \nu^{\#} = \max\{\mu^{\#}, \nu^{\#}\}.$$

Thus,

$$\lambda^{\#} \leq \max\{\mu^{\#}, \nu^{\#}\}. \tag{14.10}$$

The conclusion follows from (14.5) and (14.10). □

15. Open problems

This section is devoted to questions related to the present work. The first open problem concerns a possible extension of Theorem 1.4.

Open Problem 1. *Given $q \geq \frac{N}{N-2}$, let $(\mu_k) \subset \mathcal{G}^q$ be a bounded nonnegative sequence with weak* limit μ. For every $k \geq 1$, let u_k be such that*

$$\begin{cases} -\Delta u_k + |u_k|^{q-1} u_k = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

If (μ_k) is equidiffuse with respect to $\mathrm{cap}_{W^{2,q'}}$ and if $u_k \to 0$ in $L^1(\Omega)$, does $\mu = 0$?

In terms of reduced limits, this problem is equivalent to the question of whether $\mu^{\#} = 0$ implies $\mu = 0$. More generally, we would like to know whether the measure μ is absolutely continuous with respect to the reduced limit $\mu^{\#}$. By Theorem 1.4, if one makes the stronger assumption that (μ_k) is bounded in $W^{-2,q}(\Omega)$, then indeed $\mu \ll \mu^{\#}$.

We recall that by a result of Boccardo-Gallouët-Orsina [3] (see also [6, Theorem 4.3]) every finite measure μ in Ω, diffuse relative to capacity cap_{H^1}, can be written as $\mu = f + S$, where $f \in L^1(\Omega)$ and $S \in H^{-1}(\Omega)$. In connection with this decomposition, it would be interesting to have the following counterpart for equidiffuse sequences.

Open Problem 2. *Let $(\mu_k) \subset \mathcal{M}(\Omega)$ be a bounded sequence converging weakly* to μ. Assume that, for every $k \geq 1$, μ_k is diffuse with respect to cap_{H^1}. If (μ_k) is equidiffuse with respect to cap_{H^1}, is it possible to find sequences $(f_k) \subset L^1(\Omega)$ and $(S_k) \subset H^{-1}(\Omega)$ such that, for every $k \geq 1$,*

$$\mu_k = f_k + S_k \quad \text{in } \Omega, \tag{15.1}$$

where (f_k) converges strongly in $L^1(\Omega)$ and (S_k) is bounded in $H^{-1}(\Omega)$?

Let $q \geq \frac{N}{N-2}$. By a result of Baras-Pierre [2], every finite measure μ in Ω, diffuse relative to $\mathrm{cap}_{W^{2,q'}}$ can be written as $\mu = f + S$, where $f \in L^1(\Omega)$ and $S \in W^{-2,q}(\Omega)$. One can pose a similar question with respect to this capacity:

Open Problem 3. *Let $q \geq \frac{N}{N-2}$. Let $(\mu_k) \subset \mathcal{M}(\Omega)$ be a bounded sequence converging weakly* to μ. Assume that, for every $k \geq 1$, μ_k is diffuse with respect to $\mathrm{cap}_{W^{2,q'}}$. If (μ_k) is equidiffuse with respect to $\mathrm{cap}_{W^{2,q'}}$, is it possible to find sequences $(f_k) \subset L^1(\Omega)$ and $(S_k) \subset W^{-2,q}(\Omega)$ such that, for every $k \geq 1$,*

$$\mu_k = f_k + S_k \quad \text{in } \Omega, \tag{15.2}$$

where (f_k) converges strongly in $L^1(\Omega)$ and (S_k) is bounded in $W^{-2,q}(\Omega)$?

If one replaces the assumption of boundedness of (S_k) in $W^{-2,q}(\Omega)$ by the condition that (S_k) converges strongly in this space, then the answer is negative. In fact, if such decomposition were true, then by Theorem 12.1 we would have $\mu^\# = \mu$ for every equidiffuse sequence, but this is impossible by Theorem 9.2.

In this paper we present some conditions that assure that the reduced limit and the weak* limit of a given sequence $(\mu_k) \subset \mathcal{G}$ coincide. It would be interesting to fully investigate what happens in other cases, for instance with the sequence of convolutions $(\rho_n * \mu)$ for some given measure μ.

Open Problem 4. *Given $\mu \in \mathcal{G}$ and a sequence of smooth mollifiers (ρ_k), let $\mu^\#$ be the reduced limit associated to the sequence $(\rho_n * \mu)$. Does $\mu^\# = \mu$?*

The answer is known to be yes if g^+ and g^- are both convex (see [6]). If the answer to Open Problem 4 is negative for some nondecreasing nonlinearity g, then is it possible to find *some* sequence of smooth functions $(\psi_k) \subset C^\infty(\overline{\Omega})$ such that

$$\psi_k \overset{*}{\rightharpoonup} \mu \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega),$$

and (ψ_k) possesses a reduced limit $\mu^\#$ equal to μ?

Appendix A. $\mathcal{G} = \mathcal{G}_0$

In this appendix we prove the following result:

Theorem A.1. *For each nonlinearity g, let $\mathcal{G}(g)$ and $\mathcal{G}_0(g)$ be defined as in the Introduction. Then,*

$$\mathcal{G}(g) = \mathcal{G}_0(g).$$

The proof is based on two lemmas.

Lemma A.1. *If $\mu \in \mathcal{G}_0(g)$, then $\mu^+ \in \mathcal{G}_0(g)$ and $-\mu^- \in \mathcal{G}_0(g)$.*

Proof. First we show that $\mu \in \mathcal{G}_0(g^+)$. Since $u \in \mathcal{G}_0(g)$ problem (1.2) possesses a (unique) solution u. It follows that u is a supersolution of the problem

$$\begin{cases} -\Delta v + g^+(v) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Next w be such that

$$\begin{cases} -\Delta w = -\mu^- & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.2}$$

Then, $w \le 0$, hence $g^+(w) = 0$. Consequently, w is a subsolution of (A.1). By [16, Corollary 5.4], this implies the existence of a solution of (A.1).
Let ν^* denote the reduced limit of a measure $\nu \in \mathcal{M}(\Omega)$ relative to the nonlinearity g^+ (for the definition of reduced limit see [6]). Since $\mu \le \mu^+$ it follows that $\mu^* \le (\mu^+)^*$ (see [6, Proposition 4.4]). As $\mu \in \mathcal{G}_0(g^+)$, $\mu = \mu^*$. On the other hand, for any finite measure ν, $\nu^* \le \nu$. In particular $(\mu^+)^* \le \mu^+$. We thus have

$$\mu = \mu^* \le (\mu^+)^* \le \mu^+.$$

Since the measure $(\mu^+)^*$ is nonnegative (see [6, Corollary 4.1]), this implies that

$$\mu^+ \le (\mu^+)^* \le \mu^+.$$

Thus, $\mu^+ = (\mu^+)^* \in \mathcal{G}_0(g^+)$. But if v is a solution of (A.1) with μ replaced by μ^+, then v is positive and consequently satisfies

$$\begin{cases} -\Delta u + g(u) = \mu^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.3}$$

Therefore, $\mu^+ \in \mathcal{G}_0(g)$.

Observe that the function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{g}(t) = -g(-t)$ is a nonlinearity possessing the same properties as g. Furthermore, $\mu \in \mathcal{G}_0(g)$ if and only if $-\mu \in \mathcal{G}_0(\tilde{g})$. Hence, by the first part of the proof, $\mu^- \in \mathcal{G}_0(\tilde{g})$, which in turn implies that $-\mu^- \in \mathcal{G}_0(g)$. □

Lemma A.2. *$\mathcal{G}_0(g) + L^1(\Omega) = \mathcal{G}_0(g)$.*

Proof. Clearly, $\mathcal{G}_0(g) + L^1(\Omega) \supset \mathcal{G}_0(g)$. In order to prove the reverse inclusion, let $\nu \in \mathcal{G}_0(g)$ and $f \in L^1(\Omega)$. We have to show that $\nu + f \in \mathcal{G}_0(g)$. Let u and v denote the solutions of (1.2) with $\mu = \nu$ and $\mu = f$ respectively. If both ν and f are nonnegative, then u and v are nonnegative functions. Therefore, u and v satisfy the problem

$$\begin{cases} -\Delta v + g^+(v) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.4}$$

with $\mu = \nu$ and $\mu = f$, respectively. By [6, Corollary 4.7], $\nu + f \in \mathcal{G}_0(g^+)$ and therefore $\nu + f \in \mathcal{G}_0(g)$ since $\nu + f$ is nonnegative. Similarly, one verifies that if ν and f are nonpositive then $\nu + f \in \mathcal{G}_0(g)$.

In the general case, we observe that by Lemma A.1, ν^+ and $-\nu^-$ belong to $\mathcal{G}_0(g)$ and therefore, by the first part of the proof, $\nu^+ + f^+$ and $-\nu^- - f^-$ belong to $\mathcal{G}_0(g)$. Since

$$-\nu^- - f^- \leq \nu + f \leq \nu^+ + f^+$$

the existence of a solution of (A.1) for $\mu = \nu + f$ follows from the existence of a supersolution and a subsolution for the problem (see [16]). □

Proof of Theorem A.1. We only need to establish the inclusion $\mathcal{G}(g) \subset \mathcal{G}_0(g)$. We first prove that if $\mu \in \mathcal{G}(g)$ and if $\varphi \in C_0^\infty(\Omega)$ is such that $0 \leq \varphi \leq 1$, then

$$\varphi\mu \in \mathcal{G}_0(g).$$

Indeed, let u be a solution of (1.1). We first observe that $|g(\varphi u)| \leq |g(u)|$. Since $g(u) \in L^1_{\text{loc}}(\Omega)$ and φ has compact support in Ω, $g(\varphi u) \in L^1(\Omega)$. Next,

$$-\Delta(\varphi u) + g(\varphi u) = \varphi\mu + h \quad \text{in } \Omega,$$

where

$$h = g(\varphi u) - \Big(u\Delta\varphi + 2\nabla\varphi \cdot \nabla u + \varphi g(u)\Big).$$

Since φ has compact support, $h \in L^1(\Omega)$. Thus, $\varphi\mu + h \in \mathcal{G}_0(g)$ and consequently, by Lemma A.2, $\varphi\mu \in \mathcal{G}_0$.

Now let (φ_k) be a sequence of nonnegative functions in $C_0^\infty(\Omega)$ such that $0 \leq \varphi_k \leq 1$ and $\varphi_k \nearrow 1$ locally uniformly in Ω. It follows by dominated convergence that $\varphi_k\mu \to \mu$ in $\mathcal{M}(\Omega)$. Consequently, if u_k is the solution of (1.2) with μ replaced by $\varphi\mu_k$, then (u_k) converges in $L^1(\Omega)$ to a solution u of (1.2). Thus, $\mu \in \mathcal{G}_0(g)$. □

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