A comparison of approximation algorithms for the joint spectral radius

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Abstract

Let Σ be a set of $n \times n$ matrices and let us consider the following linear iteration:

$$x(t+1) = A_t x(t), \quad A_t \in \Sigma \text{ for all } t.$$

These kinds of *switched linear systems* arise in many applications such as asynchronous systems, hybrid systems, switching control, ...

The stability under arbitrary switchings of this system depends on a quantity called the *joint spectral radius (JSR)*, which represents the maximal growth rate of such a discrete linear system. More precisely, the JSR of a set Σ of matrices is defined by the following expression:

$$\rho(\Sigma) = \lim_{k\to\infty} \max\{\|A_{i_1} \dots A_{i_k}\|^{1/k} \mid A_i \in \Sigma\},$$

independently of the matrix norm used. For bounded sets Σ, the JSR is also equal to the so-called generalized spectral radius $\bar{\rho}$, defined by the following equation:

$$\bar{\rho}(\Sigma) = \limsup_{k\to\infty} \max\{\rho(A_{i_1} \dots A_{i_k})^{1/k} \mid A_i \in \Sigma\}.$$

For the linear iterations we consider, all trajectories converge thus to the origin if and only if the JSR of the corresponding set of matrices is strictly less than 1.

Another quantity of interest is the *joint spectral subradius (JSS)*, also called lower spectral radius, which represents the minimal achievable growth rate:

$$\check{\rho}(\Sigma) = \lim_{k\to\infty} \min\{\|A_{i_1} \dots A_{i_k}\|^{1/k} \mid A_i \in \Sigma\}.$$

The JSS is related to the mortality problem, where we ask if the zero matrix can be expressed as a finite product of matrices in Σ.

The joint spectral quantities are notoriously difficult to compute. Indeed, it is NP-Hard to approximate the JSR ; moreover, the problem of checking whether $\rho \leq 1$ is undecidable, and the decidability of the question $\rho < 1$ is currently unknown. Furthermore, the problem of approximating the JSS is undecidable in the general case (see [1] for a survey on the joint spectral quantities). Despite these negative theoretical results, several algorithms have been designed in order to approximate the JSR. Indeed, in the case of the JSR, the following easy property holds:

$$\rho(\Sigma) = \inf_{\|\cdot\|} \max\{\|A\| \mid A \in \Sigma\}.$$

Thus, one way to compute the JSR is to try to find a norm such that $\rho(\Sigma) = \max\{\|A\| \mid A \in \Sigma\}$, i.e., an *extremal* norm. Several methods have been designed using this fact (see for example [2], [3], [4]). In this paper, we will study and compare several techniques for the computation of the joint spectral quantities, such as semidefinite programming that computes the infimum on a class of norms, or geometric algorithms that tries to approximate the unit ball of an extremal norm using polytopes.

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References

[1] Raphaël M. Jungers, "The joint spectral radius, theory and applications", *Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, 2009.

[2] Vincent D. Blondel, Yurii Nesterov and Jacques Theys, "On the accuracy of the ellipsoid norm approximation of the joint spectral radius", *Linear Algebra and its Applications*, 394(1):91–107, 2005.

[3] Nicola Guglielmi, Marino Zennaro, "Finding extremal complex polytope norms for families of real matrices", *SIAM Journal of Matrix Analysis and Applications*, 31(2):602–620, 2009.

[4] Vladimir Protasov, Raphaël M. Jungers, Vincent D. Blondel, "Joint spectral characteristics of matrices: a conic programming approach", *SIAM Journal of Matrix Analysis and Applications*, to appear.