Complément au chapitre 2, « Le moteur synchrone autopiloté : modèle dynamique avec prise en compte du comportement des amortisseurs et de l'empiètement », de « Commandes d'actionneurs électriques synchrones et spéciaux » (éditeur Jean-Paul Louis), Traité EGEM, série génie électrique, Hermes, Lavoisier

Démonstration des formules [2.22]

De [2.21], on tire

$$i^D = \frac{1}{L_R}[\psi_D - M\cos(p\theta_m)\,i^a - M\cos(p\theta_m - \frac{2\pi}{3})\,i^b - M\cos(p\theta_m + \frac{2\pi}{3})\,i^c - M_{Rf}\,I^f\,] \qquad [2.90a]$$

$$i^Q = \frac{1}{L_R}[\psi_Q + M\sin(p\theta_m)\,i^a + M\sin(p\theta_m - \frac{2\pi}{3})\,i^b + M\sin(p\theta_m + \frac{2\pi}{3})\,i^c\,] \qquad [2.90b]$$

Donc [2.20] devient

$$\begin{aligned}
\psi_a = {} & [L_S - \frac{M^2}{L_R}\cos^2(p\theta_m) - \frac{M^2}{L_R}\sin^2(p\theta_m)]\,i^a \\
& + [M_S - \frac{M^2}{L_R}\cos(p\theta_m)\cos(p\theta_m - \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin(p\theta_m)\sin(p\theta_m - \frac{2\pi}{3})]\,i^b \\
& + [M_S - \frac{M^2}{L_R}\cos(p\theta_m)\cos(p\theta_m + \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin(p\theta_m)\sin(p\theta_m + \frac{2\pi}{3})]\,i^c \\
& + [M_f - \frac{M\,M_{Rf}}{L_R}]\cos(p\theta_m)\,I^f \\
& + \frac{M}{L_R}\psi_D\cos(p\theta_m) - \frac{M}{L_R}\psi_D\sin(p\theta_m)
\end{aligned} \qquad [2.91a]$$

$$\psi_b = [L_S - \frac{M^2}{L_R}\cos(p\theta_m - \frac{2\pi}{3})\cos(p\theta_m) - \frac{M^2}{L_R}\sin(p\theta_m - \frac{2\pi}{3})\sin(p\theta_m)]\, i^a$$
$$+ [M_S - \frac{M^2}{L_R}\cos^2(p\theta_m - \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin^2(p\theta_m - \frac{2\pi}{3})]\, i^b \quad [2.91b]$$
$$+ [M_S - \frac{M^2}{L_R}\cos(p\theta_m - \frac{2\pi}{3})\cos(p\theta_m + \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin(p\theta_m - \frac{2\pi}{3})\sin(p\theta_m + \frac{2\pi}{3})]\, i^c$$
$$+ [M_f - \frac{M\,M_{Rf}}{L_R}]\cos(p\theta_m - \frac{2\pi}{3})\, I^f$$
$$+ \frac{M}{L_R}\psi_D\cos(p\theta_m - \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D\sin(p\theta_m - \frac{2\pi}{3})$$

$$\psi_c = [L_S - \frac{M^2}{L_R}\cos(p\theta_m + \frac{2\pi}{3})\cos(p\theta_m) - \frac{M^2}{L_R}\sin(p\theta_m + \frac{2\pi}{3})\sin(p\theta_m)]\, i^a$$
$$+ [M_S - \frac{M^2}{L_R}\cos(p\theta_m + \frac{2\pi}{3})\cos(p\theta_m - \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin(p\theta_m + \frac{2\pi}{3})\sin(p\theta_m - \frac{2\pi}{3})]\, i^b \quad [2.91c]$$
$$+ [M_S - \frac{M^2}{L_R}\cos^2(p\theta_m + \frac{2\pi}{3}) - \frac{M^2}{L_R}\sin^2(p\theta_m + \frac{2\pi}{3})]\, i^c$$
$$+ [M_f - \frac{M\,M_{Rf}}{L_R}]\cos(p\theta_m + \frac{2\pi}{3})\, I^f$$
$$+ \frac{M}{L_R}\psi_D\cos(p\theta_m + \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D\sin(p\theta_m + \frac{2\pi}{3})$$

ou encore

$$\psi_a = [L_S - \frac{M^2}{L_R}]\, i^a + [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^b + [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^c \quad [2.92a]$$
$$+ [\,(M_f - \frac{M\,M_{Rf}}{L_R})I^f + \frac{M}{L_D}\psi_D\,]\cos(p\theta_m) - \frac{M}{L_R}\psi_D\sin(p\theta_m)$$

$$\psi_b = [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^a + [L_S - \frac{M^2}{L_R}]\, i^b + [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^c \quad [2.92b]$$
$$+ [\,(M_f - \frac{M\,M_{Rf}}{L_R})I^f + \frac{M}{L_D}\psi_D\,]\cos(p\theta_m - \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D\sin(p\theta_m - \frac{2\pi}{3})$$

$$\psi_c = [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^a + [M_S + \frac{1}{2}\frac{M^2}{L_R}]\, i^b + [L_S - \frac{M^2}{L_R}]\, i^c \quad [2.92c]$$
$$+ [\,(M_f - \frac{M\,M_{Rf}}{L_R})I^f + \frac{M}{L_D}\psi_D\,]\cos(p\theta_m + \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D\sin(p\theta_m + \frac{2\pi}{3})$$

ou encore, sachant que $i^a + i^b + i^c = 0$,

$$\psi_a = [L_S - M_S - \frac{3}{2}\frac{M^2}{L_R}] i^a$$
$$+ [(M_f - \frac{M\,M_{Rf}}{L_R}) I^f + \frac{M}{L_D}\psi_D]\cos(p\theta_m) - \frac{M}{L_R}\psi_D \sin(p\theta_m) \quad [2.93a]$$

$$\psi_b = [L_S - M_S - \frac{3}{2}\frac{M^2}{L_R}]\, i^b$$
$$+ [(M_f - \frac{M\,M_{Rf}}{L_R}) I^f + \frac{M}{L_D}\psi_D]\cos(p\theta_m - \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D \sin(p\theta_m - \frac{2\pi}{3}) \quad [2.93b]$$

$$\psi_c = [L_S - M_S - \frac{3}{2}\frac{M^2}{L_R}] i^c$$
$$+ [(M_f - \frac{M\,M_{Rf}}{L_R}) I^f + \frac{M}{L_D}\psi_D]\cos(p\theta_m + \frac{2\pi}{3}) - \frac{M}{L_R}\psi_D \sin(p\theta_m + \frac{2\pi}{3}) \quad [2.93c]$$

qui correspond bien à [2.22] compte tenu de [2.23] [2.25].

Vérification de [2.26] [2.28]

Négligeant pour l'instant les variations de Ψ_{DC} et Ψ_{QC}, la dérivée temporelle de [2.22a] est :

$$\frac{d\psi_a}{dt} = L_C \frac{di^a}{dt} - p\,\omega_m [\psi_{QC}\cos(p\theta_m) + \psi_{DC}\sin(p\theta_m)] \quad [2.94]$$

Posons [2.28] :

$$\delta = arc\,tg \frac{\psi_{QC}}{\psi_{DC}} \quad [2.95]$$

Alors

$$\frac{d\psi_a}{dt} = L_C \frac{di^a}{dt} - p\,\omega_m \sqrt{\psi_{DC}{}^2 + \psi_{QC}{}^2}\,[\sin(\delta)\cos(p\theta_m) + \cos(\delta)\sin(p\theta_m)]$$

$$= L_C \frac{di^a}{dt} - p\,\omega_m \sqrt{\psi_{DC}{}^2 + \psi_{QC}{}^2}\,\sin(p\theta_m + \delta) \qquad [2.96]$$

$$= L_C \frac{di^a}{dt} + p\,\omega_m \sqrt{\psi_{DC}{}^2 + \psi_{QC}{}^2}\,\sin(-p\theta_m - \delta)$$

$$= L_C \frac{di^a}{dt} + p\,\omega_m \sqrt{\psi_{DC}{}^2 + \psi_{QC}{}^2}\,\cos(\frac{\pi}{2} + p\theta_m + \delta)$$

soit, en définissant [2.26] et [2.27],

$$\frac{d\psi_a}{dt} = L_C \frac{di^a}{dt} + E_C \cos(p\theta_m + \psi_C) \qquad [2.97]$$

qui correspond bien au schéma de la figure 1.5.

Calcul de l'expression [2.57]

On a, en dérivant [2.36],

$$\begin{aligned}\frac{d\Psi}{dt} &= [L_{DC} + L_C < Z >]\frac{dI}{dt} + < X > \dot{\Psi}_{DC} - < Y > \dot{\Psi}_{QC} \\ &+ \dot{\rho}[L_C \frac{\partial < Z >}{\partial\rho} I + \frac{\partial < X >}{\partial\rho}\Psi_{DC} - \frac{\partial < Y >}{\partial\rho}\Psi_{QC}] \\ &+ \dot{\mu}[L_C \frac{\partial < Z >}{\partial\mu} I + \frac{\partial < X >}{\partial\mu}\Psi_{DC} - \frac{\partial < Y >}{\partial\mu}\Psi_{QC}] \\ &+ \dot{\rho}[L_C \frac{\partial < Z >}{\partial\nu} I + \frac{\partial < X >}{\partial\nu}\Psi_{DC} - \frac{\partial < Y >}{\partial\nu}\Psi_{QC}]\end{aligned} \qquad [2.98]$$

soit, compte tenu de [2.41] et [2.45],

$$\begin{aligned}\frac{d\Psi}{dt} &= [L_{DC} + L_C < Z >]\frac{dI}{dt} + < X > \dot{\Psi}_{DC} - < Y > \dot{\Psi}_{QC} \\ &+ \dot{\rho}[< Y > \Psi_{DC} + < X > \Psi_{QC}] \\ &+ \dot{\mu}[L_C \frac{\partial < Z >}{\partial \mu} I + \frac{\partial < X >}{\partial \mu}\Psi_{DC} - \frac{\partial < Y >}{\partial \mu}\Psi_{QC}] \\ &+ \dot{\nu}[L_C \frac{\partial < Z >}{\partial \nu} I + \frac{\partial < X >}{\partial \nu}\Psi_{DC} - \frac{\partial < Y >}{\partial \nu}\Psi_{QC}]\end{aligned} \quad [2.99]$$

On en déduit, usant de [2.54],

$$\begin{aligned}E &= U_{DC} - \frac{d\Psi}{dt} \\ &= -(p\omega_m + \dot{\rho})[< Y > \Psi_{DC} + < X > \Psi_{QC}] \\ &- \dot{\mu}[\frac{1}{2}L_C \frac{\partial < Z >}{\partial \mu} I + \frac{\partial < X >}{\partial \mu}\Psi_{DC} - \frac{\partial < Y >}{\partial \mu}\Psi_{QC}] \\ &- \dot{\nu}[\frac{1}{2}L_C \frac{\partial < Z >}{\partial \nu} I + \frac{\partial < X >}{\partial \nu}\Psi_{DC} - \frac{\partial < Y >}{\partial \nu}\Psi_{QC}] \\ &+ [R_{DC} + R_S < Z > + R_{add}]I\end{aligned} \quad [2.100]$$

qui n'est autre que [2.57].

Valeur moyenne de X (Annexe 1)

Sur l'intervalle d'empiètement [2.7], on a conformément à [2.13] et [2.30a] :

$$X = \frac{1}{D}\left\{\begin{aligned} &\sin(\nu + \frac{\mu}{2})[\cos(p\theta_m) - \cos(p\theta_m - \frac{2\pi}{3})] \\ &+ \sin(p\theta_m + \rho - \nu + \frac{\pi}{3})[\cos(p\theta_m) - \cos(p\theta_m + \frac{2\pi}{3})] \\ &+ \sin(\nu - \frac{\mu}{2})[\cos(p\theta_m - \frac{2\pi}{3}) - \cos(p\theta_m + \frac{2\pi}{3})]\end{aligned}\right\} \quad [2.101]$$

où l'on a posé

$$D = \sin(\nu + \frac{\mu}{2}) - \sin(\nu - \frac{\mu}{2}) = 2\sin(\frac{\mu}{2})\cos(\nu) \qquad [2.102]$$

On calcule successivement

$$X = \frac{1}{D}\left\{\begin{array}{c} -2\sin(\nu + \frac{\mu}{2})\sin(p\theta_m - \frac{\pi}{3})\sin(\frac{\pi}{3}) \\ +2\sin(p\theta_m + \rho - \nu + \frac{\pi}{3})\sin(p\theta_m + \frac{\pi}{3})\sin(\frac{\pi}{3}) \\ +2\sin(\nu - \frac{\mu}{2})\sin(p\theta_m)\sin(\frac{2\pi}{3}) \end{array}\right\} \qquad [2.103]$$

$$X = \frac{\sqrt{3}}{D}\left\{\begin{array}{c} \frac{1}{2}\cos(\rho - \nu) \\ -\sin(\nu + \frac{\mu}{2})\sin(p\theta_m - \frac{\pi}{3}) \\ +\sin(\nu - \frac{\mu}{2})\sin(p\theta_m) \\ -\frac{1}{2}\cos(2p\theta_m + \rho - \nu + \frac{2\pi}{3}) \end{array}\right\} \qquad [2.104]$$

L'intégrale de X sur [2.7] vaut donc

$$\int_{[2.7]} X d(p\theta_m) = \frac{\sqrt{3}}{D}\left\{\begin{array}{c} \frac{1}{2}\cos(\rho - \nu)(p\theta_m) \\ +\sin(\nu + \frac{\mu}{2})\cos(p\theta_m - \frac{\pi}{3}) \\ -\sin(\nu - \frac{\mu}{2})\cos(p\theta_m) \\ -\frac{1}{4}\sin(2p\theta_m + \rho - \nu + \frac{2\pi}{3}) \end{array}\right\}\Bigg|_{-\rho - \frac{\pi}{3} - \frac{\mu}{2}}^{-\rho - \frac{\pi}{3} + \frac{\mu}{2}} \qquad [2.105]$$

soit

$$\int_{[2.7]} Xd(p\theta_m) = \frac{\sqrt{3}}{D}\left\{\begin{array}{c} \frac{1}{2}\mu\cos(\rho-\nu) \\ +\sin(\nu+\frac{\mu}{2})[\cos(-\rho-\frac{2\pi}{3}+\frac{\mu}{2})-\cos(-\rho-\frac{2\pi}{3}-\frac{\mu}{2})] \\ -\sin(\nu-\frac{\mu}{2})[\cos(-\rho-\frac{\pi}{3}+\frac{\mu}{2})-\cos(-\rho-\frac{\pi}{3}-\frac{\mu}{2})] \\ -\frac{1}{4}[\sin(-\rho-\nu+\mu)-\sin(-\rho-\nu-\mu)] \end{array}\right\} \quad [2.106]$$

$$\int_{[2.7]} Xd(p\theta_m) = \frac{\sqrt{3}}{D}\left\{\begin{array}{c} \frac{1}{2}\mu\cos(\rho-\nu) \\ +2\sin(\nu+\frac{\mu}{2})\sin(\rho+\frac{2\pi}{3})\sin(\frac{\mu}{2}) \\ -2\sin(\nu-\frac{\mu}{2})\sin(\rho+\frac{\pi}{3})\sin(\frac{\mu}{2}) \\ -\frac{1}{2}\cos(\rho+\nu)\sin(\mu) \end{array}\right\} \quad [2.107]$$

ou encore, utilisant [2.102],

$$\int_{[2.7]} Xd(p\theta_m) = \frac{\sqrt{3}}{2\cos(\nu)}\left\{\begin{array}{c} \frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu) \\ +2\sin(\nu+\frac{\mu}{2})\sin(\rho+\frac{2\pi}{3}) \\ -2\sin(\nu-\frac{\mu}{2})\sin(\rho+\frac{\pi}{3}) \\ -\cos(\rho+\nu)\cos(\frac{\mu}{2}) \end{array}\right\} \quad [2.108]$$

Par ailleurs, sur l'intervalle hors empiètement [2.14], on a

$$\begin{aligned} X &= \cos(p\theta_m) - \cos(p\theta_m - \frac{2\pi}{3}) \\ &= -2\sin(p\theta_m - \frac{\pi}{3})\sin(\frac{\pi}{3}) \\ &= -\sqrt{3}\sin(p\theta_m - \frac{\pi}{3}) \end{aligned} \quad [2.109]$$

L'intégrale sur cet intervalle vaut donc

$$\int_{[2.14]} Xd(p\theta_m) = \sqrt{3}\cos(p\theta_m - \frac{\pi}{3})\Bigg|_{-\rho-\frac{\pi}{3}+\frac{\mu}{2}}^{-\rho-\frac{\mu}{2}}$$

$$= \sqrt{3}[\cos(-\rho - \frac{\mu}{2} - \frac{\pi}{3}) - \cos(-\rho - \frac{2\pi}{3} + \frac{\mu}{2})] \qquad [2.110]$$

$$= \sqrt{3}[\cos(\rho + \frac{\mu}{2} + \frac{\pi}{3}) - \cos(\rho + \frac{2\pi}{3} - \frac{\mu}{2})]$$

En additionnant [2.108] et [2.110] et en divisant par $\pi/3$, on obtient

$$< X >= \frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\begin{array}{c} \frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu) \\ +2\cos(\rho+\frac{\mu}{2}+\frac{\pi}{3})\cos(\nu) \\ +2\sin(\nu+\frac{\mu}{2})\sin(\rho+\frac{2\pi}{3}) \\ -2\cos(\rho-\frac{\mu}{2}+\frac{2\pi}{3})\cos(\nu) \\ -2\sin(\nu-\frac{\mu}{2})\sin(\rho+\frac{\pi}{3}) \\ -\cos(\rho+\nu)\cos(\frac{\mu}{2}) \end{array}\right\} \qquad [2.111]$$

soit

$$< X >= \frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\begin{array}{c} \frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu) \\ +2\cos(\nu)\cos(\rho+\frac{\mu}{2})\cos(\frac{\pi}{3})-2\cos(\nu)\sin(\rho+\frac{\mu}{2})\sin(\frac{\pi}{3}) \\ +2\sin(\nu+\frac{\mu}{2})\sin(\rho)\cos(\frac{2\pi}{3})+2\sin(\nu+\frac{\mu}{2})\cos(\rho)\sin(\frac{2\pi}{3}) \\ -2\cos(\nu)\cos(\rho-\frac{\mu}{2})\cos(\frac{2\pi}{3})+2\cos(\nu)\sin(\rho-\frac{\mu}{2})\sin(\frac{2\pi}{3}) \\ -2\sin(\nu-\frac{\mu}{2})\sin(\rho)\cos(\frac{\pi}{3})-2\sin(\nu-\frac{\mu}{2})\cos(\rho)\sin(\frac{\pi}{3}) \\ -\cos(\rho+\nu)\cos(\frac{\mu}{2}) \end{array}\right\} \qquad [2.112]$$

$$< X >=\frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\begin{array}{c}\frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu)\\ +\cos(\nu)\cos(\rho+\frac{\mu}{2})-\sqrt{3}\cos(\nu)\sin(\rho+\frac{\mu}{2})\\ -\sin(\nu+\frac{\mu}{2})\sin(\rho)+\sqrt{3}\sin(\nu+\frac{\mu}{2})\cos(\rho)\\ +\cos(\nu)\cos(\rho-\frac{\mu}{2})+\sqrt{3}\cos(\nu)\sin(\rho-\frac{\mu}{2})\\ -\sin(\nu-\frac{\mu}{2})\sin(\rho)-\sqrt{3}\sin(\nu-\frac{\mu}{2})\cos(\rho)\\ -\cos(\rho+\nu)\cos(\frac{\mu}{2})\end{array}\right\} \quad [2.113]$$

ou

$$< X >=\frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\begin{array}{c}\frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu)\\ +2\cos(\nu)\cos(\rho)\cos(\frac{\mu}{2})-2\sqrt{3}\cos(\nu)\cos(\rho)\sin(\frac{\mu}{2})\\ -2\sin(\nu)\sin(\rho)\cos(\frac{\mu}{2})+2\sqrt{3}\cos(\nu)\cos(\rho)\sin(\frac{\mu}{2})\\ -\cos(\rho+\nu)\cos(\frac{\mu}{2})\end{array}\right\} \quad [2.114]$$

soit

$$< X >=\frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\begin{array}{c}\frac{\mu/2}{\sin(\mu/2)}\cos(\rho-\nu)\\ +2\cos(\rho+\nu)\cos(\frac{\mu}{2})\\ -\cos(\rho+\nu)\cos(\frac{\mu}{2})\end{array}\right\} \quad [2.115]$$

On a finalement

$$< X >=\frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\cos(\rho-\nu)\frac{\mu/2}{\sin(\mu/2)}+\cos(\rho+\nu)\cos(\frac{\mu}{2})\right\} \quad [2.116]$$

qui n'est autre que [2.75].

Valeur moyenne <Y> (Annexe 1)

En appliquant [2.41a] à [2.116], on obtient

$$< Y > = -\frac{3\sqrt{3}}{2\pi\cos(\nu)}\left\{\sin(\rho-\nu)\frac{\mu/2}{\sin(\mu/2)}+\sin(\rho+\nu)\cos(\frac{\mu}{2})\right\} \quad [2.117]$$

qui n'est autre que [2.76].

A titre de vérification, on notera que l'on retrouve [2.116] en appliquant [2.41b] à [2.117].

Valeur moyenne <Z> (Annexe 1)

Sur l'intervalle [2.7], on a par [2.13] appliqué à [2.35],

$$Z = 1+\frac{1}{D^2}\left\{\begin{array}{c} \sin^2(\nu+\frac{\mu}{2})+\sin^2(\nu-\frac{\mu}{2}) \\ +2[\sin(\nu+\frac{\mu}{2})+\sin(\nu-\frac{\mu}{2})]\sin(p\theta_m+\rho-\nu+\frac{\pi}{3}) \\ +2\sin^2(p\theta_m+\rho-\nu+\frac{\pi}{3}) \end{array}\right\} \quad [2.118]$$

On obtient successivement

$$Z = 1+\frac{1}{D^2}\left\{\begin{array}{c} D^2+2\sin(\nu+\frac{\mu}{2})\sin(\nu-\frac{\mu}{2}) \\ +4\sin(\nu)\cos(\frac{\mu}{2})\sin(p\theta_m+\rho-\nu+\frac{\pi}{3}) \\ +1-\cos(2p\theta_m+2\rho-2\nu+\frac{2\pi}{3}) \end{array}\right\} \quad [2.119]$$

$$Z = 2+\frac{1}{D^2}\left\{\begin{array}{c} 1+2\sin(\nu+\frac{\mu}{2})\sin(\nu-\frac{\mu}{2}) \\ +4\sin(\nu)\cos(\frac{\mu}{2})\sin(p\theta_m+\rho-\nu+\frac{\pi}{3}) \\ -\cos(2p\theta_m+2\rho-2\nu+\frac{2\pi}{3}) \end{array}\right\} \quad [2.120]$$

Puisque Z est égal à 2 sur l'intervalle hors empiètement [2.14], on a

$$<Z>=2+\frac{3}{\pi D^2}\int_{[2.7]}\left\{\begin{array}{c}1\\ +2[\sin^2(\nu)\cos^2(\frac{\mu}{2})-\cos^2(\nu)\sin^2(\frac{\mu}{2})]\\ +4\sin(\nu)\cos(\frac{\mu}{2})\sin(p\theta_m+\rho-\nu+\frac{\pi}{3})\\ -\cos(2p\theta_m+2\rho-2\nu+\frac{2\pi}{3})\end{array}\right\}d(p\theta_m) \quad [2.121]$$

$$<Z>=2+\frac{3}{\pi D^2}\left\{\begin{array}{c}\mu\\ +2[\sin^2(\nu)\cos^2(\frac{\mu}{2})-\cos^2(\nu)\sin^2(\frac{\mu}{2})]\mu\\ -4\sin(\nu)\cos(\frac{\mu}{2})\cos(p\theta_m+\rho-\nu+\frac{\pi}{3})\Big|_{-\rho-\frac{\pi}{3}-\frac{\mu}{2}}^{-\rho-\frac{\pi}{3}+\frac{\mu}{2}}\\ -\frac{1}{2}\sin(2p\theta_m+2\rho-2\nu+\frac{2\pi}{3})\Big|_{-\rho-\frac{\pi}{3}-\frac{\mu}{2}}^{-\rho-\frac{\pi}{3}+\frac{\mu}{2}}\end{array}\right\} \quad [2.122]$$

soit

$$<Z>=2+\frac{3}{\pi D^2}\left\{\begin{array}{c}\mu\\ +2[\sin^2(\nu)\cos^2(\frac{\mu}{2})-\cos^2(\nu)\sin^2(\frac{\mu}{2})]\mu\\ -4\sin(\nu)\cos(\frac{\mu}{2})[\cos(-\nu+\frac{\mu}{2})-\cos(-\nu-\frac{\mu}{2})]\\ -\frac{1}{2}[\sin(-2\nu+\mu)-\sin(-2\nu-\mu)]\end{array}\right\} \quad [2.123]$$

$$<Z>=2+\frac{3}{\pi D^2}\left\{\begin{array}{c}\mu\\ +2[\sin^2(\nu)\cos^2(\frac{\mu}{2})-\cos^2(\nu)\sin^2(\frac{\mu}{2})]\mu\\ -8\sin(\nu)\cos(\frac{\mu}{2})\sin(\nu)\sin(\frac{\mu}{2})\\ -\cos(2\nu)\sin(\mu)\end{array}\right\} \quad [2.124]$$

$$< Z > = 2 + \frac{3}{\pi D^2} \left\{ \begin{array}{c} \mu \\ + 2[\sin^2(\nu)\cos^2(\frac{\mu}{2}) - \cos^2(\nu)\sin^2(\frac{\mu}{2})]\mu \\ - 4\sin^2(\nu)\sin(\mu) \\ - \sin(\mu) + 2\sin^2(\nu)\sin(\mu) \end{array} \right\} \quad [2.125]$$

$$< Z > = 2 + \frac{3}{\pi D^2} \left\{ \begin{array}{c} \mu - \sin\mu \\ + 2[\sin^2(\nu)\cos^2(\frac{\mu}{2}) - \cos^2(\nu)\sin^2(\frac{\mu}{2})]\mu \\ - 2\sin^2(\nu)\sin(\mu) \end{array} \right\} \quad [2.126]$$

$$< Z > = 2 + \frac{3}{\pi D^2} \left\{ \begin{array}{c} \mu - \sin\mu \\ + 2\sin^2(\nu)[\cos^2(\frac{\mu}{2})\mu - \sin(\mu)] \\ - 2\cos^2(\nu)\sin^2(\frac{\mu}{2})\mu \end{array} \right\} \quad [2.127]$$

et finalement

$$< Z > = 2 + \frac{3}{\pi D^2} \left\{ \begin{array}{c} \sin^2(\nu)[2\cos^2(\frac{\mu}{2})\mu + \mu - 3\sin(\mu)] \\ + \cos^2(\nu)[\mu - \sin\mu - 2\sin^2(\frac{\mu}{2})\mu] \end{array} \right\} \quad [2.128]$$

Il reste à remplacer D par sa valeur [2.102] et à simplifier pour obtenir

$$< Z > = 2 + \frac{3}{4\pi} \left\{ \begin{array}{c} tg^2(\nu)[2\cos^2(\frac{\mu}{2})\mu + \mu - 3\sin(\mu)] / \sin^2(\frac{\mu}{2}) \\ + [\mu - \sin\mu - 2\sin^2(\frac{\mu}{2})\mu] / \sin^2(\frac{\mu}{2}) \end{array} \right\} \quad [2.129]$$

On notera que <Z> ne dépend pas de ρ , ce qui est conforme à [2.46].

Calcul de $<X>^2 + <Y>^2$

Partant de [2.75] et [2.76], on obtient

$$< X >^2 + < Y >^2 = \frac{27}{4\pi^2 cos^2(\nu)} \{ [\frac{\mu/2}{sin(\mu/2)}]^2 + cos^2(\frac{\mu}{2}) + 2[cos(\rho-\nu)cos(\rho+\nu) + sin(\rho-\nu)sin(\rho+\nu)] \frac{\mu/2}{sin(\mu/2)} cos(\frac{\mu}{2}) \} \quad [2.130]$$

soit

$$< X >^2 + < Y >^2 = \frac{27}{4\pi^2 cos^2(\nu)} \{ [\frac{\mu/2}{sin(\mu/2)}]^2 + cos^2(\frac{\mu}{2}) + 2cos(2\nu) \frac{\mu/2}{sin(\mu/2)} cos(\frac{\mu}{2}) \} \quad [2.131]$$

$$< X >^2 + < Y >^2 = \frac{27}{4\pi^2 cos^2(\nu)} \{ cos^2(\nu)[(\frac{\mu/2}{sin(\mu/2)})^2 + cos^2(\frac{\mu}{2}) + 2 \frac{\mu/2}{sin(\mu/2)} cos(\frac{\mu}{2})] + sin^2(\nu)[(\frac{\mu/2}{sin(\mu/2)})^2 + cos^2(\frac{\mu}{2}) - 2 \frac{\mu/2}{sin(\mu/2)} cos(\frac{\mu}{2})] \} \quad [2.132]$$

$$< X >^2 + < Y >^2 = \frac{27}{4\pi^2} \{ [\frac{\mu/2}{sin(\mu/2)} + cos(\frac{\mu}{2})]^2 + tg^2(\nu)[\frac{\mu/2}{sin(\mu/2)} - cos(\frac{\mu}{2})]^2 \} \quad [2.133]$$

Comme prévu, cette expression ne dépend pas de ρ.

Calcul de R_{add}

Partant des définitions [2.30], on a, de façon générale,

$$
\begin{aligned}
X^2 = & (N^a)^2 \cos^2(p\theta_m) \\
& + (N^b)^2 \cos^2(p\theta_m - \frac{2\pi}{3}) \\
& + (N^c)^2 \cos^2(p\theta_m + \frac{2\pi}{3}) \\
& + 2N^a N^b \cos(p\theta_m)\cos(p\theta_m - \frac{2\pi}{3}) \\
& + 2N^b N^c \cos(p\theta_m - \frac{2\pi}{3})\cos(p\theta_m + \frac{2\pi}{3}) \\
& + 2N^c N^a \cos(p\theta_m + \frac{2\pi}{3})\cos(p\theta_m)
\end{aligned}
\qquad [2.134]
$$

$$
\begin{aligned}
Y^2 = & (N^a)^2 \sin^2(p\theta_m) \\
& + (N^b)^2 \sin^2(p\theta_m - \frac{2\pi}{3}) \\
& + (N^c)^2 \sin^2(p\theta_m + \frac{2\pi}{3}) \\
& + 2N^a N^b \sin(p\theta_m)\sin(p\theta_m - \frac{2\pi}{3}) \\
& + 2N^b N^c \sin(p\theta_m - \frac{2\pi}{3})\sin(p\theta_m + \frac{2\pi}{3}) \\
& + 2N^c N^a \sin(p\theta_m + \frac{2\pi}{3})\sin(p\theta_m)
\end{aligned}
\qquad [2.135]
$$

donc

$$\begin{aligned}X^2+Y^2 &= (N^a)^2+(N^b)^2+(N^c)^2\\ &\quad +2N^aN^b\cos(-\frac{2\pi}{3})\\ &\quad +2N^bN^c\cos(\frac{4\pi}{3})\\ &\quad +2N^cN^a\cos(\frac{2\pi}{3})\\ &= (N^a)^2+(N^b)^2+(N^c)^2\\ &\quad -N^aN^b-N^bN^c-N^cN^a\end{aligned} \qquad [2.136]$$

Or, en élevant [2.18] au carré, on obtient

$$\begin{aligned}&(N^a)^2+(N^b)^2+(N^c)^2\\ &\quad +2N^aN^b+2N^bN^c+2N^cN^a=0\end{aligned} \qquad [2.137]$$

En combinant [2.136] et [2.137], on voit que le terme $<X^2> + <Y^2>$ qui figure dans [2.55] peut s'écrire

$$\begin{aligned}<X^2+Y^2> &= <\frac{3}{2}[(N^a)^2+(N^b)^2+(N^c)^2]>\\ &= \frac{3}{2}<Z>\end{aligned} \qquad [2.138]$$

En utilisant [2.138], on tire de [2.55] l'expression [2.56] de R_{add} .

Expressions à la limite des μ petits

Nous allons décomposer en série de Taylor selon μ les expressions des annexes 1 et 3. Nous utilisons pour ce faire

$$\sin\mu=\mu-\frac{1}{6}\mu^3+\ldots \qquad [2.139]$$

$$\sin\frac{\mu}{2}=\frac{\mu}{2}-\frac{1}{6}(\frac{\mu}{2})^3+\ldots \qquad [2.140]$$

$$sin^2 \frac{\mu}{2} = (\frac{\mu}{2})^2 - \frac{1}{3}(\frac{\mu}{2})^4 + \ldots \qquad [2.141]$$

$$cos \frac{\mu}{2} = 1 - \frac{1}{2}(\frac{\mu}{2})^2 + \ldots \qquad [2.142]$$

$$cos^2 \frac{\mu}{2} = 1 - (\frac{\mu}{2})^2 + \ldots \qquad [2.143]$$

On obtient ainsi, pour [2.75] :

$$< X > \approx \frac{3\sqrt{3}}{2\pi\, cos(\nu)} [cos(\rho - \nu) \frac{\mu/2}{\mu/2 + \ldots\mu^3} + cos(\rho + \nu)(1 + \ldots\mu^2)] \qquad [2.144]$$

soit, au premier ordre en μ :

$$\begin{aligned} < X > &\approx \frac{3\sqrt{3}}{2\pi\, cos(\nu)} [cos(\rho - \nu) + cos(\rho + \nu)] \\ &= \frac{3\sqrt{3}}{2\pi\, cos(\nu)} [2 cos(\rho) cos(\nu)] \qquad [2.145] \\ &= \frac{3\sqrt{3}}{\pi} cos(\rho) \end{aligned}$$

qui n'est autre que [2.46a].

Demême, on obtient pour [2.76] :

$$< Y > \approx -\frac{3\sqrt{3}}{2\pi\, cos(\nu)} [sin(\rho - \nu) \frac{\mu/2}{\mu/2 + \ldots\mu^3} + sin(\rho + \nu)(1 + \ldots\mu^2)] \qquad [2.146]$$

soit, au premier ordre en μ :

$$\begin{aligned} < Y > &\approx -\frac{3\sqrt{3}}{2\pi\, cos(\nu)} [sin(\rho - \nu) + sin(\rho - \nu)] \\ &= -\frac{3\sqrt{3}}{2\pi\, cos(\nu)} [2 sin(\rho) cos(\nu)] \qquad [2.147] \\ &= -\frac{3\sqrt{3}}{\pi} sin(\rho) \end{aligned}$$

qui n'est autre que [2.46b].

En combinant [2.46a] et [2.46b], on obtient, toujours au premier ordre en μ :

$$< X >^2 + < Y >^2 = \frac{27}{\pi^2} \qquad [2.148]$$

A la limite de μ petit, [2.77] devient :

$$\begin{aligned} < Z > &\approx 2 - \frac{3}{4\pi[(\mu/2)^2 + ..\mu^4]} \begin{Bmatrix} tg^2(\nu)[3\mu - \frac{1}{2}\mu^3 - 2\mu + 2\mu(\frac{\mu}{2})^2 - \mu] \\ + \mu - \frac{1}{6}\mu^3 + 2(\frac{\mu}{2})^2\mu - \mu + ...\mu^4 \end{Bmatrix} \\ &= 2 - \frac{3}{4\pi[(\mu/2)^2 + ..\mu^4]} \left\{ -\frac{1}{6}\mu^3 + \frac{\mu^3}{2} + ...\mu^4 \right\} \\ &= 2 - \frac{3}{4\pi[(\mu/2)^2 + ..\mu^4]} \left\{ \frac{\mu^3}{3} + ...\mu^4 \right\} \end{aligned} \qquad [2.149]$$

soit, au premier ordre en μ :

$$< Z > \approx 2 - \frac{\mu}{\pi} \qquad [2.150]$$

qui n'est autre que [2.85].