Representing and aggregating preferences using a stochastic interpretation

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Outline

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- Markov chains
- Preference relations

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Markov chains

Definitions

$\mathbf{E}$ = set of n states = $\{E_i\}$

$\mathbf{X}$ = set of random variables = $\{X_t\}$ where t is discrete time index and $X_t \in \mathbf{E}$

$\mathbf{X}$ is a finite discrete-time *Markov chain* iff

- X_t depends only on X_{t-1} (*markovian*, memoryless process)
- This dependence is time-invariant (*homogeneous* process)

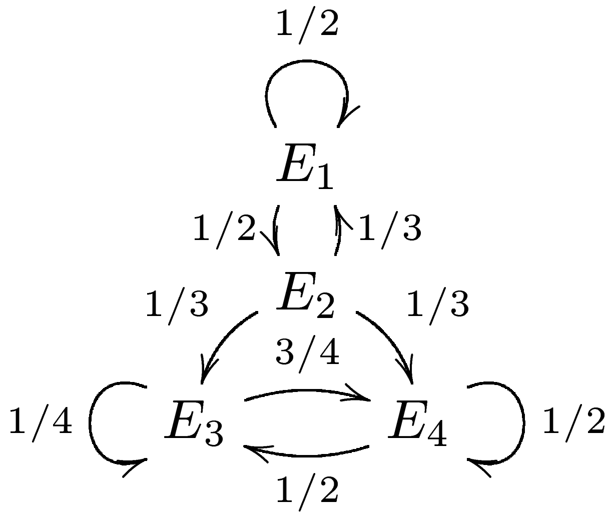
$\mathbf{X}$ completely defined by matrix $\mathbf{M} = \{p_{i,j}\}$ where

$$p_{i,j} = \mathbb{P}(X_t = E_j \mid X_{t-1} = E_i)$$

Representation as a *valued directed graph* (nodes $\equiv$ states, arcs $\equiv$ transitions)

Example and properties

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$\mathbf{M}$ is a stochastic matrix (row sum equal to 1)

Let a_0 be the initial probability distribution (corresponding to X_0)

1. It is possible to compute a_i (distribution of X_i) recursively (from a_{i-1}) $\forall i$
2. Under a *regularity* assumption, the sequence $\{a_i\}$ tends to a limit a independent of initial distribution a_0
3. This limit a is called *stationary distribution*, and is efficiently computable (left eigenvector or linear system)

In the example, $a = \begin{pmatrix} 0 & 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}$

Preference relations

Definitions

$\mathbf{A}$ = set of n alternatives = $\{A_i\}$

$\mathcal{R}$ = binary relation on $\mathbf{A}$ ($\subseteq \mathbf{A} \times \mathbf{A}$)

Let $a, b, c \in \mathbf{A}$. Relation $\mathcal{R}$ is

- *reflexive* if $a \mathrel{\mathcal{R}} a$
- *complete* if $a \mathrel{\mathcal{R}} b$ or $b \mathrel{\mathcal{R}} a$
- *symmetric* if $a \mathrel{\mathcal{R}} b \Rightarrow b \mathrel{\mathcal{R}} a$
- *asymmetric* if $a \mathrel{\mathcal{R}} b \Rightarrow b \neg\mathcal{R}\, a$
- *transitive* if $a \mathrel{\mathcal{R}} b, b \mathrel{\mathcal{R}} c \Rightarrow a \mathrel{\mathcal{R}} c$

Relation $\mathcal{R}$ is understood as $\leq$, which means $a \mathrel{\mathcal{R}} b \Leftrightarrow a$ is not strictly preferred to b

$\mathcal{I} \equiv$ indifference part of $\mathcal{R}$, defined by $a \mathrel{\mathcal{I}} b \Leftrightarrow a \mathrel{\mathcal{R}} b$ and $b \mathrel{\mathcal{R}} a$

$\mathcal{S} \equiv$ strict preference part of $\mathcal{R}$, defined by $a \mathrel{\mathcal{S}} b \Leftrightarrow a \mathrel{\mathcal{R}} b$ and $b \neg\mathcal{R}\, a$

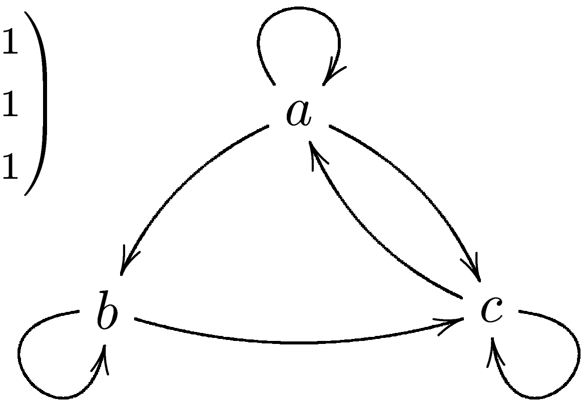
$\mathcal{I}$ is symmetric and reflexive, $\mathcal{S}$ is asymmetric, $\mathcal{R} = \mathcal{I} \cup \mathcal{S}$

Representations and examples

- Matrix representation $\{r_{i,j}\}$ with $r_{i,j}$ equal to 1 if $A_i \ \mathcal{R} \ A_j$, 0 otherwise
- Representation as a *directed graph* (nodes $\equiv$ alternatives, arcs $\equiv$ relation $\mathcal{R}$)

For example, $\mathbf{A} = \{a, b, c\}$ and $\mathcal{R} = \{(a, c), (c, a), (a, b), (b, c), (a, a), (b, b), (c, c)\}$ is reflexive and complete

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Some preference relations

Let $\mathcal{R}$ be a complete and reflexive relation

- If $\mathcal{S}$ and $\mathcal{I}$ are both transitive, $\mathcal{R}$ is a *complete preorder* (weak order). It is possible to assign a number v_i to each alternative such that $A_i \; \mathcal{R} \; A_j \Leftrightarrow v_i \leq v_j$
- If $\mathcal{S}$ is transitive et $\mathcal{I} = \{(a,a)\} \; \forall a \in \mathcal{R}$, $\mathcal{R}$ is a *total order*. It is possible to assign a *distinct* number v_i to each alternative such that $A_i \; \mathcal{R} \; A_j \Leftrightarrow v_i \leq v_j$
- If it is possible to assign a real interval $U_i = [v_i \; v_i']$ to each alternative such that $A_i \; \mathcal{I} \; A_j \Leftrightarrow U_i \cap U_j \neq \emptyset$ and $A_i \; \mathcal{S} \; A_j \Leftrightarrow v_i' < v_j$, $\mathcal{R}$ is a (total) *interval order*. $\mathcal{S}$ is transitive but $\mathcal{I}$ isn't.
- If it is possible to assign a real interval $U_i = [v_i v_i']$ to each alternative such that $A_i \; \mathcal{I} \; A_j \Leftrightarrow U_i \cap U_j \neq \emptyset$, $A_i \; \mathcal{S} \; A_j \Leftrightarrow v_i' < v_j$ and no interval is strictly included in another one, $\mathcal{R}$ is a (total) *semiorder*. $\mathcal{S}$ is transitive but $\mathcal{I}$ isn't.

Some preference relations

- Total order $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$\text{———}\ a\ \text{———}\ b\ \text{———}\ c\ \text{———}\!\!\longrightarrow$$

- Total preorder $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\text{———}\ a\ \text{———}\ b, c\ \text{————————}\!\!\longrightarrow$$

- Interval order $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$$- \{ - \overset{a}{[-]} - \overset{b}{[-]} - \overset{c}{[-]} - \underset{d}{\}} \longrightarrow$$

- Semiorder $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

$$- \overset{a}{[-]} - \overset{b}{[} - \{ -] - [- \underset{d}{\}} \overset{c}{-}] \longrightarrow$$

Theorems

Let $\mathcal{R}$ a preference relation without *equivalent alternatives* (a and b are equivalent iff $a \mathcal{R} x \Leftrightarrow b \mathcal{R} x$ and $x \mathcal{R} a \Leftrightarrow x \mathcal{R} b \; \forall x \in \mathbf{A}$)

- If $\mathcal{R}$ is a semiorder, there is a *unique total order* underlying $\mathcal{R}$. It is induced by the numbers $\{v_i\}$ (or $\{v'_i\}$). Moreover, if rows and columns of the matrix representation of $\mathcal{R}$ are sorted in that order, we get a step-type matrix.
- If $\mathcal{R}$ is an interval order, there are *two* total orders underlying $\mathcal{R}$. Moreover, if rows (and columns) of the matrix representation of $\mathcal{R}$ are sorted in the first (and second) order, we also get a step-type matrix.
- $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ is a *step-type matrix*

Stochastic method

Single preference relation

Let $\mathcal{R}$ be a complete and reflexive binary preference relation. We proceed as follows :

1. *Build a Markov chain* : Associate to each alternative A_i a state E_i
2. *Choose the transition probabilities*, requiring the following conditions to hold
   - $p_{i,j} > 0$ when $A_i \; \mathcal{R} \; A_j$
   - $p_{i,j} = 0$ when $A_i \; \neg\mathcal{R} \; A_j$
   - Matrix $\{p_{i,j}\}$ is stochastic
3. *Compute the stationary distribution* and rank the alternatives according to the resulting probabilities

Justification

Intuitively, the process always moves from an alternative to a *better or equivalent* alternative. It is thus sensible to expect a high probability for the best alternatives in the stationary distribution.

Choice of $\{p_{i,j}\}$

We want the value of $p_{i,j}$ to be independent from other alternatives than A_i and A_j

If $A_i \mathcal{R} A_j$, let $p_{i,j} = \frac{1}{n}$, otherwise let $p_{i,j} = 0$

But $\mathbf{M}$ is not stochastic $\Rightarrow$ set $p_{i,i}$ according to

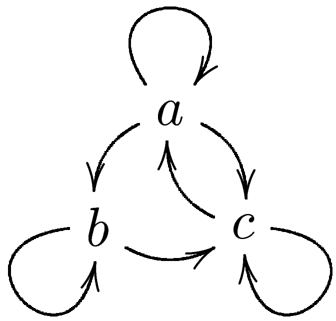
$$p_{i,i} = 1 - \sum_{j \neq i} p_{i,j}$$

With this choice, the Markov process can be described as :

1. I am currently in state E_i
2. Choose randomly a state E_j (uniformly)
3. If $A_i \mathcal{R} A_j$, move to state E_j, otherwise stay in E_i

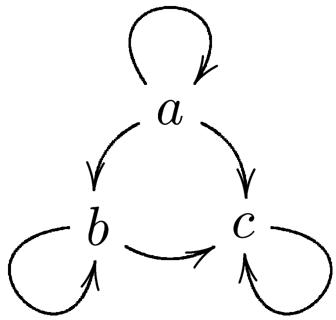
Examples

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ gives } M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$



Stationary distribution is $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$: c is the best, a and b follow.

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ gives } M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$



Stationary distribution is $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$: c is the best *but* we have no information about the other alternatives (no way to escape from c)

Better choice of $\{p_{i,j}\}$

Let's add a *neutral state* E_0, representing no alternative, such that $p_{0,0} = 0$ and $p_{0,i} = \frac{1}{n}$. Transitions from this state do not favor any alternative. We keep the rule

$$\boxed{\text{If } A_i \mathcal{R} A_j \text{, let } p_{i,j} = \frac{1}{n} \text{, otherwise let } p_{i,j} = 0}$$

but choose the $p_{i,0}$ to make the matrix stochastic, i.e.

$$p_{i,0} = 1 - \sum_{j \neq 0} p_{i,j}$$

With this choice, the Markov process can be described as

1. I am currently in state E_i
2. Choose randomly a state E_j (uniformly)
3. If $A_i \mathcal{R} A_j$, move to state E_j, otherwise move to neutral state E_0

Examples

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ gives } M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Stationary distribution is $\begin{pmatrix} \frac{7}{27} & \frac{6}{27} & \frac{9}{27} & \frac{5}{27} \end{pmatrix}$: c is the best, b is the worst

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ gives } M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Stationary distribution is $\begin{pmatrix} \frac{4}{27} & \frac{6}{27} & \frac{9}{27} & \frac{8}{27} \end{pmatrix}$: c is the best, a is the worst

Property

If $\mathcal{R}$ is a total preorder, the probabilities from the stationary distribution are ranked according to $\mathcal{R}$

Going multicriteria

$\mathbf{R}$ = set of preference relations $\{\mathcal{R}_i\}$ where $\mathcal{R}_i$ is complete and reflexive. These relations are weighted by $\{w_i\}$ whose sum is equal to 1.

$\mathbf{M}_i$ = stochastic matrix for each $\mathcal{R}_i$ (computed as above). Let

$$\mathbf{M} = \sum_i w_i \mathbf{M}_i$$

$\mathbf{M}$ is also stochastic and describes a Markov chain. We rank the alternatives according to the probabilities from the resulting stationary distribution.

1. I am currently in state E_i
2. Choose randomly a relation $\mathcal{R}_p$ (according to the $\{w_i\}$)
3. Choose randomly a state E_j (uniformly)
4. If $A_i \ \mathcal{R}_p \ A_j$, move to state E_j, otherwise move to neutral state E_0

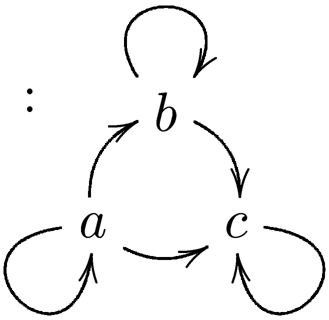
Examples

$r = 2, w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{3}$

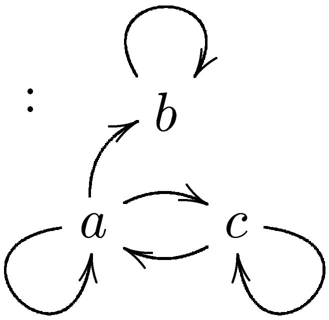
$$M_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$\mathcal{R}_1$:



$\mathcal{R}_2$:



Stationary distribution is $\begin{pmatrix} \frac{16}{79} & \frac{24}{79} & \frac{21}{79} & \frac{18}{79} \end{pmatrix}$: b is the best, a is the worst

Expressing incomparability

Method

In light of *Arrow's theorem*, we shouldn't expect good properties from a preference analysis method producing a total preorder.

$\Rightarrow$ Weaken the relation produced by our method

No distinction between $A_i \; \mathcal{S} \; A_j$ et $A_i \; \mathcal{I} \; A_j$ (indifference and strict preference)

When A_i and A_j are indifferent, choose something less radical than $p_{i,j} = \frac{1}{n}$:

- If $A_i \; \mathcal{S} \; A_j$, let $p_{i,j} = \frac{1}{n}$
- If $A_i \; \mathcal{I} \; A_j$, let $p_{i,j} = \frac{\alpha}{n}$
- Otherwise let $p_{i,j} = 0$

where α is a *parameter* to choose between 0 and 1

Now, the Markov process can be described as

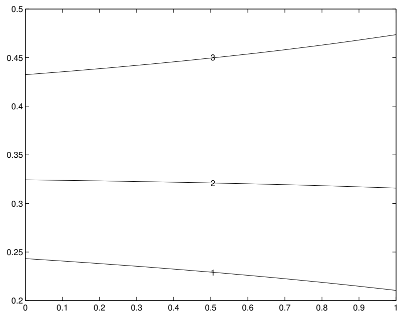
1. I am currently in state E_i
2. Choose randomly a relation $\mathcal{R}_p$ (according to the $\{w_i\}$)
3. Choose randomly a state E_j
   - If $A_i \ \mathcal{S}_p \ A_j$, move to state E_j
   - If $A_i \ \mathcal{I}_p \ A_j$, move to state E_j with probability α and to neutral state E_0 with probability $1 - \alpha$
   - Otherwise, move to state E_0

Use the family of total preorders parameterized by α to deduce a *partial preorder* :

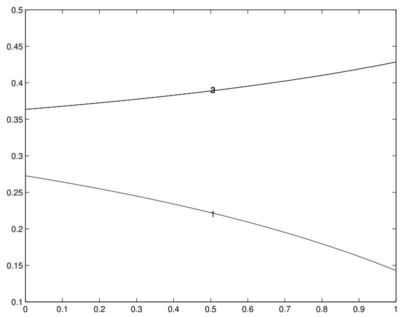
- If A_i is better than A_j for all values of α, declare $A_i > A_j$
- If A_i is worse than A_j for all values of α, declare $A_i < A_j$
- Otherwise, A_i and A_j are not comparable

Examples

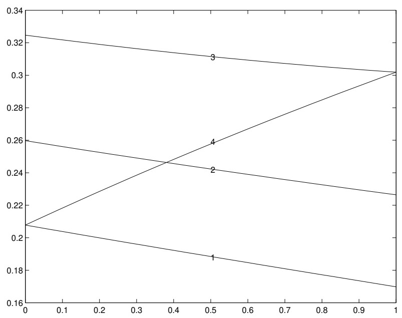
Total order



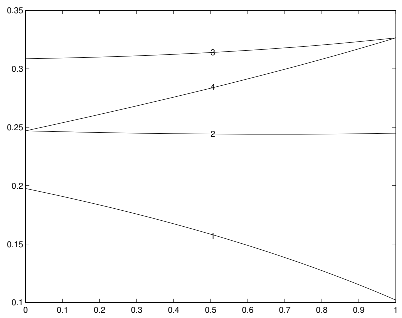
Total preorder



Interval order



Semiorder



Properties

If $\mathcal{R}$ is a total preorder, the rankings obtained match $\mathcal{R}$ $\forall \alpha \in [0\ 1]$

If $\mathcal{R}$ is a semiorder, the rankings obtained match the unique total order underlying $\mathcal{R}$ $\forall \alpha \in]0\ 1[$

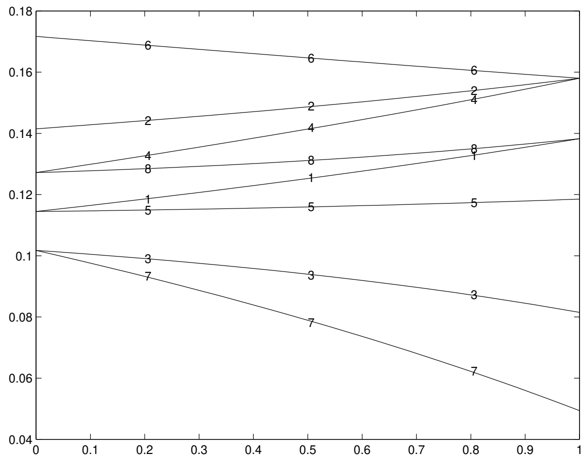
If $\mathcal{R}$ is an interval order, the rankings obtained letting α tend to 0 and 1 match the two total orders underlying $\mathcal{R}$ $\Rightarrow$ there is at least one case of incomparability

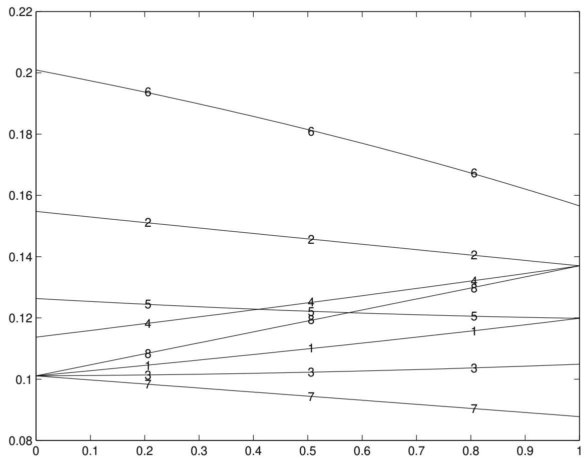
Usage

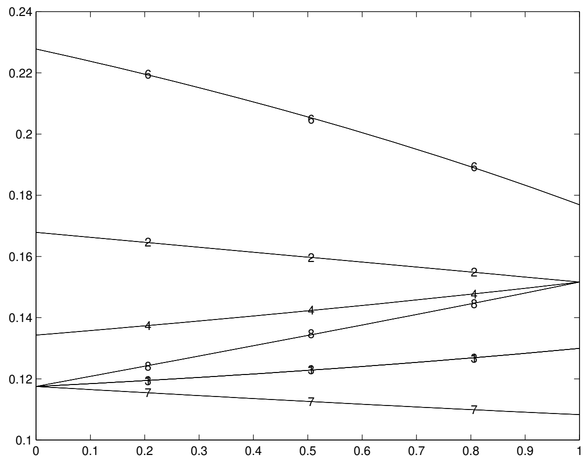
To select the best alternative, choose among those being ranked first for at least one value of α (these ones being incomparable)

The more a *crossing* is close to 0 or 1, the less the associated incomparability is strong

Illustrations - Semiorders

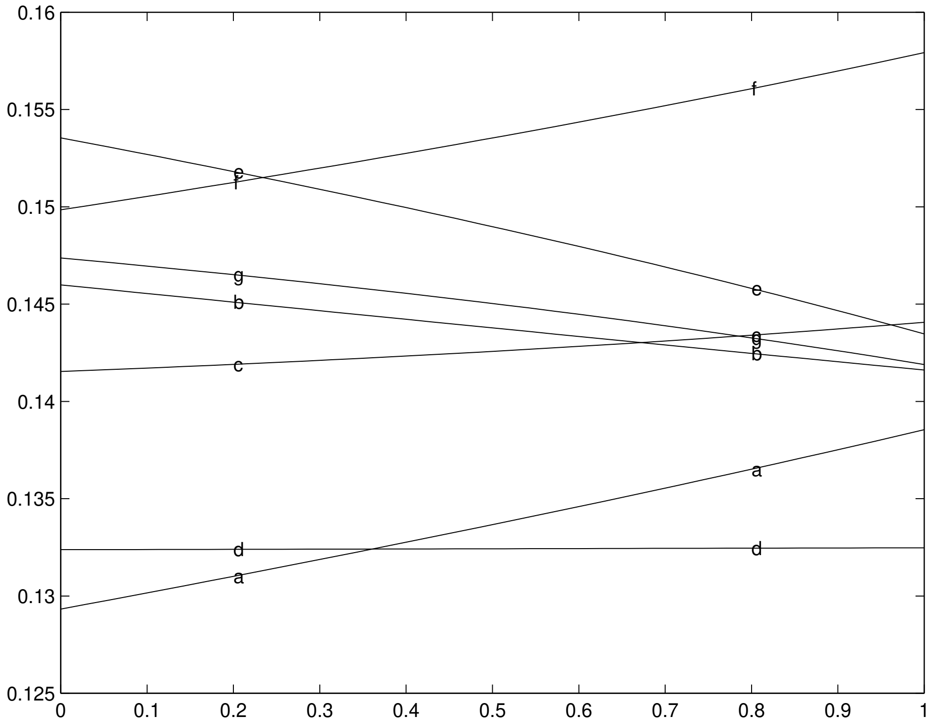






Factory location

- 5 numerical criteria : Price, Transport, Environment, Residential, Competition
- Use of thresholds to compute a semiorder for each criterion



Conclusions

Stochastic method

- Intuitive principle
- Graphical results
- No parameter value to choose
- Easy to implement

Multicriteria analysis

- Natural generalization
- Sensitivity analysis easy to perform
- Cardinal information allowed

Further research

- Theoretical properties (independence)
- Valued relations