Analyzing conic problems involving a single second-order cone constraint

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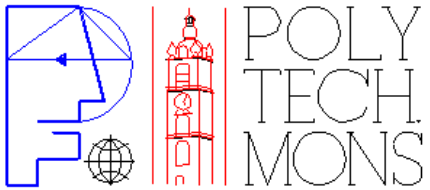

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Outline

Introduction

- ◇ Convex, conic and second-order cone optimization
- ◇ Two easy problems

Second-order cone feasibility problem

- ◇ Main idea and preliminary discussion
- ◇ Our algorithm: a three-case discussion

Second-order cone optimization problem

- ◇ Using the feasibility problem as a subproblem

Concluding remarks

- ◇ Summary, complexity and generalizations

Introduction

Convex optimization

Let $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ a convex function and $C \subseteq \mathbb{R}^n$ a convex set

$$\inf_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad x \in C$$

Properties

- ◇ Local optima ⇒ global, form a convex optimal set
- ◇ Lagrange duality ⇒ related (asymmetric) dual problem
- ◇ Efficient *interior-point* methods (*self-concordant barriers*)

Conic optimization

Let $\mathcal{C} \subseteq \mathbb{R}^n$ a *solid, pointed, closed convex cone* :

$$\inf_{x \in \mathbb{R}^n} c^{\mathrm{T}} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C} \quad \Rightarrow \textbf{Equivalent} \text{ setting}$$

Primal-dual pair

Dual cone is also a solid pointed closed convex cone

$$\mathcal{C}^* = \left\{ x^* \in \mathbb{R}^n \mid x^{\mathrm{T}} x^* \geq 0 \text{ for all } x \in \mathcal{C} \right\}$$

$\Rightarrow$ pair of primal-dual problems

$$\inf_{x \in \mathbb{R}^n} \; c^{\mathrm{T}} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C}$$

$$\sup_{(y,s) \in \mathbb{R}^{m+n}} \; b^{\mathrm{T}} y \quad \text{s.t.} \quad A^{\mathrm{T}} y + s = c \text{ and } s \in \mathcal{C}^*$$

Several cones: $x^1 \in \mathcal{C}^1, \ldots, x^r \in \mathcal{C}^r \Leftrightarrow (x^1, \ldots, x^r) \in \mathcal{C}^1 \times \cdots \times \mathcal{C}^r$

Advantages over classical formulation

- ◇ Remarkable primal-dual symmetry
- ◇ Special handling of (*easy*) linear equality constraints

Examples

$\mathcal{C} = \mathbb{R}^n_+ = \mathcal{C}^* \Rightarrow$ linear optimization

$\mathcal{C} = \mathbb{S}^n_+ = \mathcal{C}^* \Rightarrow$ semidefinite optimization

Both cones are *self-dual*.

A single $\mathbb{R}^n_+/\mathbb{S}^n_+$ cone can be considered w.l.o.g.

Second-order cone optimization

$$\mathbb{L}^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \|(x_1, \ldots, x_n)\| \leq x_0\} \subset \mathbb{R}^{n+1}$$

Second-order or Lorentz cone $\mathbb{L}^n$ is self-dual

But a list of several constraints $x^i \in \mathbb{L}^{n_i}, i = 1, \ldots, r$ cannot be concatenated into a single second-order cone

Goal of this talk: Study the problem with a single second-order cone

Single-constraint second-order cone problem

$$\inf_{x_0 \in \mathbb{R},\ x \in \mathbb{R}^n} c_0 x_0 + c^{\mathrm{T}} x \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $c_0 \in \mathbb{R}$, $c \in \mathbb{R}^n$; $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

Known results

It is *well-known* that this problem can be solved *analytically*

[Alizadeh and Goldfarb 2001]: solve primal-dual optimality conditions

$$\begin{aligned} a_0 x_0 + A_0 x = b \quad &\text{and} \quad (x_0, x) \in \mathbb{L}^n \\ a_0^{\mathrm{T}} y + z_0 = c_0,\ A^{\mathrm{T}} y + z = c \quad &\text{and} \quad (z_0, z) \in \mathbb{L}^n \\ (x_0, x) \circ (z_o, z) = 0 \quad &(\ \Leftrightarrow\ c_0 x_0 + c^{\mathrm{T}} x = b^{\mathrm{T}} y) \end{aligned}$$

where $(x_0, x) \circ (z_o, z) = (x_0 y_0 + x^{\mathrm{T}} y, x_0 y + y_0 x)$

But only works when *strong duality* holds !

Analytical solution of optimality conditions

Define $\bar{A} = (a_0 \ A)$, $\bar{c} = (c_0, c)$, $P = P_{\text{null } \bar{A}} = I - \bar{A}^\dagger \bar{A}$.

Assuming $x_0 > 0$ and $z_0 > 0$, one obtains after some linear algebra

$$
\begin{aligned}
\gamma &= 1 - 2a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}a_0 \\
\alpha &= \frac{z_0}{x_0} = \sqrt{\frac{-\gamma \bar{c}^{\mathrm{T}} P \bar{c} + 2(e^{\mathrm{T}} P \bar{c})^2}{\gamma b^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b + 2(a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b)^2}} \\
\delta &= \frac{\bar{c}^{\mathrm{T}} P \bar{c} + \alpha^2 b^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b}{e^{\mathrm{T}} P \bar{c} + \alpha a_0^{\mathrm{T}}(\bar{A}\bar{A}^{\mathrm{T}})^{-1}b} \\
y &= (\bar{A}\bar{A}^{\mathrm{T}})^{-1}(\bar{A}\bar{c} + \alpha b - \delta a_0) \\
(z_0, z) &= \bar{c} - \bar{A}^{\mathrm{T}} y = P\bar{c} - \bar{A}^\dagger(\alpha b - \delta a_0) \\
(x_0, x) &= (z_0/\alpha, -z/\alpha)
\end{aligned}
$$

Technical derivations, little intuition on why this works, special cases?

Classification of conic convex problems

Feasibility status of conic problem: define $\mathcal{L} = \{x \in \mathbb{R}^n \mid Ax = b\}$

$$\inf_{x \in \mathbb{R}^n} c^{\mathrm{T}} x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C} \quad \Leftrightarrow \quad x \in \mathcal{L} \cap \mathcal{C}$$

- ◇ Feasible iff $\mathcal{L} \cap \mathcal{C} \neq \emptyset$. Moreover, in this case,
  - *Strictly* feasible (s.f.) if $\mathcal{L} \cap \operatorname{int} \mathcal{C} \neq \emptyset$
  - *Weakly* feasible (w.f.) otherwise
- ◇ Infeasible iff $\mathcal{L} \cap \mathcal{C} = \emptyset$. Moreover, in this case,
  - *Strictly* infeasible (s.i.) if $\operatorname{dist}(\mathcal{L}, \mathcal{C}) > 0$
  - *Weakly* infeasible (w.i.) otherwise, i.e. $\operatorname{dist}(\mathcal{L}, \mathcal{C}) = 0$
    $\Leftrightarrow \exists \{x^k\} \mid x^k \in \mathcal{L}$ and $\lim_{k \to \infty} \operatorname{dist}(x^k, \mathcal{C}) = 0$
    (which implies then that $\{x^k\}$ is unbounded)

All these cases already arise when $\mathcal{C} = \mathbb{L}^2$!

Two easy problems

Smallest-norm point on a line

Let $\alpha, \beta \in \mathbb{R}^n$ and $\beta \neq 0$

$$\min_{t \in \mathbb{R}} \phi(t) = \|\alpha - t\beta\|$$

$\phi(t)^2 = t^2 \|\beta\|^2 - 2t\alpha^{\mathrm{T}}\beta + \|\alpha\|^2$ is easily minimized $\rightarrow t_* = \frac{\alpha^{\mathrm{T}}\beta}{\|\beta\|^2}$

Minimum distance is $\phi(t_*)^2 = \|\alpha\|^2 - t_*^2 \|\beta\|^2 = \|\alpha\|^2 - \frac{(\alpha^{\mathrm{T}}\beta)^2}{\|\beta\|^2}$

Intersecting a ball with an affine subspace

Decide the feasibility of the following convex problem

$$\text{Find } x \in \mathbb{R}^n \quad \text{s.t.} \quad Ax = b \text{ and } \|x\| \leq 1$$

with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Assume that A has full row rank ($\Rightarrow AA^{\mathrm{T}} \succ 0$ and $Ax = b$ is feasible)

Idea: compute minimum-norm solution $\hat{x}$ of $Ax = b$

We have thus $A\hat{x} = b$ and $\hat{x} \perp \operatorname{null} A \Leftrightarrow \hat{x} \in \operatorname{range} A^{\mathrm{T}} \Leftrightarrow \hat{x} = A^{\mathrm{T}}\hat{y}$

which implies $AA^{\mathrm{T}}\hat{y} = b$, $\hat{y} = (AA^{\mathrm{T}})^{-1}b$ and finally $\hat{x} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$

($A^{\dagger} = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ is the Moore-Penrose generalized inverse of A)

Discuss according to $\|\hat{x}\|^2 = \left\|A^{\dagger}b\right\|^2 = b^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$

- $\|\hat{x}\|^2 > 1 \Rightarrow$ problem is strictly infeasible
- $\|\hat{x}\|^2 = 1 \Rightarrow$ problem is weakly feasible, $\hat{x}$ is unique solution
- $\|\hat{x}\|^2 < 1 \Rightarrow$ problem is strictly feasible, $\hat{x}$ is among the solutions

(obviously problem cannot be weakly infeasible)

Forming $AA^{\mathrm{T}} \in \mathbb{R}^{m\times m} \rightarrow \mathcal{O}\left(m^2 n\right)$ operations
Factorizing $AA^{\mathrm{T}} = LL^{\mathrm{T}}$, $L \in \mathbb{R}^{m\times m}$ triangular $\rightarrow \mathcal{O}\left(m^3\right)$ operations
$\Rightarrow$ Computing $x^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y$ for any $x, y \in \mathbb{R}^m$ is $\mathcal{O}\left(m^2 n\right)$ operations.

Second-order cone feasibility problem

Our problem

Decide the feasibility of the following convex problem

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume w.l.o.g. that $a_0 x_0 + Ax = b$ is feasible and $(a_0 \ A)$ full row rank $(a_0 a_0^{\mathrm{T}} + AA^{\mathrm{T}} \succ 0)$

Main idea

Use the fact that $\mathbb{L}^n$ and $\mathbb{B}^n$ are strongly related:

$$(x_0, x) \in \mathbb{L}^n \quad \Leftrightarrow \quad (x/x_0) \in \mathbb{B}^n \text{ and } x_0 > 0 \quad \text{or} \quad (x_0, x) = (0, 0)$$

$\rightarrow$ use *homogenization* to get rid of x_0 variable

Our procedure: outline

Let $t > 0$ be a homogenizing variable. Problem

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

becomes *equivalent* to problem

$$\text{Find } (x_0, x, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad a_0 x_0 + Ax = b\,t \text{ and } (x_0, x) \in \mathbb{L}^n$$

Each point (x_0, x) becomes a ray $(tx_0, tx, t),\ t > 0$, **except** for $(0, 0)$

Problem is completely *homogeneous* $\rightarrow$ arbitrarily fix $x_0 = 1$

$\Rightarrow$ constraint $(x_0, x) \in \mathbb{L}^n$ becomes equivalent to $x \in \mathbb{B}^n \quad \rightarrow$

$$\text{Find } (x, t) \in \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad Ax = b\,t - a_0 \text{ and } x \in \mathbb{B}^n$$

which is the second easy problem with a parameter $t \in \mathbb{R}_{++}$

Soln (x, t) to this problem $\rightarrow$ soln $(1/t, x/t) \in \mathbb{L}^n$ to original problem

Preliminary discussion

Do the linear constraints $a_0 x_0 + Ax = b$ imply that x_0 is constant?

This would mean $(x_0, x) \in \operatorname{null} A \Rightarrow x_0 = 0$, which is equivalent to $(1, 0) \in \operatorname{range} A^{\mathrm{T}}$ or to the existence of $y \mid A^{\mathrm{T}} y = 0$ and $a_0^{\mathrm{T}} y = 1$

$\rightarrow A$ has not full row rank ($\Leftrightarrow \operatorname{rank} A = m - 1$ and $a_0 \notin \operatorname{range} A$)

We have then $x_0 = y^{\mathrm{T}} b = C$ for all feasible solutions

- $C < 0 \rightarrow$ problem strictly infeasible
- $C = 0 \rightarrow (0, 0)$ unique potential solution
  - $b = 0 \rightarrow$ problem weakly feasible
  - $b \neq 0 \rightarrow$ problem strictly infeasible
- $C > 0$ problem becomes $A(x/x_0) = b/x_0 - a_0$ with $(x/x_0) \in \mathbb{B}^n$ which is *easy* (look at $\left\| A^{\dagger}(b/x_0 - a_0) \right\| \rightarrow$ s.i., w.f. or s.f.)

Main case

We can thus assume A has full row rank. Problem is equivalent to

$$\text{Find } (x,t) \in \mathbb{R}^n \times \mathbb{R}_{++} \quad \text{s.t.} \quad Ax = bt - a_0 \text{ and } x \in \mathbb{B}^n$$

This suggests to look at $\left\|A^\dagger(bt - a_0)\right\|^2 = \|\alpha - t\beta\|^2$ with

$$\alpha = A^\dagger a_0 = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}a_0 \text{ and } \beta = A^\dagger b = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$$

$(\|\alpha\|^2 = a_0^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}a_0,\ \|\beta\|^2 = b^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b,\ \alpha^{\mathrm{T}}\beta = a_0^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b)$

However the possibility $x_0 = 0$ was left out!

In this case, we have $(x_0, x) = (0,0)$ which implies $b = 0$ and $\beta = 0$

We therefore have to distinguish two cases: $b = 0$ and $b \neq 0$

and add soln $(0,0)$ to homogenized solutions $(1/t, x/t)$ when $b = 0$

Find $(x,t) \in \mathbb{R}^n \times \mathbb{R}_{++}$ s.t. $Ax = bt - a_0$ and $x \in \mathbb{B}^n$

Case A: $b = 0$

In this case, $\|\alpha - t\beta\|^2 = \|\alpha\|$ does not depend from t

- $\diamond$ $\|\alpha\| > 1 \rightarrow$ no solution except $(0,0) \rightarrow$ problem w.f.
- $\diamond$ $\|\alpha\| = 1 \rightarrow$ ray $(t, t\alpha), t \in \mathbb{R}_+$ is solution $\rightarrow$ problem w.f.
- $\diamond$ $\|\alpha\| < 1 \rightarrow$ interior solutions $\rightarrow$ problem s.f.

Case B: $b \neq 0$

We can here safely ignore solutions with $x_0 = 0$

In theory, minimum value of $\|\alpha - t\beta\|^2$ is attained for $t_* = \frac{\alpha^{\mathrm{T}}\beta}{\|\beta\|^2}$

But t is required to be positive

$\rightarrow$ distinguish whether t_* is positive or not $\Rightarrow$ discuss the sign of $\alpha^{\mathrm{T}}\beta$

Case B_1: $\alpha^T\beta \leq 0$

Minimum t_* cannot be reached

$\rightarrow$ actual minimum attained for $t \rightarrow 0^+$

This means we have to look at $\|\alpha\|^2$

- $\|\alpha\| > 1 \rightarrow$ no solution $\rightarrow$ problem s.i.
- $\|\alpha\| < 1 \rightarrow \exists t > 0$ such that $\|\alpha - t\beta\|^2 < 1 \rightarrow$ problem s.f.
- $\|\alpha\| = 1 \rightarrow$ no solution since $t = 0$ is forbidden

  But $\text{dist}(\beta t - \alpha, \mathbb{B}^n) \rightarrow 0$. Let $x^t = (1/t, \beta - \alpha/t)$.

  What about $\text{dist}(x^t, \mathbb{L}^n)$ as $t \rightarrow 0$?

  One has $\text{dist}(x^t, \mathbb{L}^n) \approx (1/t)\,\text{dist}(\beta t - \alpha, \mathbb{B}^n)$ and thus

  - $\alpha^T\beta = 0 \rightarrow \text{dist}(x^t, \mathbb{L}^n) \rightarrow 0 \rightarrow$ w.i.
  - $\alpha^T\beta < 0 \rightarrow \text{dist}(x^t, \mathbb{L}^n) > \epsilon \rightarrow$ s.i.

Case B_2: $\alpha^T\beta > 0$

Minimum t_* is achieved. The smallest-norm solution is then

$$t_*\beta - \alpha \text{ with } \|t_*\beta - \alpha\|^2 = \|\alpha\|^2 - t_*^2\|\beta\|^2 = \|\alpha\|^2 - \frac{(\alpha^T\beta)^2}{\|\beta\|^2} = \delta$$

- $\diamond$ $\delta > 1 \rightarrow$ no solution $\rightarrow$ problem s.i.
- $\diamond$ $\delta < 1 \rightarrow$ there are interior solutions $\rightarrow$ problem s.f.
- $\diamond$ $\delta = 1 \rightarrow$ only one solution $(1/t_*, \alpha/t_* - \beta) \rightarrow$ problem w.f.

  Note this solution is easy to compute: (use $\mathcal{O}\left(m^2 n\right)$ operations)

$$(x_0, x) = \left(1/t_*, A^T(AA^T)^{-1}(b - a_0/t_*)\right) \text{ with } t_* = \frac{a_0^T(AA^T)^{-1}b}{b^T(AA^T)^{-1}b}$$

Geometrical interpretation:

Study intersection of half-line $\{\alpha - \mathbb{R}_{++}\beta\} \subset \mathbb{R}^n$ with ball $\mathbb{B}^n$

(values of β leading to intersection form a cone)

Summarizing table

- A not full row rank $\rightarrow x_0 = C$
  - $C < 0 \rightarrow$ s.i.
  - $C = 0$ and $b = 0 \rightarrow$ w.f. ; $C = 0$ and $b \neq 0 \rightarrow$ s.i.
  - $C > 0 \rightarrow$ s.f., w.f. or s.i. when $\left\| A^{\dagger}(b/x_0 - a_0) \right\| <, =, > 1$
- A has full row rank, define α, β and $\delta = \|\alpha\|^2 - \frac{(\alpha^{\mathrm{T}}\beta)^2}{\|\beta\|^2}$

	$\|\alpha\| < 1$	$\|\alpha\| = 1$	$\|\alpha\| > 1$
$\alpha^{\mathrm{T}}\beta < 0$	s.f.	s.i.	s.i.
$b \neq 0$, $\alpha^{\mathrm{T}}\beta = 0$	s.f.	w.i.	s.i.
$b = 0$, $\alpha^{\mathrm{T}}\beta = 0$	s.f.	w.f.	w.f.
$\alpha^{\mathrm{T}}\beta > 0$	s.f.	s.f.	s.f., w.f., s.i. $(\delta <, =, > 1)$

Notes: "Loose" dependence from a_0 and b ; $\|\beta\|$ plays a role in

only a single case

Second-order cone optimization problem

Problem definition

$$\inf_{x_0 \in \mathbb{R},\ x \in \mathbb{R}^n} c_0 x_0 + c^{\mathrm{T}} x \quad \text{s.t.} \quad a_0 x_0 + Ax = b \text{ and } (x_0, x) \in \mathbb{L}^n$$

with $c_0 \in \mathbb{R}$, $c \in \mathbb{R}^n$; $a_0 \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume w.l.o.g. that $a_0 x_0 + Ax = b$ is feasible and $(a_0\ A)$ full row rank

Using feasibility problem as subproblem

Idea: add $c_0 x_0 + c^{\mathrm{T}} x = \gamma$ as a constraint with $\gamma \in \mathbb{R}$ as a parameter

$\rightarrow$ test whether γ is a *feasible* objective value

$$\text{Find } (x_0, x) \in \mathbb{R} \times \mathbb{R}^n \text{ s.t. } \begin{pmatrix} c_0 \\ a_0 \end{pmatrix} x_0 + \begin{pmatrix} c^{\mathrm{T}} \\ A \end{pmatrix} x = \begin{pmatrix} \gamma \\ b \end{pmatrix} \text{ and } (x_0, x) \in \mathbb{L}^n$$

which is a feasibility second-order cone problem with new data

Data of the subproblem

We have

$$\tilde{a}_0 = \begin{pmatrix} c_0 \\ a_0 \end{pmatrix}, \; \tilde{A} = \begin{pmatrix} c^{\mathrm{T}} \\ A \end{pmatrix}, \tilde{b} = \begin{pmatrix} \gamma \\ b \end{pmatrix}$$

What are the new quantities α and β?

We need to evaluate $(\tilde{A}\tilde{A}^{\mathrm{T}})^{-1} = \begin{pmatrix} c^{\mathrm{T}}c & c^{\mathrm{T}}A^{\mathrm{T}} \\ Ac & AA^{\mathrm{T}} \end{pmatrix}^{-1}$

Possible to compute this as a function of $(AA^{\mathrm{T}})^{-1}$ and c but tedious

Better approach

We can actually suppose *without loss of generality* that $c \in \operatorname{null} A$

Indeed, decomposing $c = c_1 + c_2$ with $c_1 \in \operatorname{null} A$ and $c_2 \in \operatorname{range} A^{\mathrm{T}}$ implies $c_2 = A^{\mathrm{T}} d$ and then

$$\begin{aligned} c_0 x_0 + c^{\mathrm{T}} x &= c_0 x_0 + c_1^{\mathrm{T}} x + c_2^{\mathrm{T}} x = c_0 x_0 + c_1^{\mathrm{T}} x + d^{\mathrm{T}} A x \\ &= c_0 x_0 + c_1^{\mathrm{T}} x + d^{\mathrm{T}} b - d^{\mathrm{T}} a_0 x_0 \\ &= (c_0 - d^{\mathrm{T}} a_0) x_0 + c_1^{\mathrm{T}} x + d^{\mathrm{T}} b \end{aligned}$$

$\rightarrow Ac = 0$ can be assumed if

- $\diamond$ c_0 is replaced by $(c_0 - d^{\mathrm{T}} a_0)$
- $\diamond$ the objective value is shifted by a constant equal to $d^{\mathrm{T}} b$

For brevity, we only consider here the case where $\tilde{A}$ has full row rank

$\Rightarrow$ we have $c \neq 0$ and one can further assume that $\|c\| = 1$ and $c_0 \geq 0$

We have now

$$(\tilde{A}\tilde{A}^{\mathrm{T}})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & AA^{\mathrm{T}} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (AA^{\mathrm{T}})^{-1} \end{pmatrix}$$

$$\begin{aligned} \|\tilde{\alpha}\|^2 &= \|\alpha\|^2 + c_0^2 \\ \|\tilde{\beta}\|^2 &= \|\beta\|^2 + \gamma^2 \\ \tilde{\alpha}^{\mathrm{T}}\tilde{\beta} &= \alpha^{\mathrm{T}}\beta + \gamma c_0 \end{aligned}$$

and we can easily apply results for the feasibility problem.

We note that $\tilde{\alpha}^{\mathrm{T}}\tilde{\beta}$ can take all real values unless $c_0 = 0$

Special case: $c_0 = 0$

In this case, one can simply read the values from the feasibility table

	$\|\alpha\| < 1$	$\|\alpha\| = 1$	$\|\alpha\| > 1$
$\alpha^T\beta < 0$	s.f.	s.i.	s.i.
$b \neq 0,\ \alpha^T\beta = 0$	s.f.	w.i.	s.i.
$b = 0,\ \alpha^T\beta = 0$	s.f.	w.f.	w.f.
$\alpha^T\beta > 0$	s.f.	s.f.	s.f., w.f., s.i. $(\delta <, =, > 1)$

Observations

s.i. $\rightarrow$ s.i, s.f. and w.f. $\rightarrow$ problem unbounded from above and below

w.i. $\rightarrow$ problem asymptotically unbounded from above and below

Exception: Case $(\alpha^T\beta > 0, \|\alpha\| > 1)$ where $\tilde{\delta}$ increases from δ to $\|\alpha\|^2$ as γ^2 increases from 0 to $+\infty$: s.i. implies s.i., w.f. implies $\gamma = 0$ while s.f. implies a bounded objective: $\gamma^2 \leq M(\alpha, \beta)$

General case: $c_0 \neq 0$

Sign of $\tilde{\alpha}^{\mathrm{T}}\tilde{\beta}$ has to be discussed:

- $\gamma \leq -\alpha^{\mathrm{T}}\beta/c_0$

  Again situation will not depend from γ

  - $\|\alpha\|^2 + c_0^2 < 1 \rightarrow$ problem unbounded from below
  - $\|\alpha\|^2 + c_0^2 = 1 \rightarrow$ non-attainment of objective $-\alpha^{\mathrm{T}}\beta/c_0$
  - $\|\alpha\|^2 + c_0^2 > 1 \rightarrow \gamma$ objective cannot be lower than $-\alpha^{\mathrm{T}}\beta/c_0$

- $\gamma > -\alpha^{\mathrm{T}}\beta/c_0$

  One has to check here when $\tilde{\delta} = 1$

  This leads to a quadratic equation whose solutions determine the potential extremal values of the objective function.

$$\tilde{\delta} = \|\alpha\| + c_0^2 - \frac{(\alpha^{\mathrm{T}}\beta + c_0\gamma)^2}{\|\beta\|^2 + \gamma^2} <, =, > 1$$

which is equivalent to

$$(\|\beta\|^2 + \gamma^2)(\|\alpha\| + c_0^2 - 1) - (\alpha^{\mathrm{T}}\beta + c_0\gamma)^2 <, =, > 0$$

and optimal solutions are easy to compute.

Notes: No dependence on (the direction of) c !

Conclusions

Summary and perspectives

- ◇ Conic problems with a single second-order cone constraint can be solved analytically

  Bad cases (when strong duality fails) are detected and handled

- ◇ Interior-point method complexity for this problem is $\mathcal{O}\left(n^3 \log \frac{1}{\varepsilon}\right)$ this procedure is $\mathcal{O}\left(m^2 n\right)$ $\rightarrow$ dependence on accuracy removed

- ◇ Application of the same technique to dual problem (study of gap)

- ◇ Possible generalizations
  - Allow additional free variables $\rightarrow$ still possible
  - Allow linear inequality constraints $\rightarrow$ quadratic programming
  - Use of this result for several second-order cones (relaxation?)