Linearization of Second-Order Cone Optimization Problems and Application to Limit Analysis in Mechanical Engineering

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- Linearization vs. direct resolution

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Introduction

Convex optimization

Let $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ a convex function and $C \subseteq \mathbb{R}^n$ a convex set

$$\inf_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad x \in C$$

Why ?

General nonlinear problems: too hard to solve

Linear optimization: efficiently solvable but limited modelling

$\Rightarrow$ Generalize linear optimization while keeping its good properties

- ◇ Local optima $\Rightarrow$ global, form a convex optimal set
- ◇ Lagrange duality $\Rightarrow$ related (asymmetric) dual problem
- ◇ Efficient *interior-point* methods (*self-concordant barriers*)

Conic optimization

Let $\mathcal{C} \subseteq \mathbb{R}^n$ a *solid, pointed, closed convex cone* :

$$\inf_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C} \qquad \Rightarrow \textbf{Equivalent} \text{ setting}$$

Solved with interior-point methods with ϵ relative accuracy using

$$\mathcal{O}\left(\sqrt{\nu} \log \frac{1}{\epsilon}\right)$$

iterations where ν depends only on the structure of $\mathcal{C}$ *[NN94]*.

Duality

Dual cone is also a solid pointed closed convex cone

$$\mathcal{C}^* = \left\{x^* \in \mathbb{R}^n \mid x^T x^* \geq 0 \text{ for all } x \in \mathcal{C}\right\}$$

$\Rightarrow$ pair of primal-dual problems

Primal-dual pair

Similar conic structure for the dual

$$\inf_{x\in\mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b \text{ and } x \in \mathcal{C}$$

$$\sup_{(y,s)\in\mathbb{R}^{m+n}} b^T y \quad \text{s.t.} \quad A^T y + s = c \text{ and } s \in \mathcal{C}^*$$

Weak duality holds – Strong duality holds with a *Slater* condition

Advantages over classical formulation

- ◇ Remarkable primal-dual symmetry
- ◇ Special handling of (*easy*) linear equality constraints

Three special cases

- ◇ $\mathcal{C} = \mathbb{R}^n_+ = \mathcal{C}^* \Rightarrow$ linear optimization
- ◇ $\mathcal{C} = \mathbb{S}^n_+ = \mathcal{C}^* \Rightarrow$ semidefinite optimization
- ◇ $\mathcal{C} = \mathbb{L}^n = \mathcal{C}^* \Rightarrow$ second-order cone optimization

Second-order (Lorentz) cone $\mathbb{L}^n$:

$$\mathbb{L}^n := \left\{(x_0, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq x_0^2\right\}$$

These are self-scaled cones for which there are

- ◇ very efficient primal-dual interior-point algorithms
- ◇ that also perform very well in practice

Second-Order Cone Optimization

General SOCO problem (equivalent to conic formulation):

$$\min c^T x \quad \text{s.t.} \quad \begin{cases} b^l \le Ax \le b^u, \\ x^l \le x \le x^u, \\ x_{I_k} \in \mathbb{L}^{n_k} \ \forall k \end{cases}$$

with $x \in \mathbb{R}^n$, $I_k \subseteq \{1, \cdots, n\}$, $\# I_k = n_k + 1$

Generalizes

- Linear optimization
- (Convex) Quadratic optimization
- (Convex) Quadratically constrained quadratic optimization

Many applications (robust LO, engineering, portfolio, etc.)

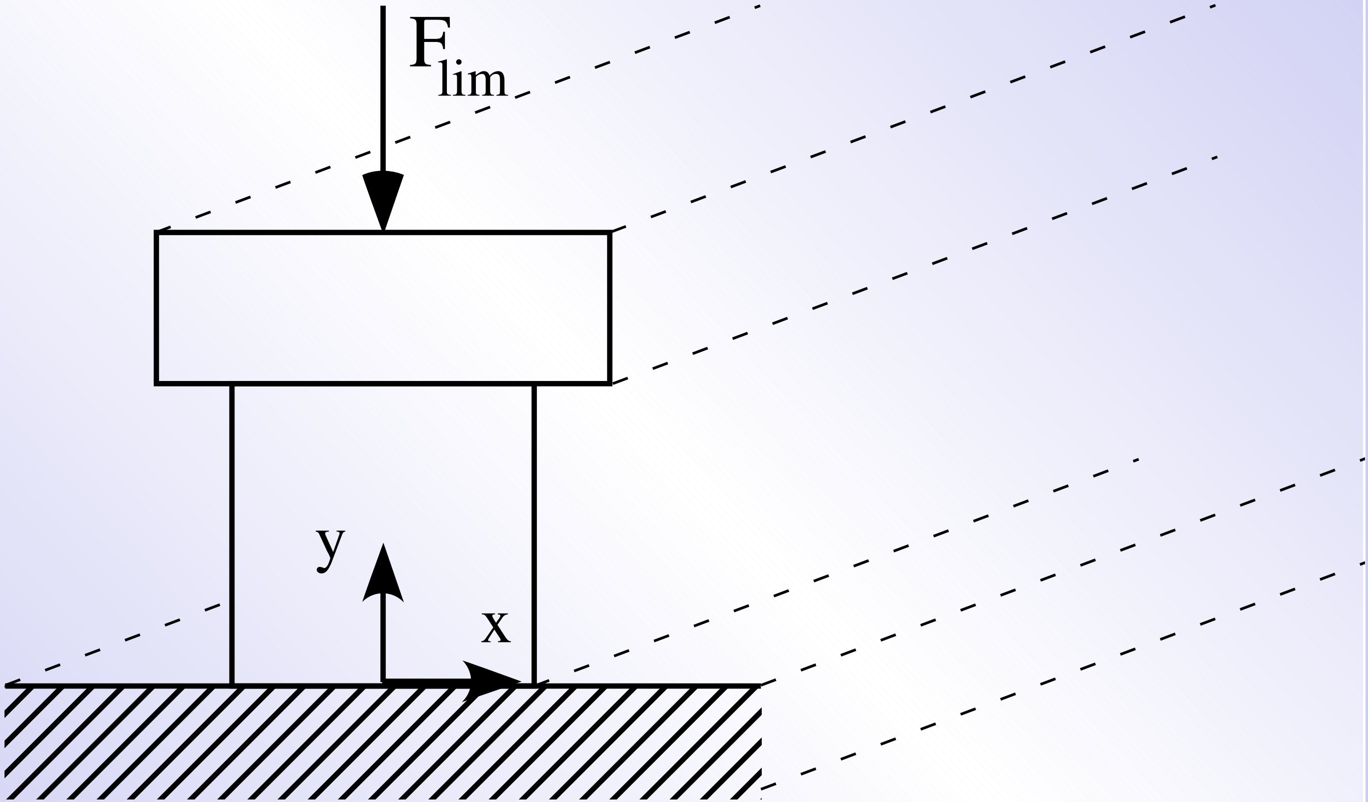
Limit analysis

Objective: compute the ruin load of a mechanical structure
Here: metallic slab, constant section, sufficient length $\Rightarrow$ 2D problem
What's the maximum load one can apply on its upper surface ?

Two approaches

- Kinematic approach: find a kinematically admissible velocity field $\equiv$ a failure mechanism satisfying the flow rule and velocity boundary conditions $\Rightarrow$ upper bound (not considered here)
- Static approach: find a stress field satisfying equilibrium, boundary conditions and plasticity criterion $\Rightarrow$ lower bound

This problem is nonlinear: model it as a second-order cone (quadratic) optimization problem



F_{lim}
y
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Principle

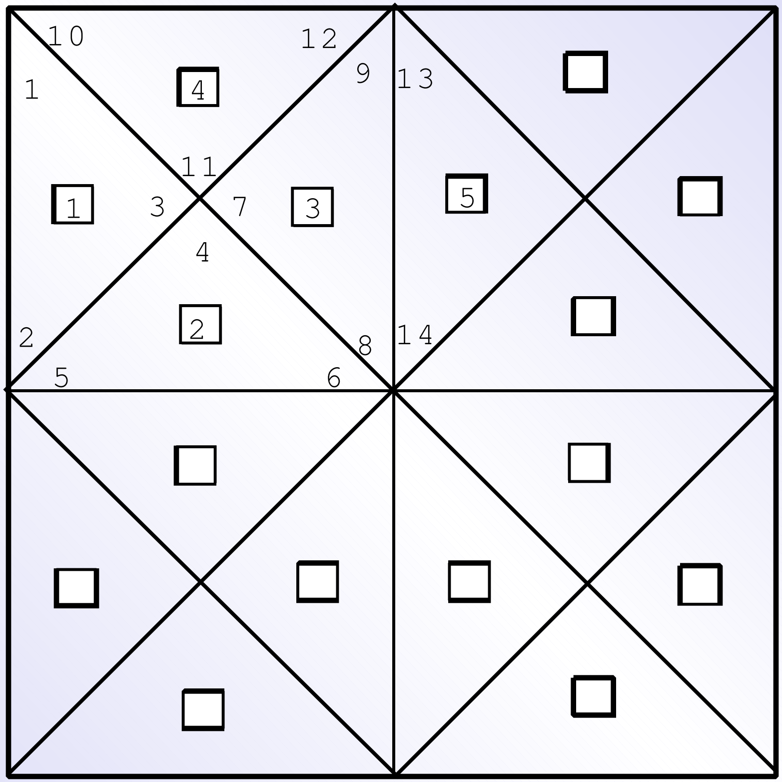
Consider a stress field σ (three components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) that is both

- ◇ statically admissible: equilibrium equations, stress continuities, boundary conditions, vertical load
- ◇ plastically admissible: Tresca-von Mises plasticity crit. $f(\sigma) \leq 0$

and maximize the corresponding load

Discretization

- ◇ Solid is discretized (finite-element modeling) into a mesh composed of right-angled triangles
- ◇ Variables $\equiv$ stress field at each vertex (node) of each triangle
- ◇ Stress field is interpolated linearly within each triangle
- ◇ Vertex shared by two triangles $\Rightarrow$ different stress fields values !



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Objective function

Load $\equiv \int_{\text{upper surface}} \sigma_{yy} \,\mathrm{d}S \quad \longrightarrow \quad$ linear function of the variables

Constraints

- ◇ Equilibrium: $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$ and $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$
- ◇ Continuity: if triangles (i) and (j) share a side with normal n: $\sigma^{(i)} n = \sigma^{(j)} n$
- ◇ Boundary: $\sigma^{(k)} n = 0$ for all lateral triangles (k) with normal n
- ◇ Plasticity: $(\sigma_{xx} - \sigma_{yy})^2 + (2\sigma_{xy})^2 - 4k^2 \leq 0$

All these constraints are linear **except** the plasticity constraints, which can be modelled using a three-variable second-order cone:

$$(2k, \sigma_{xx} - \sigma_{yy}, 2\sigma_{xy}) \in \mathbb{L}^2$$

Linearization schemes

Second-order cone constraint $(k, u, v) \in \mathbb{L}^2$ with constant k defines a disc of radius k in the (u, v)-plane

Standard scheme

Use m linear constraints to define a *regular m-sided polygon*

Relative accuracy of the approximation is

$$\varepsilon = \cos(\frac{\pi}{m})^{-1} - 1 \approx \frac{\pi^2}{2m^2}$$

$\Rightarrow$ very expensive ($m > 2000$ for $\varepsilon = 10^{-6}$)

but it is possible to do much better ...

Ben-Tal & Nemirovsky's improved scheme

Key idea: use additional variables and project i.e.

$$(u, v) \in \text{approximated disc in } \mathbb{R}^2 \Leftrightarrow \exists y \in \mathbb{R}^m \mid (u, v, y) \in \mathcal{P}$$

where $\mathcal{P}$ is a polytope in the higher-dimensional space $\mathbb{R}^{2+m}$

Let $q \geq 1$ a positive parameter and consider the following system

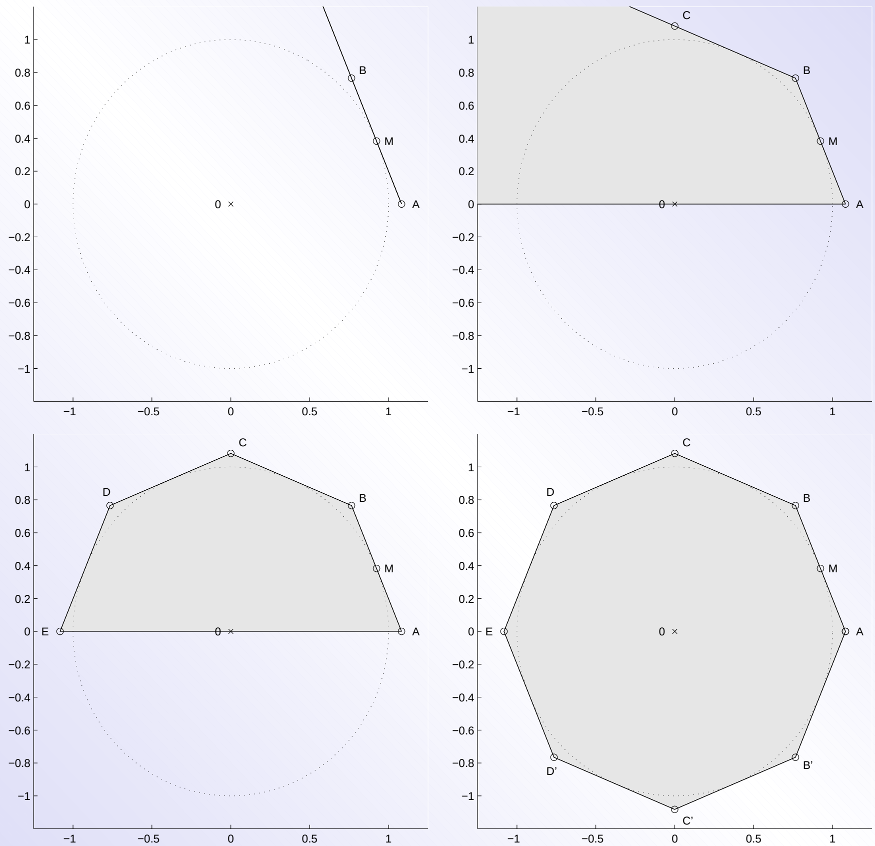
$$\left\{ \begin{array}{rcl} \alpha_{i+1} & = & \alpha_i \cos \frac{\pi}{2^i} + \beta_i \sin \frac{\pi}{2^i} \\ \beta_{i+1} & \geq & \left| -\alpha_i \sin \frac{\pi}{2^i} + \beta_i \cos \frac{\pi}{2^i} \right| \end{array} \right., \quad 0 \leq i < q$$

$$\left\{ \begin{array}{rcl} \beta_q & \leq & 2 \sin \frac{\pi}{2^q} \\ 1 & = & \alpha_q \cos \frac{\pi}{2^q} + \beta_q \sin \frac{\pi}{2^q} \end{array} \right.$$

Its projection on (α_0, β_0) is a regular 2^q-sided polygon!

$$\varepsilon = \cos(\frac{\pi}{2^q})^{-1} - 1 \approx \frac{\pi^2}{2^{2q+1}}$$

at the cost of $2q + 1$ inequalities and $2q$ additional variables
($q = 12$ is enough to obtain $\varepsilon < 10^{-6}$)



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Numerical experiments

Objective: compare the three different approaches

- ◇ Standard linearization of the second-order cone problem
- ◇ Ben-Tal & Nemirovsky's improved linearization scheme
- ◇ Direct resolution using a second-order cone interior-point solver

Standard vs. improved linearization schemes

Experiments with a 3528-triangle mesh on a Power Mac G4
Linear problems solved using XA interior-point solver

	m=16 / q=4	m=64 / q=6	m=256 / q=8	m=1024 / q=10
Standard	3.7947 / 404s	3.7869 / 722s	3.7854 / 1430s	
Row×Col	31,576 × 98,786	31,576 × 268,130	31,576 × 945,506	Out of memory
Nonzeros	374,502	1,051,428	3,760,932	
B-T. & N.	3.7947 / 585s	3.7869 / 648s	3.7854 / 855s	3.78527 / 804s
Row×Col	70,562 × 74,091	91,730 × 88203	112,898 × 102,315	134,066 × 116,427
Nonzeros	308,701	372,205	435,709	499,213

- ◇ Both schemes give exactly the same maximum load
- ◇ B-T. & N. scheme solves problems faster
- ◇ B-T. & N. scheme uses less memory

Improved linearization vs. direct resolution

Several refinements of a triangle mesh on a 500 MHz PC
Linear and second-order cone optimization problems solved using the MOSEK interior-point solver developed by E. Andersen

Grid size	SOCO	B-T. & N. $q = 5$		$q = 8$
4-2	0.1s	0.2s	3k $\times$ 5k	0.4s
8-4	0.5s	1.2s	4k $\times$ 7k	2.1s
16-8	3.7s	9.8s	11k $\times$ 19k	13.5s
24-12	10.2s	30.4s	24k $\times$ 43k	42.0s

- ◇ Direct SOCO resolution is more accurate and
- ◇ Direct SOCO resolution solves problems faster
- ◇ B-T. & N. with simplex: $\approx 80\times$ slowdown! $\rightarrow$ highly degenerate

Concluding remarks

- ◇ Limit analysis problems in mechanical engineering can be successfully modelled and solved as second-order cone optimization problems
- ◇ Actual value of the stress field at the optimum can help understand how the structure is likely to collapse under excessive load
- ◇ Ben-Tal & Nemirovsky's improved linearization scheme is computationally superior to the standard linearization scheme that is traditionally used in the field, especially when high accuracy is required ...
- ◇ ... but direct resolution using a second-order cone interior-point solver is the fastest solution method currently available

Analysis of 3D structures

- 3D plasticity criterion can still be modelled as a second-order cone constraint using $\mathbb{L}^3$
- Resulting problems should be efficiently solvable using a second-order cone solver (using 3D discretization of the stress field)
- Linearization leads to the approximation of a sphere in $\mathbb{R}^3$
- Standard linearization scheme requires a prohibitively high number of constraints, even for a modest accuracy
- Improved linearization scheme can still be used due to the fact that a four-variable second-order cone $\mathbb{L}^3$ can be expressed as the projection of two three-variable $\mathbb{L}^2$ second-order cones

$$(k, u, v, w) \in \mathbb{L}^3 \Leftrightarrow (\gamma, u, v) \in \mathbb{L}^2 \text{ and } (k, \gamma, w) \in \mathbb{L}^2$$