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Boundary control for exact cancellation of boundary disturbances in hyperbolic systems of conservation laws

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***Abstract*— We propose a boundary control methodology based on Riemann invariants to control the state of an hyperbolic system of conservation laws at a boundary, when a known disturbance is applied at the other boundary. The resulting control is an explicit feedforward expression, function of the disturbance at past time instants. The method is illustrated with the control of the water level at the downstream boundary of an open channel, when the inflow rate at the upstream boundary is submitted to a known disturbance.**

***Index Terms*— Partial differential equations, Hyperbolic systems, Boundary control, Disturbance rejection, Conservation laws.**

I. INTRODUCTION

In this paper, we are concerned with systems of conservation laws that are described by hyperbolic partial differential equations, with an independent time variable $t \in [0, +\infty)$ and an independent space variable on a finite time interval $x \in [0, L]$. For such systems, the boundary control problem that we consider is the problem of finding a control law to be applied at the right boundary (i.e. at $x = L$) which cancels the effect of a measurable disturbance applied at the left boundary (i.e. at $x = 0$). Using the Riemann invariants and the properties of the associated characteristic form, it is shown that such a feedforward control law can be derived as an explicit infinite series function of the known disturbance at past time instants. The control design method is illustrated with an hydraulic application : the control of the water level at the downstream boundary of a reach of an open channel which is disturbed by a time-varying inflow rate at the upstream boundary. For the sake of simplicity, our presentation is limited to second order systems (i.e. systems of two scalar conservation laws). More precisely, we present in Sections II and III, a control design method for a generic *homogeneous* system of two *linear* conservation laws. But, the hydraulic application presented in Section IV clearly shows that the control can be robustly applied to *nonhomogeneous* systems of *nonlinear* conservation laws.

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II. STATEMENT OF THE CONTROL PROBLEM

We consider a system of two linear conservation laws of the general form :

$$\partial_t h(t, x) + \partial_x q(t, x) = 0 \tag{1}$$

$$\partial_t q(t, x) + \alpha\beta\partial_x h(t, x) + (\alpha - \beta)\partial_x q(t, x) = 0 \tag{2}$$

where :

- t and x are the two independent variables : a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ on a finite interval;
- $(h, q); [0, +\infty) \times [0, L] \to \Omega \subset I\!R^2$ is the vector of the two dependent variables (i.e. $h(t, x)$ and $q(t, x)$ are the two states of the system);
- α and β are two real positive constants.

The first equation (1) can be interpreted as a mass conservation law with h the conserved quantity and q the flux. The second equation can then be interpreted as a momentum conservation law.

We are concerned with the solutions of the Cauchy problem for the system (1)-(2) over $[0, +\infty) \times [0, L]$ under an initial condition :

$$h(0, x), q(0, x) \qquad x \in [0, L]$$

and two boundary conditions of the form :

$$q(t, 0) = q_d(t) \qquad t \in [0, +\infty) \tag{3}$$

$$q(t, L) = q_c(t) \qquad t \in [0, +\infty) \tag{4}$$

The left boundary function $q_d(t)$ is supposed to be a measurable time-varying disturbance. The right boundary condition $q_c(t)$ is a control function that can be freely manipulated by the operator.

The *boundary control problem* that we consider in this paper is the problem of finding a *feedforward control law* $q_c(t)$, function of the past measured disturbances $q_d(s), s \leq t$, such that the state $h(t, L)$ at the right boundary is identically zero ($h(t, L) \equiv 0 \ \forall t$) along the solution of the Cauchy problem.

In other and less technical terms, we want a control law which exactly cancels the influence of the left boundary disturbance $q_d(t)$ on the right boundary state $h(t, L)$. The relevance of this control problem will be motivated by the application presented in Section IV.

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III. DESIGN OF THE CONTROL LAW

In order to solve this boundary control problem, we introduce the *Riemann coordinates* (see e.g. [1] p. 79) defined by the following change of coordinates :

$$a(t,x) = q(t,x) + \beta h(t,x) \quad (5)$$
$$b(t,x) = q(t,x) - \alpha h(t,x) \quad (6)$$

With these coordinates, the system (1)-(2) is rewritten under the following diagonal form :

$$\partial_t a(t,x) + \alpha \partial_x a(t,x) = 0 \quad (7)$$
$$\partial_t b(t,x) - \beta \partial_x b(t,x) = 0 \quad (8)$$

The two solutions $a(t,x)$ and $b(t,x)$ of the system (7)-(8) are constant along the characteristic lines $x_a(t)$ and $x_b(t)$ respectively defined by the two differential equations :

$$\frac{dx_a}{dt} = \alpha > 0 \qquad \frac{dx_b}{dt} = -\beta < 0$$

The Rieman coordinates $a(t,x)$ and $b(t,x)$ are therefore also called *Riemann invariants*.

The change of coordinates (5)-(6) is inverted as follows :

$$h(t,x) = \frac{a(t,x) - b(t,x)}{\alpha + \beta} \quad (9)$$
$$q(t,x) = \frac{\alpha a(t,x) + \beta b(t,x)}{\alpha + \beta} \quad (10)$$

It follows that, in order to satisy the control objective $h(t,L) = 0 \ \forall t$, we need to impose :

$$a(t,L) = b(t,L) \quad \forall t \quad (11)$$

Moreover, since $b(t,x)$ is invariant along its characteristic line, we have :

$$b(t,0) = b(t - \frac{L}{\beta}, L)$$

and from (11) :

$$b(t,0) = b(t - \frac{L}{\beta}, L) = a(t - \frac{L}{\beta}, L)$$

Similarly, since $a(t,x)$ is invariant along its characteristic line, we have :

$$b(t,0) = a(t - \frac{L}{\beta}, L) = a(t - \frac{L}{\beta} - \frac{L}{\alpha}, 0)$$

Now from the inverse coordinate change (10) and the boundary condition (3) we have :

$$q_d(t) = q(t,0) = \frac{\alpha a(t,0) + \beta b(t,0)}{\alpha + \beta} = \frac{\alpha a(t,0) + \beta a(t - (L/\alpha) - (L/\beta), 0)}{\alpha + \beta}$$

which implies :

$$a(t,0) = (1+k) q_d(t) - k a(t - \tau, 0) \quad (12)$$

with the following notations :

$$k = \frac{\beta}{\alpha} \qquad \tau = \frac{L}{\alpha} + \frac{L}{\beta}$$

Expression (12) is obiously valid for any t and can be shifted in time by any time delay. Hence we have :

$$a(t-\tau, 0) = (1+k) q_d(t-\tau) - k a(t - 2\tau, 0)$$

Substituting this expression in (12), we get :

$$a(t,0) = (1+k) q_d(t) - k(1+k) q_d(t-\tau) + k^2 a(t - 2\tau, 0).$$

By continuing the recurrence for successive time delays $n\tau$ $n = 2, 3, 4....$, we can establish the following expression for $a(t,0)$ under the form of an infinite series :

$$a(t,0) = \sum_{n=0}^{+\infty} (-1)^n k^n (1+k) q_d(t - n\tau) \quad (13)$$

Using an identical line of reasoning, we can also establish a similar expression for $b(t,0)$:

$$b(t,0) = \sum_{n=0}^{+\infty} (-1)^n k^n (1+k) q_d(t - (n+1)\tau) \quad (14)$$

Moreover, we have from (10) :

$$q_c(t) = q(t,L) = \frac{1}{\alpha + \beta} \left[\alpha a(t,L) + \beta b(t,L) \right]$$

Since $a(t,x)$ and $b(t,x)$ are invariant along their characteristic lines, $q_c(t)$ can be rewritten as :

$$q_c(t) = \frac{1}{\alpha + \beta} \left[\alpha a(t - \frac{L}{\alpha}, 0) + \beta b(t + \frac{L}{\beta}, 0) \right]$$

Using (13) and (14), we then finally get the expression of a **feedforward boundary control law** $q_c(t)$ in function of the boundary disturbance $q_d(t)$:

$$q_c(t) = \sum_{n=0}^{+\infty} (-1)^n k^n (1+k) q_d(t - \frac{L}{\alpha} - n\tau)$$

IV. APPLICATION TO LEVEL CONTROL IN AN OPEN CHANNEL

In the field of hydraulics, the flow in open-channels is generally represented by the so-called Saint Venant equations which are a typical example of a system of conservation laws.

We consider the special case of a reach of an open channel delimited by two overflow spillways as represented in Figure 1.

We assume that :

1) the channel is horizontal,
2) the channel is prismatic with a constant rectangular section and a unit width,
3) the friction effects are neglected.

The flow dynamics are described by a system of two laws of conservation (Saint-Venant or shallow water equations), namely a law of mass conservation :

$$\partial_t H + \partial_x Q = 0, \quad (15)$$

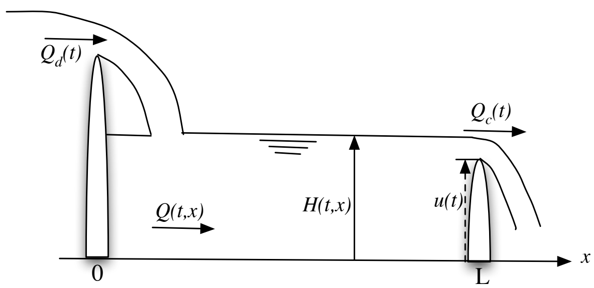

Fig. 1. A reach of an open channel delimited by two overflow spillways : a fixed spillway on the left and an adjustable spillway on the right

and a law of momentum conservation :

$$\partial_t Q + \partial_x(\frac{Q^2}{H} + g\frac{H^2}{2}) = 0. \qquad (16)$$

where $H(t,x)$ represents the water level and $Q(t,x)$ the water flow rate in the reach while g denotes the gravitation constant.

The control action is the vertical position $u(t)$ of the spillway located at the right extremity of the pool and related to the state variables H et Q by the following expression :

$$Q_c(t) = Q(t,L) = \gamma(H(t,L)_L - u(t))^{3/2} \qquad (17)$$

where γ is the characteristic constant of the spillway.

For a constant spillway position $u(t) = \bar{u}\ \forall t$ and a constant inflow rate $Q_d(t) = \bar{Q}\ \forall t$, there is a unique steady state solution which satisfies the following relations :

$$\begin{aligned} H(t,x) &= \bar{u} + (\gamma^{-1}\bar{Q})^{2/3} \\ Q(t,x) &= \bar{Q} \qquad\qquad \forall x \in [0,L] \text{ and } \forall t \end{aligned}$$

The control objective is to regulate the level $H(t,L)$ at the set point $\bar{H}$, by acting on the spillway position $u(t)$. More precisely, it is requested to adjust the control $u(t)$ in order to have $H(t,L) = \bar{H}\ \forall t$ in spite of the variations of the disturbing inflow rate $Q_d(t)$.

Let us consider the deviations of $H(t,x)$ and $Q(t,x)$ with respect to the steady-state values $\bar{H}$ and $\bar{Q}$:

$$\begin{aligned} h(t,x) &= H(t,x) - \bar{H} \\ q(t,x) &= Q(t,x) - \bar{Q}. \end{aligned}$$

By linearising the model equations (15)-(16) around the steady-state $(\bar{H},\bar{Q})$, we get the linear model :

$$\partial_t h(t,x) + \partial_x q(t,x) = 0$$

$$\partial_t q(t,x) + \left(g\bar{H} - \frac{\bar{Q}^2}{\bar{H}^2}\right)\partial_x h(t,x) + 2\frac{\bar{Q}}{\bar{H}}\partial_x q(t,x) = 0$$

This model is exactly in the form (1)-(2) with :

$$\alpha = \sqrt{g\bar{H}} + \frac{\bar{Q}}{\bar{H}} \quad \text{and} \quad \beta = \sqrt{g\bar{H}} - \frac{\bar{Q}}{\bar{H}}$$

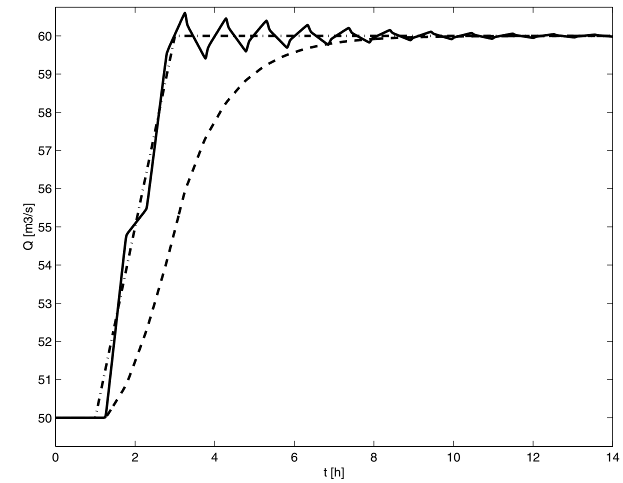

Fig. 2. Flow rates : inflow $Q_d(t)$ (*dash-dot*), outflow $Q_c(t)$ with feedforward control (*solid*), outflow without control (*dotted*).

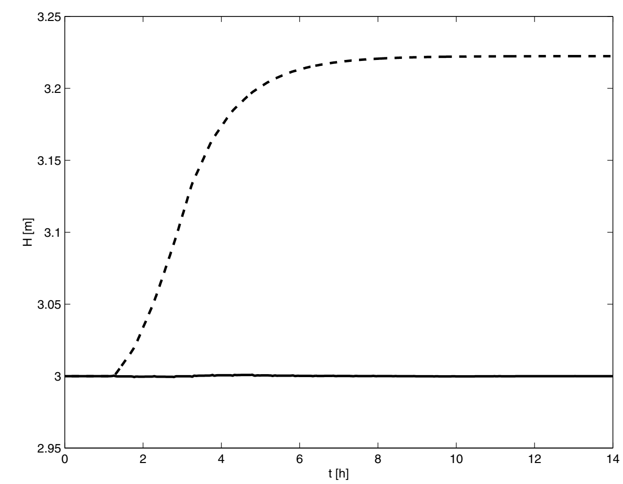

Fig. 3. Level $H(t,L)$: with feedforward control (*solid*), without control (*dotted*).

Furthermore, we have $\alpha > 0$ and $\beta > 0$ under subcritical (or fluvial) flow conditions.

Then, by a straightforward application of the control design presented in Section III, we readily get the following **feedforward boundary control law** :

$$u(t) = \bar{H} - (\gamma^{-1}\bar{Q}_c(t))^{2/3}$$

with :

$$Q_c(t) = \bar{Q} + \sum_{n=0}^{+\infty}(-1)^n k^n(1+k)\left[Q_d(t - \frac{L}{\alpha} - n\tau) - \bar{Q}\right]$$

We shall now illustrate the efficiency of this control law with two numerical simulation experiments. We consider a pool with length $L = 5000\ m$ and width $\ell = 40\ m$. The steady state values are respectively $\bar{H} = 3\ m$ and $\bar{Q} = 50\ m^3/sec$. The system is initially at steady-state : $H(0,x) = \bar{H}$, $Q(0,x) = \bar{Q}\ \forall x \in [0,L]$. A unit-step

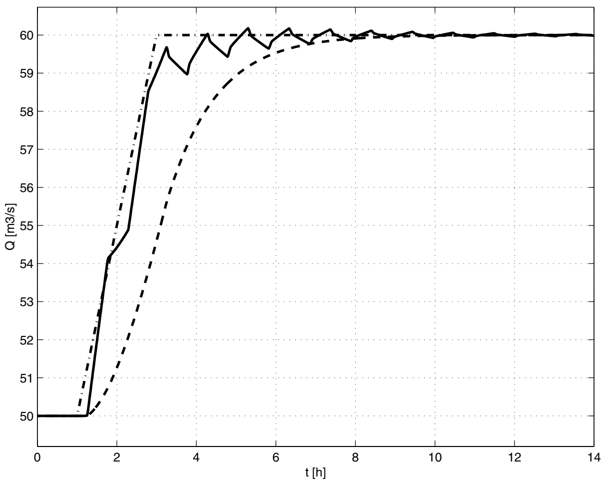

Fig. 4. Simulation result with friction; Flow rates : inflow $Q_d(t)$ (*dash-dot*), outflow $Q_c(t)$ with feedforward control (*solid*), outflow without control (*dotted*).

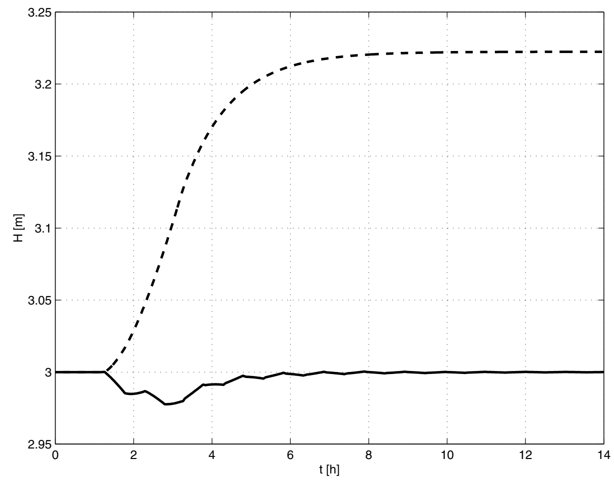

Fig. 5. Simulation result with friction; Level $H(t, L)$: with feedforward control (*solid*), without control (*dotted*).

disturbance with an amplitude of $10\ m^3/sec$ is applied to to the boundary flow rate $Q_d(t)$. The system is represented by the nonlinear shallow water equations (15)-(16) which are solved with a standard Preisman numerical scheme. The experimental results are shown in Fig. 2 and Fig. 3. The effectiveness of the controller is well illustrated since it appears clearly that the level $H(t, L)$ is totally insensitive to a 20 % variation of the disturbance inflow rate $Q_d(t)$. In order to assess the robustness of the control with respect to modelling uncertainties, we also consider the more realistic situation where the friction effects are not negligible. The momentum equation is then modified as :

$$\partial_t V + \partial_x\left(\frac{Q^2}{H} + g\frac{H^2}{2}\right) + gC_f\frac{Q^2}{H^2} = 0 \qquad (18)$$

where C_f denotes a friction parameter. The same experiment as above is performed and the results are shown in Fig. 4 and Fig. 5. Here, however, the channel is simulated with the model equation (18) and the numerical value of the friction parameter C_f is set to a value corresponding to a Strickler coefficient $K = 33\ m^{1/3}/sec$ which is a typical value for a standard canalised waterway. Obviously, in this case, the disturbance cancellation is no longer exact but it clearly remains rather acceptable as it can be seen in Fig. 5.

V. CONCLUSIONS

In this paper, we have proposed an approach for finding a feedforward control law which exactly cancels the influence of a left boundary disturbance on a right boundary state in an hyperbolic system of two conservation laws. The relevance of this control problem has been motivated and illustrated with a simulation experiment regarding the water level control in an open channel disturbed by a time varying inflow rate.

This research is part of a series of continuing research regarding the boundary control of systems of conservation laws with applications to level and flow control in open channels ([2], [3], [4], [5]).

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