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BOUNDARY CONTROL OF SYSTEMS OF CONSERVATION LAWS : LYAPUNOV STABILITY WITH INTEGRAL ACTIONS

Valérie Dos Santos * **Georges Bastin** **
Jean-Michel Coron *** **Brigitte d'Andréa-Novel** ****

* *Université de Lyon, Lyon, F-69003, France; Université de Lyon 1, CNRS UMR 5007, LAGEP, ESCPE, Villeurbanne, F-69622, France*

** *CESAME, Université Catholique de Louvain, 4, Avenue G. Lemaitre, 1348 Louvain-La-Neuve, Belgium.*

*** *Department of Mathematics, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France.*

**** *Centre de Robotique (CAOR), ENSMP, 60, Boulevard Saint Michel, 75272 Paris Cedex 06, France.*

Abstract: A boundary control law with integral actions is proposed for a generic class of two-by-two homogeneous systems of linear conservation laws. Sufficient conditions on the tuning parameters are stated that guarantee the asymptotic stability of the closed-loop system. The closed-loop stability is analysed with an appropriate Lyapunov function. The control design method is validated with an experimental application to the regulation of water depth and flow rate in a pilot open-channel described by Saint-Venant equations. This hydraulic application shows that the control can be robustly implemented on nonhomogeneous systems of nonlinear conservation laws.

Keywords: Lyapunov stability, Proportional-Integral control, Saint-Venant equations, Systems of conservation laws,

1. INTRODUCTION

In this paper, we are concerned with two-by-two systems of conservation laws that are described by hyperbolic partial differential equations, with one independent time variable $t \in [0, \infty)$ and one independent space variable on a finite interval $x \in [0, L]$. Such systems are used to model many physical situations and engineering problems. A famous example is that of Saint-Venant (or shallow water) equations which describe the flow of water in irrigation channels and waterways. This example will be presented in Section 4. Other typical examples include gas and fluid transportation networks, packed bed and plug-flow reactors, drawing processes in glass and polymer industries, road traffic etc. For such systems, the boundary control problem that we address is the problem of designing feedback control actions at the boundaries (i.e. at $x = 0$ and $x = L$) in order to ensure that the smooth solution of the Cauchy problem converges to a desired steady-state.

The present paper is in the direct continuation of our previous paper (Coron et al. (2007)) where a static feedback control law was presented and the closed-loop stability analysed with an appropriate Lyapunov function. But obviously, a static control law may be subject to steady-state regulation errors in case of constant disturbances or model inaccuracies. In the present paper, we show how additional integral actions can be introduced in the control law in order to cancel the static errors and how the Lyapunov function can be modified in order to prove the asymptotic stability of the closed-loop system. The statement of the control law and the Lyapunov stability analysis are developed in Sections 2 and 3 for a generic *homogeneous* system of two *linear* conservation laws. In Section 4, we present an experimental validation on a laboratory pilot plant. This hydraulic application clearly shows that the control can be robustly implemented on *nonhomogeneous* systems of *nonlinear* conservation laws.

2. STATEMENT OF THE CONTROL LAW

We consider the class of two-by-two systems of linear conservation laws of the general form:

$$\partial_t h(t,x) + \partial_x q(t,x) = 0 \tag{1}$$

$$\partial_t q(t,x) + \alpha\beta\partial_x h(t,x) + (\alpha-\beta)\partial_x q(t,x) = 0 \tag{2}$$

where

- t and x are the two independent variables : a time variable $t \in [0,+\infty)$ and a space variable $x \in [0,L]$ on a finite interval;
- $(h,q) : [0,+\infty) \times [0,L] \to \mathbb{R}^2$ is the vector of the two dependent variables (i.e. $h(t,x)$ and $q(t,x)$ are the two states of the system);
- α and β are two real positive constants:

$$\alpha > \beta > 0.$$

The first equation (1) can be interpreted as a mass conservation law with h the density and q the flux. The second equation can then be interpreted as a momentum conservation law. As usual in control design, the model (1)-(2) must be viewed as a linear approximation of the system dynamics around a steady-state. This will be illustrated with the application of Section 4.

We are concerned with the solutions of the Cauchy problem for system (1)-(2) over $[0,+\infty) \times [0,L]$ under an initial condition

$$h(0,x), q(0,x) \qquad x \in [0,L].$$

Furthermore, it is assumed that the system is subject to physical boundary conditions that can be assigned by an external operator and are written in the following general abstract form:

$$g_0(h(t,0), q(t,0), u_0(t)) = 0 \quad t \in [0,+\infty) \tag{3a}$$

$$g_L(q(t,L), h(t,L), u_L(t)) = 0 \quad t \in [0,+\infty) \tag{3b}$$

with $g_0, g_L : \mathbb{R}^3 \to \mathbb{R}$. The functions u_0, u_L: $[0,+\infty) \to \mathbb{R}$ represent the boundary control actions that can be manipulated by the operator. A concrete illustration of such boundary conditions will be given in Section 4.

In order to define the feedback control laws, it is convenient to introduce the *Riemann coordinates* (see e.g. Lax (1973)) defined by the following change of coordinates:

$$a(t,x) = q(t,x) + \beta h(t,x) \tag{4a}$$

$$b(t,x) = q(t,x) - \alpha h(t,x). \tag{4b}$$

With these coordinates, the system (1)-(2) is rewritten under the following diagonal form:

$$\partial_t a(t,x) + \alpha\partial_x a(t,x) = 0 \tag{5a}$$

$$\partial_t b(t,x) - \beta\partial_x b(t,x) = 0. \tag{5b}$$

The change of coordinates (4) is inverted as follows:

$$h(t,x) = \frac{a(t,x) - b(t,x)}{\alpha+\beta} \tag{6a}$$

$$q(t,x) = \frac{\alpha a(t,x) + \beta b(t,x)}{\alpha+\beta}. \tag{6b}$$

In the Riemann coordinates, the control problem can be restated as the problem of designing the control laws in such a way that the solutions $a(t,x)$ and $b(t,x)$ converge to zero. We shall show that this problem can be solved by selecting the boundary control laws $u_0(t)$ and $u_L(t)$ such that the Riemann coordinates satisfy *linear* boundary conditions of the following form:

$$a(t,0) + k_0 b(t,0) + m_0 y_0(t) = 0 \tag{7a}$$

$$b(t,L) + k_L a(t,L) + m_L y_L(t) = 0 \tag{7b}$$

where k_0, k_L, m_0, m_L are constant tuning parameters while y_0 and y_L are integrals of the flow $q(t,0)$ and the density $h(t,L)$ respectively:

$$y_0(t) = \int_0^t q(s,0)ds = \int_0^t \frac{\alpha a(s,0) + \beta b(s,0)}{\alpha+\beta} ds \tag{8a}$$

$$y_L(t) = \int_0^t h(s,L)ds = \int_0^t \frac{a(s,L) - b(s,L)}{\alpha+\beta} ds. \tag{8b}$$

Remarks

1) Conditions (7) give only an implicit definition of the control laws. The derivation of explicit expressions obviously requires an explicit knowledge of the functions g_0 and g_L in (3). In the special case where the boundary conditions (3) are linear, u_0 and u_L reduce to standard Proportional-Integral (PI) control laws. This point will be further illustrated in Section 4.

2) In our previous paper (Coron et al. (2007)), we have dealt with the special case without integral actions, i.e. m_0=m_L=0 in (7). We have shown

with an appropriate Lyapunov function that inequality $|k_0 k_L| < 1$ is a sufficient condition to guarantee the closed-loop stability and the exponential convergence of $a(t,x)$ and $b(t,x)$ to zero. Our contribution in the present paper is to extend this Lyapunov stability analysis to the case where integral terms are introduced in the control law and to validate the methodology with experimental results.

3. LYAPUNOV STABILITY ANALYSIS

Let us define the following candidate Lyapunov function:

$$\begin{aligned} U(t) &= \frac{A}{\alpha}\int_0^L a^2(t,x)e^{-(\mu/\alpha)x}dx \\ &+ \frac{B}{\beta}\int_0^L b^2(t,x)e^{+(\mu/\beta)x}dx \\ &+ \frac{\alpha+\beta}{2}\left[N_0 y_0^2(t) + N_L y_L^2(t)\right] \end{aligned} \tag{9}$$

and the following norm:

$$\begin{aligned} \psi(t) = \int_0^L &\left[a^2(t,x) + b^2(t,x)\right] dx \\ &+ |y_0(t)|^2 + |y_L(t)|^2. \end{aligned}$$

We have the following stability result.

Theorem. If the four constant tuning parameters k_0, k_L, m_0, m_L satisfy the following inequalities:

$$|k_0| < 1 \qquad |k_L| < \frac{\alpha}{\beta} \qquad |k_0 k_L| < 1$$
$$m_0 > 0 \qquad m_L < 0,$$

there exist six positive constants A, B, μ, N_0, N_L, C such that

$$\dot{U} \leqslant -\mu U \quad \text{and} \quad \psi(t) \leqslant C\psi(0)e^{-\mu t}$$

for every solution $a(t,x)$, $b(t,x)$, $t \geqslant 0$, $x \in [0,L]$ of (5)-(7).

Proof : The time derivative of the function $U(t)$ along the solutions of the system (5)-(7) is

$$\dot{U} = -\mu U + \dot{U}_0 + \dot{U}_L$$

with

$$\begin{aligned} \dot{U}_0 &= \left[Ak_0^2 - B\right] b_0^2 \\ &+ \left[2Ak_0 m_0 + N_0(\beta - \alpha k_0)\right] b_0 y_0 \\ &+ \left[Am_0^2 + \mu N_0 \frac{\alpha+\beta}{2} - N_0 m_0 \alpha\right] y_0^2 \end{aligned} \tag{10}$$

and

$$\begin{aligned} \dot{U}_L &= \left[\tilde{B}k_L^2 - \tilde{A}\right] a_L^2 \\ &+ \left[2\tilde{B}k_L m_L + N_L(1 + k_L)\right] a_L y_L \\ &+ \left[\tilde{B}m_L^2 + N_L m_L + \mu\frac{\alpha+\beta}{2} N_L\right] y_L^2 \end{aligned} \tag{11}$$

with $\tilde{A} = Ae^{-\mu L/\alpha}$, $\tilde{B} = Be^{\mu L/\beta}$.

We first consider the special case where $\mu = 0$ and (10)-(11) reduce to:

$$\begin{aligned} \left[Ak_0^2 - B\right] b_0^2 &+ \left[2Ak_0 m_0 + N_0(\beta - \alpha k_0)\right] b_0 y_0 \\ &+ \left[Am_0^2 - N_0 m_0 \alpha\right] y_0^2, \end{aligned} \tag{12}$$

$$\begin{aligned} \left[Bk_L^2 - A\right] a_L^2 &+ \left[2Bk_L m_L + N_L(1 + k_L)\right] a_L y_L \\ &+ \left[Bm_L^2 + N_L m_L\right] y_L^2. \end{aligned} \tag{13}$$

We are going to prove that there exist $A > 0$, $B > 0$, $N_0 > 0$, $N_L > 0$ such that (12)-(13) are negative definite quadratic (NDQ) forms in b_0, y_0 and a_L, y_L respectively. We set $B = 1$. Then (12) is a NDQ form if (and only if)

$$A < \frac{1}{k_0^2} \tag{14}$$

and

$$P_0 = 4Am_0^2 + 4N_0(Ak_0\beta - \alpha)m_0 + N_0^2(\beta - \alpha k_0)^2 < 0.$$

P_0 is a polynomial of degree 2 in N_0 with discriminant

$$\Delta_0 = 16(\alpha^2 - A\beta^2)(1 - Ak_0^2)m_0^2.$$

In view of (14), $\Delta_0 > 0$ if

$$A < \frac{\beta^2}{\alpha^2}. \tag{15}$$

Then, if inequalities (14)-(15) hold, P_0 has two real roots and we will have $N_0 > 0$ and $P_0 < 0$ if the greatest root is positive, that is if (and only if) the following inequality holds:

$$(Ak_0\beta - \alpha)m_0 < \sqrt{(\alpha^2 - A\beta^2)(1 - Ak_0^2)m_0^2}.$$

Under conditions (14)-(15), it can be shown that this inequality holds for any $m_0 > 0$. Let us summarize : there exist $A > 0$ and $N_0 > 0$ such that P_0 is an NDQ form if A can be selected such that

$$0 < A < \frac{1}{k_0^2} \quad 0 < A < \frac{\beta^2}{\alpha^2}. \tag{16}$$

Similarly (13) is a NDQ form if (and only if)

$$A > k_L^2$$

and

$$P_L = 4Am_L^2 + 4N_L(k_L + A)m_L + N_L^2(1 + k_L)^2 < 0.$$

With a similar line of reasoning, we can show that there exist $A > 0$ and $N_L > 0$ such that P_L is an NDQ form if A can be selected such that

$$k_L^2 < A \quad 1 < A. \tag{17}$$

Hence, using conditions (16)-(17) together, a sufficient condition to have $A > 0$, $B > 0$, $N_0 > 0$, $N_L > 0$ such that (12)-(13) are NDQ forms is

$$\max\left\{k_L^2, 1\right\} < \min\left\{\frac{1}{k_0^2}, \frac{\alpha^2}{\beta^2}\right\}$$

which is equivalent to

$$|k_0| < 1 \qquad |k_L| < \frac{\alpha}{\beta} \qquad |k_0 k_L| < 1.$$

By continuity with respect to μ, under the same conditions, there exist $\mu > 0$, $A > 0$, $B > 0$,

$N_0 > 0$, $N_L > 0$ such that (10) and (11) are quadratic negative definite forms. This implies that U is a Lyapunov function for the system (5)-(7) such that $\dot{U} \leqslant -\mu U$ along the system solutions with $\dot{U} = 0$ if and only if $U = 0$. The inequality $\psi(t) \leqslant C\psi(0)e^{-\mu t}$ follows readily. QED.

4. EXPERIMENTAL VALIDATION

4.1 Saint-Venant equations

The Saint-Venant equations are among the most frequent application examples of two-by-two hyperbolic systems of conservation laws in engineering. They are used for a long time for simulation and design of model-based controllers in open-channels (see e.g. the recent publications by de Halleux et al. (2003), Litrico et al. (2005) and Litrico and Fromion (2006) and the references therein).

For a pool of a prismatic open channel with a constant slope, the flow dynamics are described as follows by the Saint-Venant equations:

$$\partial_t A + \partial_x Q = 0 \tag{18a}$$

$$\partial_t Q + \partial_x \left(\frac{Q^2}{A}\right) + gA\partial_x H(A) = gA\left[S - J(A,Q)\right] \tag{18b}$$

with L the pool length, $A(t,x)$ the wet area at time t and abscissa $x \in [0,L]$, $Q(t,x)$ the flow rate at time t and abscissa $x \in [0,L]$, g the gravity constant and $H(A)$ the water depth. Furthermore S is the bottom slope and

$$J(A,Q) = \frac{Q^2 n^2}{A^2\left[R(A)\right]^{4/3}}$$

is the friction slope with n the Manning friction coefficient and $R(A)$ the hydraulic radius.

A steady-state regime (A_e, Q_e) is a constant solution of equations (18), i.e. $A(t,x) = A_e$, $Q(t,x) = Q_e$ $\forall t$ and $\forall x \in [0,L]$, which satisfies the relation

$$J(A_e, Q_e) = S. \tag{19}$$

A linearized model is used to describe the variations about this equilibrium. The following notations are introduced:

$$h(t,x) = A(t,x) - A_e \quad q(t,x) = Q(t,x) - Q_e.$$

The linearized model is then written as

$$\begin{aligned} &\partial_t h(t,x) + \partial_x q(t,x) = 0 \\ &\partial_t q(t,x) + \alpha\beta\partial_x h(t,x) + (\alpha-\beta)\partial_x q(t,x) \\ &\qquad = -\gamma h(t,x) - \delta q(t,x) \end{aligned}$$

with

Fig. 1. The ESISAR pilot channel.

$$\begin{aligned} \alpha &= \sqrt{gA_e \frac{\partial H}{\partial A}(A_e)} + \frac{Q_e}{A_e} \\ \beta &= \sqrt{gA_e \frac{\partial H}{\partial A}(A_e)} - \frac{Q_e}{A_e} \\ \gamma &= gA_e \frac{\partial J}{\partial A}(A_e, Q_e) \\ \delta &= gA_e \frac{\partial J}{\partial Q}(A_e, Q_e). \end{aligned}$$

In the special case where the channel is horizontal ($S = 0$) and the friction slope is negligible ($n \approx 0$), we observe that $\gamma = \delta = 0$ and that this linearized system is exactly in the form of the linear hyperbolic system we have considered in Sections 2 and 3. Moreover, the flow in the channel is *fluvial* (as opposed to *torrential*) when $\alpha > \beta > 0$. It is therefore legitimate to apply the control with integral actions that we have analysed above to open channels having small bottom and friction slopes under fluvial flow conditions. In the next section, we shall illustrate the efficiency of the control with experimental results on a real life laboratory plant.

4.2 Experimental setup

An experimental validation has been performed with the Valence micro-channel (see Fig. 1). This pilot channel is located at ESISAR/INPG engineering school in Valence (France). It is operated under the responsability of the LCIS laboratory. The channel (overall length = 8 meters) has an adjustable slope and a rectangular cross-section. The water depth in the channel is denoted $H(t,x)$ such that:

$$A(t,x) = wH(t,x) \qquad (w = \text{channel width}).$$

The channel is furnished with three underflow control gates. Ultrasound sensors provide water level measurements at different locations of the channel. For the experiments reported hereafter the configuration is a single pool (length $L =$

7 meters, width w=0.1 meter, slope S= 0.0016) bounded by two underflow gates. The flow rates at the gates are supposed to be related to the water depths by the gate characteristics expressed as

$$Q(t,0) = c_0 U_0(t)\sqrt{g(H_{up} - H(t,0))} \qquad (21a)$$

$$Q(t,L) = c_L U_L(t)\sqrt{g(H(t,L) - H_{do})} \qquad (21b)$$

where c_0 and c_L are constant coefficients while U_0 and U_L denote the control signals at the upstream and downstream gates respectively. H_{up} is the water depth at the upstream of the upstream gate, H_{do} is the water depth at the downstream of the downstream gate.

4.3 PI control laws

In order to explicit the control laws, the gate characteristics (21) are linearised about the steady-state $(A_e = wH_e, Q_e)$:

$$q(t,0) = -\frac{gQ_e}{2K_0^2}h(t,0) + c_0 K_0 u_0(t) \qquad (22a)$$

$$q(t,L) = \frac{gQ_e}{2K_L^2}h(t,L) + c_L K_L u_L(t) \qquad (22b)$$

with

$$K_0 = \sqrt{g(H_{up} - H_e)} \qquad K_L = \sqrt{g(H_e - H_{do})}$$

$$\text{and:} \quad u_0 = U_0 - \frac{Q_e}{c_0 K_0} \qquad u_L = U_L - \frac{Q_e}{c_L K_L}.$$

Moreover, using the definition of the Riemann coordinates (4), the boundary conditions (7) are rewritten as

$$q(t,0) + \lambda_0 h(t,0) + \mu_0 \int_0^t q(s,0)ds = 0 \qquad (23a)$$

$$q(t,L) + \lambda_L h(t,L) + \mu_L \int_0^t h(s,L)ds = 0 \quad (23b)$$

with:

$$\lambda_0 = \frac{\beta - k_0\alpha}{1+k_0} \qquad \lambda_L = \frac{k_L\beta - \alpha}{1+k_L}$$

$$\mu_0 = \frac{m_0}{1+k_0} \qquad \mu_L = \frac{m_L}{1+k_L}.$$

Then, by eliminating $h(t,0)$ between (22a) and (23a), we get the following PI control law for u_0:

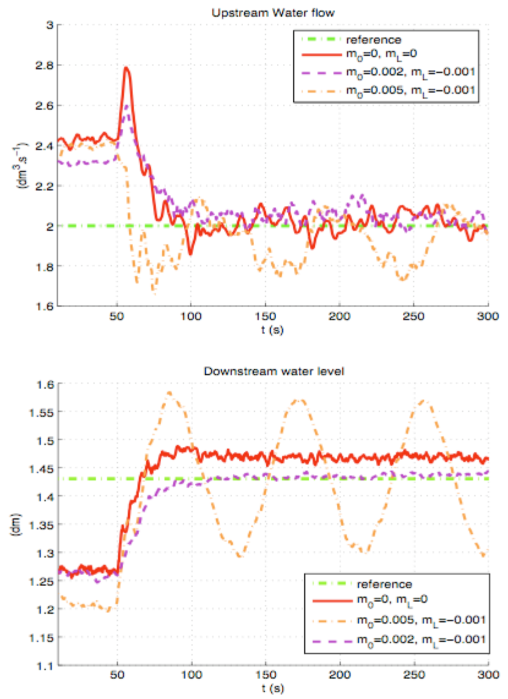
$$u_0(t) = K_{po}q(t,0) - K_{io}\int_0^t q(s,0)ds$$

with

$$K_{po} = \frac{1}{c_0 K_0}\left[1 - \frac{gQ_e}{2\lambda_0 K_0^2}\right] \qquad K_{io} = \frac{gQ_e\mu_0}{2\lambda_0 c_0 K_0^3}.$$

Similarly, by eliminating $q(t,L)$ between (22b) and (23b), we get the following PI control law for u_L:

$$u_L(t) = -K_{pL}h(t,L) - K_{iL}\int_0^t h(s,L)ds$$



a)

b)

Fig. 2. Experimental results

with

$$K_{pL} = \frac{1}{c_L K_L}\left[\lambda_L + \frac{gQ_e}{2K_L^2}\right] \qquad K_{iL} = \frac{\mu_L}{c_L K_L}.$$

Hence the control law u_0 is a PI dynamic feedback of the flow rate $q(t,0) = Q(t,0) - Q_e$ and the control law u_L is a PI dynamic feedback of the water depth $h(t,L) = w(H(t,L) - H_e)$. These control laws are implemented with direct on-line measurements of the water levels H_{up}, H_{do}, $H(t,0)$, $H(t,L)$). The flow rates at the gates are not directly measured but are computed on-line with the gate characteristics (21).

4.4 Experimental results

The experimental results are shown in Fig.2. Three experiments are presented with increasing values of the integral gains m_0 and m_L indicated in the figure captions while the parameters k_0 and k_L are set to

$$k_0 = -0.213, \quad k_L = -1.157, \quad k_0 k_L = 0.246.$$

In each experiment, the system is initially in open loop at a steady state

$$Q(0,x) \approx 2.35 \text{ lit/sec}, \quad H(0,L) \approx 0.125 \text{ m}.$$

The loop is closed at time $t = 50sec$ with a new set point given by:

$$Q_e = 2 \text{ lit/sec}, \quad H_e(L) = 0.143 \text{ m}.$$

In the first experiment without integral actions (m_0=m_L=0), there is clearly an offset of about 4 cm on the level $H(t,L)$. In the second experiment with $m_0 = 0.002$ and $m_L = -0.001$ the static error is efficiently cancelled by the integral actions. Finally a third experiment illustrates the sensitivity of the transient behaviour with respect to the choice of the gain values. For largest gain values ($m_0 = 0.005$, $m_L = -0.001$), the closed loop system starts to oscillate and becomes even unstable if the gains are further increased.

5. CONCLUSION

In this paper, we have been concerned with the boundary control of two-by-two systems of conservation laws that are described by hyperbolic partial differential equations, with one independent time variable and one independent space variable. A boundary control law with integral actions has been proposed for a generic class of homogeneous systems of two linear conservation laws. Sufficient conditions on the tuning parameters have been stated in order to guarantee the asymptotic stability of the closed-loop system which has been proved with an appropriate Lyapunov stability analysis. The control design method has been validated with an experimental application to the regulation of water depth and flow in a pilot open-channel described by Saint-Venant equations.

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