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Lyapunov exponential stability of linear hyperbolic systems of balance laws

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Abstract: Explicit boundary dissipative conditions are given for the exponential stability in L^2-norm of one-dimensional linear hyperbolic sytems of balance laws $\partial_t\boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x\boldsymbol{\xi} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ over a finite space interval, when the matrix $\mathbf{M}$ is marginally diagonally stable. The result is illustrated with an application to boundary feedback stabilisation of open channels represented by linearised Saint-Venant-Exner equations.

Keywords: Hyperbolic systems, Lyapunov function, Saint-Venant equations, Stabilisation.

1. INTRODUCTION

Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g.Serre [2001]). Many physical systems having an engineering interest are described by systems of one-dimensional hyperbolic balance laws. Typical examples are for instance the telegrapher equations for electrical lines, the shallow water (Saint-Venant) equations for open channels, the isothermal Euler equations for gas flow in pipelines or the Aw-Rascle equations for road traffic. In this paper, our concern is to analyse the stability (in the sense of Lyapunov) of the steady-states of such systems. The analysis is developped for a general class of linear systems of one-dimensional hyperbolic balance laws. As a matter of illustration, an application to linearised Saint-Venant-Exner equations for open channels with a moving sediment bed is presented.

We are concerned with $n \times n$ linear hyperbolic systems of balance laws of the form:

$$\partial_t\boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x\boldsymbol{\xi} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0} \quad t \in [0,+\infty), \quad x \in (0,L) \tag{1}$$

where $\boldsymbol{\xi} : [0,+\infty) \times [0,L] \to \mathbb{R}^n$, $\boldsymbol{\Lambda}$ and $\mathbf{M}$ are real $n \times n$ matrices. Without loss of generality, we may assume that $\boldsymbol{\Lambda}$ is diagonal with non-zero real diagonal entries such that

$$\begin{aligned} &\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \\ &\lambda_i > 0 \ \ \forall i \in \{1,\ldots,m\}, \\ &\lambda_i < 0 \ \ \forall i \in \{m+1,\ldots,n\}. \end{aligned}$$

We introduce the notations

$$\begin{cases} \boldsymbol{\xi}^+ = (\xi_1,\ldots,\xi_m) \\ \boldsymbol{\xi}^- = (\xi_{m+1},\ldots,\xi_n) \end{cases} \quad \text{such that } \boldsymbol{\xi} = (\boldsymbol{\xi}^{+T}, \boldsymbol{\xi}^{-T})$$

and

$$\begin{cases} \boldsymbol{\Lambda}^+ = \mathrm{diag}\{\lambda_1,\ldots,\lambda_m\} \\ \boldsymbol{\Lambda}^- = \mathrm{diag}\{|\lambda_{m+1}|,\ldots,|\lambda_n|\} \end{cases} \Rightarrow \boldsymbol{\Lambda} = \mathrm{diag}\{\boldsymbol{\Lambda}^+, -\boldsymbol{\Lambda}^-\}.$$

With these notations, the linear hyperbolic system (1) is writtem

$$\partial_t \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Lambda}^+ & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Lambda}^- \end{pmatrix} \partial_x \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0}. \tag{2}$$

Our concern is to analyze the exponential stability of this system under boundary conditions of the form

$$\begin{pmatrix} \boldsymbol{\xi}^+(t,0) \\ \boldsymbol{\xi}^-(t,L) \end{pmatrix} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}^+(t,L) \\ \boldsymbol{\xi}^-(t,0) \end{pmatrix} \tag{3}$$

and an initial condition of the form

$$\boldsymbol{\xi}(0,x) = \boldsymbol{\xi}^o(x), \quad x \in (0,L). \tag{4}$$

The classical definition of a solution to the Cauchy problem (2)-(3)-(4) in $L^2((0,L);\mathbb{R}^n)$ is

Definition *1.* Let $\boldsymbol{\xi}^0 \in L^2((0,L);\mathbb{R}^n)$. A map $\boldsymbol{\xi} : [0,+\infty) \times (0,L) \to \mathbb{R}^n$ is a solution of the Cauchy problem (2)-(3)-(4) if $\boldsymbol{\xi} \in C^0([0,+\infty); L^2((0,L);\mathbb{R}^n))$ is such that, for every $\varphi = (\varphi_+^T, \varphi_-^T)^T \in C^1([0,+\infty) \times [0,L];\mathbb{R}^n)$ with compact support and satisfying

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$$\begin{pmatrix} \varphi_+(t,L) \\ \varphi_-(t,0) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\Lambda}^+)^{-1} K_{00}^T \boldsymbol{\Lambda}^+ & (\boldsymbol{\Lambda}^+)^{-1} K_{10}^T \boldsymbol{\Lambda}^- \\ (\boldsymbol{\Lambda}^-)^{-1} K_{01}^T \boldsymbol{\Lambda}^+ & (\boldsymbol{\Lambda}^-)^{-1} K_{11}^T \boldsymbol{\Lambda}^- \end{pmatrix} \begin{pmatrix} \varphi_+(t,0) \\ \varphi_-(t,L) \end{pmatrix}$$

we have

$$\int_0^{+\infty} \int_0^L (\varphi_t^T + \varphi_x^T \boldsymbol{\Lambda} + \varphi^T \mathbf{M}) \boldsymbol{\xi} dx dt + \int_0^L \varphi^T(0,x) \boldsymbol{\xi}^0(x) dx = 0.$$

With this definition, we have the following classical result (see e.g. [Coron, 2007, Section 2.1 and Section 2.3] for methods to get this result).

Proposition *1.* For every $\boldsymbol{\xi}^0 \in L^2((0,L);\mathbb{R}^n)$, the Cauchy problem (2)-(3)-(4) has a unique solution. Moreover, for every $T > 0$, there exists $C(T) > 0$ such that, for every $\boldsymbol{\xi}^0 \in L^2((0,L);\mathbb{R}^n)$, the solution to the Cauchy problem (2)-(3)-(4) satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^n)} \leqslant C(T) \|\boldsymbol{\xi}^0\|_{L^2((0,L);\mathbb{R}^n)}, \forall t \in [0,T].$$

We adopt the following definition for the exponential stability of the linear hyperbolic system (2)-(3).

Definition *2.* The linear hyperbolic system (2)-(3) is exponentially stable if there exist $\nu > 0$ and $C > 0$ such that, for every $\boldsymbol{\xi}^0 \in L^2((0,L);\mathbb{R}^n)$, the solution to the Cauchy problem (2)-(3)-(4) satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^n)} \leqslant Ce^{-\nu t} \|\boldsymbol{\xi}^0\|_{L^2((0,L);\mathbb{R}^n)}, \forall t \in [0,+\infty).$$

The problem of analysing the asymptotic stability of the equilibrium $\boldsymbol{\xi} \equiv \mathbf{0}$ for systems of **conservation** laws $\partial_t \boldsymbol{\xi} + \boldsymbol{C}(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi} = \mathbf{0}$ has been considered in the literature for more than 25 years. To our knowledge, first results were published by Slemrod [1983] and by Greenberg and Li [1984] for the special case of 2×2 systems. They rely on the method of characteristics and establish the exponential convergence of the solutions in $C^1(0,L)$-norm. A generalization to $n \times n$ systems was given by Li and his collaborators (see e.g. the textbook Li [1994]).

A different approach that uses a Lyapunov function has been introduced in Coron et al. [2007, 2008] in order to prove the exponential convergence of **nonlinear** systems of **conservation** laws in $H^2(0,L)$-norm. The Lyapunov function is related to a similar function used in Coron [1999] for the stabilization of the Euler equation of incompressible fluids. It is also similar to the Lyapunov function used in Xu and Sallet [2002] to analyse the stability of a class of linear symmetric hyperbolic systems.

In the present paper, our main contribution is to explain how this Lyapunov stability analysis can be further extended to the case of **linear** hyperbolic sytems of **balance** laws $\partial_t \boldsymbol{\xi} + \boldsymbol{\Lambda} \partial_x \boldsymbol{\xi} - \mathbf{M} \boldsymbol{\xi} = \mathbf{0}$. In Theorem 1 we first give a general implicit formulation of sufficient stability conditions. Then in Theorem 2, we show that, when the matrix $\mathbf{M}$ is diagonally marginally stable, an explicit boundary dissipativity condition holds for stability with convergence in $L^2(0,L)$-norm. Finally, in Section 4, we present an application to the boundary feedback control of open channels represented by linearised Saint-Venant-Exner equations.

2. LYAPUNOV STABILITY : GENERAL SUFFICIENT CONDITIONS

The system (2)-(3)-(4) is rewritten as

$$\partial_t \boldsymbol{\xi} + \boldsymbol{\Lambda} \partial_x \boldsymbol{\xi} - \mathbf{M} \boldsymbol{\xi} = \mathbf{0} \;\; t \in [0,+\infty), \;\; x \in (0,L), \tag{5a}$$
$$\mathbf{K}_0 \boldsymbol{\xi}(t,0) + \mathbf{K}_1 \boldsymbol{\xi}(t,L) = \mathbf{0}, \;\; t \in [0,+\infty), \tag{5b}$$
$$\boldsymbol{\xi}(0,x) = \boldsymbol{\xi}^o(x), \;\; x \in (0,L) \tag{5c}$$

with

$$\mathbf{K}_0 := \begin{pmatrix} I & -K_{01} \\ \mathbf{0} & -K_{11} \end{pmatrix}, \quad \mathbf{K}_1 = \begin{pmatrix} -K_{00} & \mathbf{0} \\ -K_{10} & I \end{pmatrix}$$

The following candidate Lyapunov function is introduced:

$$V = \int_0^L \boldsymbol{\xi}^T \mathbf{P}(x) \boldsymbol{\xi} dx \tag{6}$$

The weighting matrix $\mathbf{P}(x)$ is defined as follows: $\mathbf{P}(x) \triangleq \text{diag}\{p_i e^{-\sigma_i \mu x}, i = 1, \ldots, n\}$, with $\mu > 0$, $p_i > 0$ positive real numbers and $\sigma_i = \text{sign}(\lambda_i)$.

The time derivative of V along the solutions of (5) is

$$\begin{aligned} \dot{V} &= \int_0^L \left(\partial_t \boldsymbol{\xi}^T \mathbf{P}(x) \boldsymbol{\xi} + \boldsymbol{\xi}^T \mathbf{P}(x) \partial_t \boldsymbol{\xi} \right) dx \\ &= \int_0^L \left(-\partial_x \boldsymbol{\xi}^T \boldsymbol{\Lambda} \mathbf{P}(x) \boldsymbol{\xi} - \boldsymbol{\xi}^T \mathbf{P}(x) \boldsymbol{\Lambda} \partial_x \boldsymbol{\xi} \right. \\ &\qquad \left. + \boldsymbol{\xi}^T \mathbf{M}^T \mathbf{P}(x) \boldsymbol{\xi} + \boldsymbol{\xi}^T \mathbf{P}(x) \mathbf{M} \boldsymbol{\xi} \right) dx. \end{aligned}$$

Defining the positive diagonal matrix

$$\mathbf{Q}(x) \triangleq \text{diag}\{p_i |\lambda_i| e^{-\sigma_i \mu x}, i = 1, \ldots, n\}$$

and integrating by parts, we obtain:

$$\begin{aligned} \dot{V} = &- \int_0^L \partial_x \left[\boldsymbol{\xi}^T \boldsymbol{\Lambda} \mathbf{P}(x) \boldsymbol{\xi} \right] dx \\ &+ \int_0^L \boldsymbol{\xi}^T \Big(-\mu \mathbf{Q}(x) + \mathbf{M}^T \mathbf{P}(x) + \mathbf{P}(x) \mathbf{M} \Big) \boldsymbol{\xi} \, dx \\ = &- \left[\boldsymbol{\xi}^T \boldsymbol{\Lambda} \mathbf{P}(x) \boldsymbol{\xi} \right]_0^L \\ &+ \int_0^L \boldsymbol{\xi}^T \Big(-\mu \mathbf{Q}(x) + \mathbf{M}^T \mathbf{P}(x) + \mathbf{P}(x) \mathbf{M} \Big) \boldsymbol{\xi} \, dx \\ = &- \left[\boldsymbol{\xi}^T(t,L) \boldsymbol{\Lambda} \mathbf{P}(L) \boldsymbol{\xi}(t,L) - \boldsymbol{\xi}^T(t,0) \boldsymbol{\Lambda} \mathbf{P}(0) \boldsymbol{\xi}(t,0) \right] \\ &+ \int_0^L \boldsymbol{\xi}^T \Big(-\mu \mathbf{Q}(x) + \mathbf{M}^T \mathbf{P}(x) + \mathbf{P}(x) \mathbf{M} \Big) \boldsymbol{\xi} \, dx. \end{aligned}$$

The system (5) is exponentially stable if this function $\dot{V}$ is negative definite. We have thus shown the following result.

Theorem 1. The system (5) is exponentially stable if there exist $\mu > 0$ and $p_i > 0$ $i = 1, \ldots, n$ such that

C1. The boundary quadratic form $\boldsymbol{\xi}^T(t,L) \boldsymbol{\Lambda} \mathbf{P}(L) \boldsymbol{\xi}(t,L) - \boldsymbol{\xi}^T(t,0) \boldsymbol{\Lambda} \mathbf{P}(0) \boldsymbol{\xi}(t,0)$ is positive definite under the constraint of the linear boundary condition $\mathbf{K}_0 \boldsymbol{\xi}(t,0) + \mathbf{K}_1 \boldsymbol{\xi}(t,L) = \mathbf{0}$, $\forall t \geqslant 0$ along the solutions of the system (2)-(3)-(4);
C2. The matrix $-\mu \mathbf{Q}(x) + \mathbf{M}^T \mathbf{P}(x) + \mathbf{P}(x) \mathbf{M}$ is negative definite $\forall x \in (0,L)$.

Remark 1. Boundary conditions that satisfy condition C1 are called *Dissipative Boundary Conditions*. Condition C1 is satisfied if and only if the leading principal minors of order $> 2n$ of the matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{K}_0 & \mathbf{K}_1 \\ -\mathbf{K}_0^T & \mathbf{\Lambda P}(0) & \mathbf{0} \\ -\mathbf{K}_1^T & \mathbf{0} & -\mathbf{\Lambda P}(L) \end{pmatrix}$$

are strictly positive (see Väliaho [1982]).

Remark 2. For $\mu > 0$ sufficiently small, condition C2 is satisfied if there exist $p_i > 0$ such that $\mathbf{M}^T\mathbf{P}(0) + \mathbf{P}(0)\mathbf{M}$ is a negative semi-definite matrix. A question that has attracted some attention in the literature concerns the conditions on a matrix $\mathbf{M}$ for which there exist a diagonal positive matrix $\mathbf{P}$ such that $\mathbf{M}^T\mathbf{P} + \mathbf{PM}$ is negative definite (see e.g. Barker et al. [1978] for an early reference and Shorten et al. [2009] for a recent reference). When such a matrix $\mathbf{P}$ exists, the matrix $\mathbf{M}$ is said to be *diagonally stable* (because it is stable and the associated Lyapunov equation is satisfied with a diagonal $\mathbf{P}$). Here, with condition C2, we are rather concerned with a diagonally *marginally* stable matrix $\mathbf{M}$ which means that we require only that $\mathbf{M}^T\mathbf{P}+\mathbf{PM}$ be negative semidefinite.

For general systems of the form (1), it is rather clear that more explicit stability conditions can be derived only on a case by case basis when the internal structure and the numerical values of the involved matrices $\mathbf{\Lambda}, \mathbf{M}, \mathbf{K}_0, \mathbf{K}_1$ are at least partially specified. In the next section, we investigate the special case of system (5) when $\mathbf{M}$ is diagonally marginally stable and we show that a fairly simple explicit dissipative boundary condition can be given in that case. This is of great practical interest since models with diagonally marginally stable $\mathbf{M}$ appear in many concrete physical and engineering applications as we illustrate in Section 4 with the example of Saint-Venant-Exner equations for open channels with non-uniform bathymetry.

3. DISSIPATIVE BOUNDARY CONDITION WHEN M IS DIAGONALLY MARGINALLY STABLE

In this section, we will present a variant of Theorem 1 with an explicit characterisation of a sufficient dissipative boundary condition which guarantees the system exponential stability in the case where $\mathbf{M}$ is diagonally marginally stable. We consider again the system written in the form (2)-(3)-(4) and we define the matrix

$$\mathbf{K} := \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}.$$

Let $\mathcal{D}_p$ denote the set of diagonal $p \times p$ real matrices with strictly positive diagonal entries. We define the set $\mathcal{P}$ as follows :

$$\mathcal{P} \triangleq \left\{\mathrm{P} \in \mathcal{D}_n \text{ such that } \mathbf{M}^T\mathrm{P}+\mathrm{P}\mathbf{M} \text{ is negative semidefinite}\right\}$$

With the above notations, the candidate Lyapunov function (6) is written

$$V = \int_0^L \left[(\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\xi}^+)e^{-\mu x} + (\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\xi}^-)e^{\mu x}\right] dx \qquad (7)$$

with $P_0 \in \mathcal{D}_m$, $P_1 \in \mathcal{D}_{n-m}$ and $\mu > 0$. We introduce the following norm for the matrix $\mathbf{K}$:

$$\rho(\mathbf{K}) \triangleq \inf \left\{\|\Delta\mathbf{K}\Delta^{-1}\|, \Delta \in \mathcal{S}\right\}$$

where $\| \ \|$ denotes the usual matrix 2-norm and the set $\mathcal{S}$ is defined as follows:

$$\mathcal{S} := \Big\{\Delta = \mathrm{diag}\left\{D_0, D_1\right\}, D_0^2 = P_0\Lambda^+, \qquad (8)$$

$$D_1^2 = P_1\Lambda^-, \mathrm{P} = \mathrm{diag}\left\{P_0, P_1\right\} \in \mathcal{P}\Big\}. \qquad (9)$$

We have the following Theorem.

Theorem 2. If $\mathbf{M}$ is diagonally marginally stable, if the boundary dissipative condition $\rho(\mathbf{K}) < 1$ is satisfied, then the linear hyperbolic system (2)-(3) is exponentially stable.

Proof. The time derivative of the Lyapunov function V is

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \qquad (10)$$

with

$$\dot{V}_1 \triangleq -\left[\boldsymbol{\xi}^{+T} P_0 \mathbf{\Lambda}^+ \boldsymbol{\xi}^+ e^{-\mu x}\right]_0^L + \left[\boldsymbol{\xi}^{-T} P_1 \mathbf{\Lambda}^- \boldsymbol{\xi}^- e^{\mu x}\right]_0^L$$

$$\dot{V}_2 \triangleq \int_0^L \boldsymbol{\xi}^T \left(-\mu P(x)|\mathbf{\Lambda}| + \mathbf{M}^T P(x) + P(x)\mathbf{M}\right) \boldsymbol{\xi}\, dx$$

and $P(x) \triangleq \mathrm{diag}\left\{P_0 e^{-\mu x}, P_1 e^{\mu x}\right\}$, $|\mathbf{\Lambda}| \triangleq \mathrm{diag}\left\{\mathbf{\Lambda}^+, \mathbf{\Lambda}^-\right\}$.

In order to prove that the boundary condition (3) is dissipative we will show that P_0, P_1 and μ can be selected such that $\dot{V}$ is a negative definite function. In order to prove that $\dot{V}_1$ is a negative definite quadratic form, we introduce the following notations:

$$\boldsymbol{\xi}_0^-(t) \triangleq \boldsymbol{\xi}^-(t,0) \quad \boldsymbol{\xi}_1^+(t) \triangleq \boldsymbol{\xi}^+(t,L).$$

Using the boundary condition (3), we have

$$\begin{aligned} \dot{V}_1 &= -\left[\boldsymbol{\xi}^{+T} P_0 \mathbf{\Lambda}^+ \boldsymbol{\xi}^+ e^{-\mu x}\right]_0^L + \left[\boldsymbol{\xi}^{-T} P_1 \mathbf{\Lambda}^- \boldsymbol{\xi}^- e^{\mu x}\right]_0^L \\ &= -\left(\boldsymbol{\xi}_1^{+T} P_0 \mathbf{\Lambda}^+ \boldsymbol{\xi}_1^+\, e^{-\mu L} + \boldsymbol{\xi}_0^{-T} P_1 \mathbf{\Lambda}^- \boldsymbol{\xi}_0^-\right) \\ &\quad + \left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T\right) P_0 \mathbf{\Lambda}^+ \left(K_{00}\boldsymbol{\xi}_1^+ + K_{01}\boldsymbol{\xi}_0^-\right) \\ &\quad + \left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T\right) P_1 \mathbf{\Lambda}^- \left(K_{10}\boldsymbol{\xi}_1^+ + K_{11}\boldsymbol{\xi}_0^-\right) e^{\mu L}. \end{aligned}$$

Since $\mathbf{M}$ is diagonally marginally stable and $\rho(\mathbf{K}) < 1$ by assumption, we know that the set $\mathcal{P}$ is not empty and we can select matrices P_0 and P_1 such that

$$\mathrm{P} = \mathrm{diag}\left\{P_0, P_1\right\} \in \mathcal{P}, D_0^2 = P_0\Lambda^+, D_1^2 = P_1\Lambda^-,$$

$$\Delta = \mathrm{diag}\left\{D_0, D_1\right\} \text{ and } \|\Delta\mathbf{K}\Delta^{-1}\| < 1. \qquad (11)$$

We define

$$\mathbf{z} \triangleq \begin{pmatrix} D_0\boldsymbol{\xi}_1^+ \\ D_1\boldsymbol{\xi}_0^- \end{pmatrix}.$$

Then, using inequality (11), we have

$$\begin{aligned} &\left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T\right) P_0 \mathbf{\Lambda}^+ \left(K_{00}\boldsymbol{\xi}_1^+ + K_{01}\boldsymbol{\xi}_0^-\right) \\ &+ \left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T\right) P_1 \mathbf{\Lambda}^- \left(K_{10}\boldsymbol{\xi}_1^+ + K_{11}\boldsymbol{\xi}_0^-\right) \\ &= \|\Delta\mathbf{K}\Delta^{-1}\mathbf{z}\|^2 \\ &< \|\mathbf{z}\|^2 = \boldsymbol{\xi}_1^{+T} P_0 \mathbf{\Lambda}^+ \boldsymbol{\xi}_1^+ + \boldsymbol{\xi}_0^{-T} P_1 \mathbf{\Lambda}^- \boldsymbol{\xi}_0^-. \end{aligned}$$

It follows readily that μ can be taken sufficiently small such that $\dot{V}_1$ is a negative definite quadratic form $\forall t \geqslant 0$.

Moreover, since $\mathbf{M}^T\mathrm{P} + \mathrm{P}\mathbf{M}$ is negative semidefinite (because $\mathrm{P} \in \mathcal{P}$), $\mu > 0$ can be taken sufficiently small such that $\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}$ is negative semidefinite for all x in $[0, L]$ and therefore that $-\mu\mathbf{P}(x)|\mathbf{\Lambda}| + \mathbf{M}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{M}$ is negative definite for all x in $[0, L]$. It follows that for μ sufficiently small there exist $\alpha > 0$ such that

$$\dot{V}_2 < -\alpha V \implies \dot{V} = \dot{V}_1 + \dot{V}_2 < -\alpha V \quad \forall \boldsymbol{\xi} \neq 0.$$

Consequently the solutions of the system (2)-(3)-(4) exponentially converge to $\mathbf{0}$ in L^2-norm.

4. APPLICATION TO THE SAINT-VENANT-EXNER MODEL

In the previous section, we have shown that for systems with $\mathbf{M}$ diagonally marginally stable, the dissipative boundary condition $\rho(\mathbf{K}) < 1$ is a sufficient exponential stability condition. This is true in particular for hydraulic systems described by linearised shallow-water equations as long as the subcritical flow condition is satisfied as we shall illustrate in the present section for an open channel with variable bathymetry.

We consider a pool of a prismatic sloping open channel with a rectangular cross-section, a unit width and a moving bathymetry (because of sediment transportation). The state variables of the model are: the water depth $H(t,x)$, the water velocity $V(t,x)$ and the bathymetry $B(t,x)$ which is the depth of the sediment layer above the channel bottom. The dynamics of the system are described by the coupling of Saint-Venant and Exner equations:

$$\begin{aligned}\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + H\frac{\partial V}{\partial x} &= 0,\\ \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + g\frac{\partial B}{\partial x} &= gS_B - C_f\frac{V^2}{H},\\ \frac{\partial B}{\partial t} + a|V|^{m-1}\frac{\partial V}{\partial x} &= 0.\end{aligned} \tag{12}$$

In these equations, g is the gravity constant, S_B is the bottom slope of the channel, C_f is a friction coefficient and a is a parameter that encompasses porosity and viscosity effects on the sediment dynamics.

4.1 Steady-state and Linearisation

A *steady-state* is a constant state H^*, V^*, B^* which satisfies the relation

$$gS_BH^* = C_fV^{*2}.$$

In order to linearise the model, we define the deviations of the state $H(t,x), V(t,x), B(t,x)$ with respect to the steady-state:

$$\begin{aligned}h(x,t) &= H(x,t) - H^*,\\ u(x,t) &= V(x,t) - V^*,\\ b(x,t) &= B(x,t) - B^*.\end{aligned}$$

Then the linearised Saint-Venant-Exner model (12) around a steady-state is

$$\frac{\partial h}{\partial t} + V^*\frac{\partial h}{\partial x} + H^*\frac{\partial u}{\partial x} = 0, \tag{13a}$$

$$\frac{\partial u}{\partial t} + V^*\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial b}{\partial x} = C_f\frac{V^{*2}}{H^{*2}}h - 2C_f\frac{V^*}{H^*}u, \tag{13b}$$

$$\frac{\partial b}{\partial t} + aV^{*2}\frac{\partial u}{\partial x} = 0. \tag{13c}$$

4.2 Characteristic (Riemann) coordinates

In the sequel, we set the parameter m to 3 as in Hudson and Sweby [2003]. In matrix form, the linearised model (13) can be written as

$$\frac{\partial W}{\partial t} + \mathbf{A}(W^*)\frac{\partial W}{\partial x} = \mathbf{B}(W^*)W \tag{14}$$

where

$$W = \begin{pmatrix} h \\ u \\ b \end{pmatrix}, \quad \mathbf{A}(W^*) = \begin{pmatrix} V^* & H^* & 0 \\ g & V^* & g \\ 0 & aV^{*2} & 0 \end{pmatrix},$$

$$\mathbf{B}(W^*) = \begin{pmatrix} 0 & 0 & 0 \\ C_f\dfrac{V^{*2}}{H^{*2}} & -2C_f\dfrac{V^*}{H^*} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exact, but rather complicated expressions of the eigenvalues of $\mathbf{A}(W^*)$ can be obtained by using the *Cardano-Vieta* method, see Hudson and Sweby [2003]. Once the eigenvalues are obtained, the corresponding left eigenvectors can be computed as

$$L_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} \begin{pmatrix} (V^* - \lambda_i)(V^* - \lambda_j) + gH^* \\ H^*(2V^* - \lambda_i - \lambda_j) \\ gH^* \end{pmatrix},$$
$$k \neq i \neq j \in \{1,2,3\}.$$

We multiply (14) by L_k^T in order to rewrite the model in terms of the characteristic coordinates Ψ_k ($k = 1,2,3$). Then we obtain

$$\frac{\partial \Psi_k}{\partial t} + \lambda_k\frac{\partial \Psi_k}{\partial x} = L_k^T\mathbf{B}W, \qquad k = 1,2,3, \tag{15}$$

where

$$\Psi_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\Big[\big((V^* - \lambda_i)(V^* - \lambda_j) + gH^*\big)h + \big(H^*(2V^* - \lambda_i - \lambda_j)\big)u + gH^*b\Big]. \tag{16}$$

Conversely, we can express h, u and b in terms the characteristic coordinates:

$$h = \Psi_1 + \Psi_2 + \Psi_3, \tag{17a}$$

$$u = \frac{1}{H^*}\Big[\big(\lambda_1 - V^*\big)\Psi_1 + \big(\lambda_2 - V^*\big)\Psi_2 + \big(\lambda_3 - V^*\big)\Psi_3\Big], \tag{17b}$$

$$b = \frac{1}{gH^*}\Big[\Big(\big(\lambda_1 - V^*\big)^2 - gH^*\Big)\Psi_1 + \Big(\big(\lambda_2 - V^*\big)^2 - gH^*\Big)\Psi_2 + \Big(\big(\lambda_3 - V^*\big)^2 - gH^*\Big)\Psi_3\Big]. \tag{17c}$$

Using the new variables Ψ_k, the RHS of (15) writes:

$$\begin{aligned}L_k^T\mathbf{B}W &= \gamma_1 l_2^k h + \gamma_2 l_2^k u\\ &= \sum_{s=1}^{3}\Big(\gamma_1 + \gamma_2\frac{\lambda_s - V^*}{H^*}\Big)l_2^k\Psi_s,\end{aligned} \tag{18}$$

where

$$\gamma_1 = C_f\frac{V^{*2}}{H^{*2}}, \quad \gamma_2 = -2C_f\frac{V^*}{H^*},$$

and l_2^k is the second component of L_k^T. Equation (18) can be rewritten as:

$$L_k^T\mathbf{B}W = C_f\frac{V^*}{H^*}\frac{(2V^* - \lambda_i - \lambda_j)}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\sum_{s=1}^{3}\Big(3V^* - 2\lambda_s\Big)\Psi_s,$$
$$k \neq i \neq j \in \{1,2,3\}. \tag{19}$$

For the sake of simplicity, we introduce the following notation θ_k:

$$\theta_k = C_f \frac{V^*}{H^*} \frac{(2V^* - \lambda_i - \lambda_j)}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}.$$

Then equation (15) writes:

$$\frac{\partial \xi_k}{\partial t} + \lambda_k \frac{\partial \xi_k}{\partial x} + \sum_{s=1}^{3} (2\lambda_s - 3V^*)\theta_s \xi_s = 0 \qquad (k = 1, 2, 3) \qquad (20)$$

where the characteristic coordinates are now defined as

$$\xi_k = \frac{1}{\theta_k} \Psi_k.$$

From (20), the linearised model (15) in characteristic form may now be written as

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\Lambda} \frac{\partial \boldsymbol{\xi}}{\partial x} - \mathbf{M}\boldsymbol{\xi} = 0, \qquad (21)$$

where

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T, \quad \boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \lambda_3)$$

and

$$\mathbf{M} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix},$$

with

$$\alpha_k = \Big(3V^* - 2\lambda_k\Big)\theta_k$$

From Hudson and Sweby [2003], the three eigenvalues of the matrix $\mathbf{A}$ are such that

$$\lambda_1 < 0 < \lambda_2 \ll \lambda_3 \qquad (22)$$

with λ_1 and λ_2 the characteristic velocities of the water flow and λ_2 the characteristc velocity of the sediment motion. Obviously the sediment motion is much slower than the water flow. On the basis of (22), we are now going to determine the sign of the coefficients α_k in $\mathbf{M}$.

For α_1, we have:

$$\alpha_1 = C_f \frac{V^*}{\bar{H}} \Big(3V^* - 2\lambda_1\Big) \frac{2V^* - \lambda_3 - \lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}. \qquad (23)$$

Since $\lambda_1 < 0$, we have $3V^* - 2\lambda_1 > 0$. The trace of the matrix $\mathbf{A}$ is: $tr(\mathbf{A}) = 2V^*$. Then

$$\lambda_1 + \lambda_2 + \lambda_3 = 2V^*$$

and, therefore,

$$2V^* - \lambda_2 - \lambda_3 = \lambda_1 < 0.$$

Using (22), we infer that:

$$\lambda_1 - \lambda_2 < 0 \quad \text{and} \quad \lambda_1 - \lambda_3 < 0.$$

From the aboves inequalities, we conclude that $\alpha_1 < 0$.

For α_2, we have

$$\alpha_2 = C_f \frac{V^*}{\bar{H}} \Big(3V^* - 2\lambda_2\Big) \frac{2V^* - \lambda_1 - \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}. \qquad (24)$$

Since the sediment motion is much more slower than the water flow, we may assume that $3V^* - 2\lambda_2 > 0$. Moreover, using the trace of $\mathbf{A}$, we have

$$2V^* - \lambda_1 - \lambda_3 = \lambda_2 > 0.$$

From (22), we have also

$$\lambda_2 - \lambda_1 > 0 \quad \text{and} \quad \lambda_2 - \lambda_3 < 0.$$

From these inequalities, we conclude that $\alpha_2 < 0$.

Finally, for α_3, we have

$$\alpha_3 = C_f \frac{V^*}{\bar{H}} \Big(3V^* - 2\lambda_3\Big) \frac{2V^* - \lambda_1 - \lambda_2}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}. \qquad (25)$$

From (22), we have

$$\lambda_3 - \lambda_1 > 0 \quad \text{and} \quad \lambda_3 - \lambda_2 > 0,$$

and from the trace of $\mathbf{A}$, we have

$$2V^* - \lambda_1 - \lambda_2 = \lambda_3 > 0.$$

Using again the trace of $\mathbf{A}$, we have also

$$3V^* - 2\lambda_3 = 2\lambda_1 + 2\lambda_2 - V^*.$$

Since λ_2 is small, $3V^* - 2\lambda_3$ has the same sign as $2\lambda_1 - V^*$. Since $\lambda_1 < 0$ is negative, we obtain: $3V^* - 2\lambda_3 < 0$ and consequently $\alpha_3 < 0$.

Hence all the coefficients α_k in matrix $\mathbf{M}$ are strictly negative.

4.3 Lyapunov stability under boundary feedback control

We are now going to show how Theorem 2 may be applied to analyse the stability of an open channel under boundary feedback control.

We assume that the channel is provided with hydraulic control devices (pumps, valves, mobile spillways, sluice gates, ...) which are located at both ends and allow to assign the values of the flow-rate. On-line measurements of the water levels at both ends $h(t, 0) + b(t, 0)$ and $h(t, L) + b(t, L)$ are assumed to be available for feedback control. Obviously, instead of the flow-rates, we may as well consider the velocities $u(t, 0)$ and $u(t, L)$ as being the control actions. Therefore we introduce the following boundary conditions:

$$u(t, 0) = -k_1 h(t, 0), \qquad (26a)$$
$$u(t, L) = -k_2(h(t, L) + b(t, L)), \qquad (26b)$$
$$b(t, 0) = 0. \qquad (26c)$$

Conditions (26a)-(26b) are linear feedback static control laws with the tuning parameters k_1 and k_2. The third condition is supposed to be a physical constraint. In order to invoke Theorem 2, we have

1) to find a matrix $\mathbf{P} = \text{diag}\{p_1, p_2, p_3\}$ such that $\boldsymbol{\xi}^T\big(\mathbf{M}^T\mathbf{P} + \mathbf{P}\mathbf{M}\big)\boldsymbol{\xi}$ is a negative semi definite quadratic form,

2) to find the range of admissible values of the tuning parameters k_i such that the boundary conditions are dissipative.

For the matrix $\mathbf{P}$, a straightforward choice is $p_i := |\alpha_i|$ (i = 1,2,3) since then the quadratic form is

$$\boldsymbol{\xi}^T\big(\mathbf{M}^T\mathbf{P} + \mathbf{P}\mathbf{M}\big)\boldsymbol{\xi} = -2\left(\sum_{i=1}^{3} |\alpha_i|\xi_i\right)^2.$$

In order to check the dissipativity condition $\rho(\mathbf{K}) < 1$, we have to compute the matrix $\mathbf{K}$ and the matrix Δ. It is easy to verify that, in the Riemann coordinates $\boldsymbol{\xi}$, the boundary conditions (26) can be written in the form (3) as follows:

$$\begin{pmatrix} \xi_1(t,L) \\ \xi_2(t,0) \\ \xi_3(t,0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \chi_2(k_2) & \chi_3(k_2) \\ \pi_2(k_1) & 0 & 0 \\ \pi_3(k_1) & 0 & 0 \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} \xi_1(t,0) \\ \xi_2(t,L) \\ \xi_3(t,L) \end{pmatrix} \tag{27}$$

where π_i and χ_j are homographic transformations of the tuning parameters k_1 and k_2 respectively.

Moreover, we have, by definition, that $\mathbf{P} = \operatorname{diag}\{|\alpha_1|, |\alpha_2|, |\alpha_3|\}$ and $|\mathbf{\Lambda}| = \operatorname{diag}\{|\lambda_1|, \lambda_2, \lambda_3\}$. Consequently:

$$\Delta = \operatorname{diag}\left\{\sqrt{|\lambda_1||\alpha_1|}, \sqrt{\lambda_2|\alpha_2|}, \sqrt{\lambda_3|\alpha_3|}\right\}$$

and

$$\Delta\mathbf{K}\Delta^{-1} =$$

$$\begin{pmatrix} 0 & \chi_2(k_2)\sqrt{\dfrac{|\lambda_1||\alpha_1|}{\lambda_2|\alpha_2|}} & \chi_3(k_2)\sqrt{\dfrac{|\lambda_1||\alpha_1|}{\lambda_3|\alpha_3|}} \\ \pi_2(k_1)\sqrt{\dfrac{\lambda_2|\alpha_2|}{|\lambda_1||\alpha_1|}} & 0 & 0 \\ \pi_3(k_1)\sqrt{\dfrac{\lambda_3|\alpha_3|}{|\lambda_1||\alpha_1|}} & 0 & 0 \end{pmatrix}$$

Then, it is a matter of tedious but fairly straightforward calculations to show that

$$\|\Delta\mathbf{K}\Delta^{-1}\| < 1$$

if and only if the tuning parameters k_1 and k_2 are selected such that

$$\pi_2^2(k_1)\frac{\lambda_2|\alpha_2|}{|\lambda_1||\alpha_1|} + \pi_3^2(k_1)\frac{\lambda_3|\alpha_3|}{|\lambda_1||\alpha_1|} < 1$$

and

$$\chi_2^2(k_2)\frac{|\lambda_1||\alpha_1|}{\lambda_2|\alpha_2|} + \chi_3^2(k_2)\frac{|\lambda_1||\alpha_1|}{\lambda_3|\alpha_3|} < 1.$$

5. CONCLUSIONS

We have addressed the issue of stating sufficient boundary conditions for the exponential stability of linear hyperbolic systems of balance laws. In Theorem 1 we have first given a general implicit formulation of sufficient dissipative boundary conditions. Our analysis relies on the use of an explicit Lyapunov function. The exponential weight $e^{\pm\mu x}$ is essential to get a strict Lyapunov function.

Then in Theorem 2, we have shown that the explicit dissipativity condition $\rho(\mathbf{K}) < 1$ gives a convergence in $L^2(0, L)$-norm for systems of balance laws with a diagonally marginally stable matrix $\mathbf{M}$. This theorem has been applied to give tuning conditions for boundary feedback stabilisation of an open channel represented by the Saint-Venant-Exner model.

The same Lyapunov function cannot be directly used to analyse the local stability of the steady-states in the **nonlinear case**. In order to extend the Lyapunov stability analysis to the nonlinear case, the Lyapunov function has to be augmented (as shown in detail in Coron et al. [2007, 2008]) or modified (as discussed in Gugat and Herty [2011] and in Dick et al. [2010] for the special case of gas pipelines represented by isentropic Euler equations).

REFERENCES

G. P. Barker, A. Berman, and R. J. Plemmons. Positive diagonal solutions to the Lyapunov equations. *Linear and Multilinear Algebra*, 5(3):249–256, 1978.

J-M. Coron. *Control and Nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

J-M. Coron. On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain. *SIAM Journal of Control and Optimization*, 37(6):1874–1896, 1999.

J-M. Coron, B. d'Andréa-Novel, and G. Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, January 2007.

J-M. Coron, G. Bastin, and B. d'Andréa-Novel. Dissipative boundary conditions for one dimensional nonlinear hyperbolic systems. *SIAM Journal of Control and Optimization*, 47(3):1460–1498, 2008.

M. Dick, M. Gugat, and G. Leugering. Classical solutions and feedback stabilisation for the gas flow in a sequence of pipes. *Networks and Heterogeneous Media*, 5(4):691–709, 2010.

J.M. Greenberg and T. Li. The effect of boundary damping for the quasilinear wave equations. *Journal of Differential Equations*, 52:66–75, 1984.

M. Gugat and M. Herty. Existence of classical solutions and feedback stabilisation for the flow in gas networks. *ESAIM Control Optimisation and Calculus of Variations*, 17(1):28–51, 2011.

J. Hudson and P.K. Sweby. Formulations for numerically approximating hyperbolic systems governing sediment transport. *Journal of Scientific Computing*, 19:225–252, 2003.

T-T. Li. *Global Classical Solutions for Quasi-Linear Hyperbolic Systems.* Research in Applied Mathematics. Masson and Wiley, 1994.

D. Serre. Systems of conservation laws : a challenge for the XXIst century. In B. Engquist and W. Schmid, editors, *Mathematics Unlimited*, pages 1061–1080, 2001.

R. Shorten, O. Mason, and C. King. An alternative proof of the Barker, Berman, Plemmons (BBP) result on diagonal stability and extensions. *Linear Algebra and its Applications*, 430:34–40, 2009.

M. Slemrod. Boundary feedback stabilization for a quasilinear wave equation. In *Control Theory for Distributed Parameter Systems*, volume 54 of *Lecture Notes in Control and Information Sciences*, pages 221–237. Springer Verlag, 1983.

H. Väliaho. On the definity of quadratic forms subject to linear constraints. *Journal of Optimization Theory and Applications*, 38(1):143 – 145, 1982.

C.Z. Xu and G. Sallet. Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM Control Optimisation and Calculus of Variations*, 7:421–442, 2002.