Further results on boundary feedback stabilisation of 2×2 hyperbolic systems over a bounded interval

Georges Bastin * **Jean-Michel Coron** **

* *Department of Mathematical Engineering, Université catholique de Louvain, 4, Avenue G. Lemaitre, 1348 Louvain-La-Neuve, Belgium.*

** *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Institut Universitaire de France, 4, Place Jussieu, 75252 Paris Cedex, France.*

Abstract: Conditions for boundary feedback stabilisability of linear 2×2 hyperbolic systems over a bounded interval are investigated. The main result is to show that the existence of a quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result is used to give explicit conditions for the existence of stabilising linear boundary feedback control laws. The analysis is illustrated with an application to the boundary feedback stabilisation of open channels represented by Saint-Venant equations with non-uniform steady-states. *Copyright ©IFAC 2010*

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1. INTRODUCTION

In this paper we discuss the boundary feedback stabilisation of linear 2×2 hyperbolic systems over a bounded interval and its application to nonlinear systems with non-uniform steady-states.

Conditions for boundary feedback stabilisability of linear hyperbolic systems in canonical form are established in Section 3. Our main result is to show that the existence of a quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result is then used to give explicit conditions for the existence of linear boundary feedback control laws in two cases : (i) when the control is available on both sides of the system; (ii) when the control is available only on one side of the system.

Behind this analysis, our motivation is in fact to investigate the stabilisation of non-linear hyperbolic systems with non-uniform steady-states. We are particularly interested in the stabilisation of open-channels using hydraulic control devices. In Section 4, we show how our analysis can be applied to the design of stabilising control laws for open-channels represented by Saint-Venant equations with a non-uniform steady-state.

A preliminary proposition, which is a key result for our analysis, regarding the existence of functions satisfying certain differential inequalities is first given in Section 2.

2. A PRELIMINARY PROPOSITION

Let $L > 0$, let $a \in C^0([0, L])$ and $b \in C^0([0, L])$. We are interested in the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f > 0 \text{ in } [0, L], \tag{1}$$

$$g > 0 \text{ in } [0, L], \tag{2}$$

$$f' < 0 \text{ a.e. in } [0, L], \tag{3}$$

$$g' > 0 \text{ a.e. in } [0, L], \tag{4}$$

$$-f'g' > (af + bg)^2 \text{ a.e. in } [0, L]. \tag{5}$$

A necessary and sufficient condition for the existence of (f, g) is given in the following proposition.

Proposition 1. There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1) to (5) hold if and only if the maximal solution η of the Cauchy problem

$$\eta' = \left| a + b\eta^2 \right|, \ \eta(0) = 0, \tag{6}$$

is defined on $[0, L]$.

Remark 1. The function

$$(x, s) \in [0, L] \times \mathbb{R} \mapsto \left| a(x) + b(x)s^2 \right| \in \mathbb{R}$$

is continuous in $[0, L] \times \mathbb{R}$ and locally Lipschitz with respect to s. Hence the Cauchy problem (6) has a unique maximal solution.

Proof of Proposition 1. We start with the "only if" part. Let $f \in C^1([0, L])$ and $g \in C^1([0, L])$ be such that (1) to (5) hold. Let us define $h \in C^1([0, L])$ by

$$h(x) := \frac{1}{f(x)}, \ \forall x \in [0, L]. \tag{7}$$

(Note that, by (1), $f(x) \neq 0$ for every $x \in [0, L]$.) Then (1), (3) and (5) become respectively

$$h > 0 \text{ in } [0, L], \tag{8}$$
$$h' > 0 \text{ a.e. in } [0, L], \tag{9}$$
$$g'h' > (a + bgh)^2 \text{ a.e. in } [0, L]. \tag{10}$$

Note that, by (2) and (8), $g(x)h(x) > 0$ for every $x \in [0, L]$. This allows to define $w \in C^1([0, L])$ by

$$w(x) := \sqrt{g(x)h(x)}, \forall x \in [0, L]. \tag{11}$$

We have

$$w(0) > 0. \tag{12}$$

Note that

$$w' = \frac{1}{2\sqrt{gh}} (g'h + gh'). \tag{13}$$

From (2), (4), (8), (9) and (13), we have

$$w' > 0 \text{ a.e. in } [0, L]. \tag{14}$$

From (13) we have

$$w'^2 = \frac{1}{4gh} (g'h + gh')^2 = g'h' + \frac{1}{4gh} (g'h - gh')^2. \tag{15}$$

From (10), (11), (14) and (15), we have

$$w' > \left|a + bw^2\right| \text{ a.e. in } [0, L]. \tag{16}$$

From (6), (12), (16) and a classical theorem on ordinary differential equation, we have, on the interval of definition $I \subset [0, L]$ of η,

$$\eta < w. \tag{17}$$

This shows that $I = [0, L]$ and concludes the proof of the "only if" part of Proposition 1.

Let us now turn to the "if" part of Proposition 1. We assume that the maximal solution of the Cauchy problem (6) is defined on $[0, L]$. Then, if $\varepsilon > 0$ is small enough, the solution η_ε of the Cauchy problem

$$\eta'_\varepsilon = \left|a + b\eta_\varepsilon^2\right| + \varepsilon, \, \eta_\varepsilon(0) = \varepsilon, \tag{18}$$

is defined on $[0, L]$. We choose such a $\varepsilon > 0$. Note that

$$\eta_\varepsilon > 0 \text{ in } [0, L]. \tag{19}$$

Let us define $f \in C^1([0, L])$ and $g \in C^1([0, L])$ by

$$f(x) := \frac{1}{\eta_\varepsilon(x)}, \forall x \in [0, L], \tag{20}$$
$$g(x) := \eta_\varepsilon(x), \forall x \in [0, L]. \tag{21}$$

(Note that, by (19), f is well defined.) Clearly (1) and (2) hold. From (20) and (21), we have

$$f' = -\frac{\eta'_\varepsilon}{\eta_\varepsilon^2}, \tag{22}$$
$$g' = \eta'_\varepsilon. \tag{23}$$

From (18), (22) and (23), we have (3), (4) and

$$-f'g' = \frac{\eta'^2_\varepsilon}{\eta_\varepsilon^2}. \tag{24}$$

From (20) and (21), we have

$$(af + bg)^2 = \frac{1}{\eta_\varepsilon^2} \left(a + b\eta_\varepsilon^2\right)^2. \tag{25}$$

From (18), (24) and (25), we get (5).

This concludes the proof of the "if" part of Proposition 1.

Remark 2 With the proof of the "if" part of *Proposition 1*, we have in fact proved that if the maximal solution η of the Cauchy problem $\eta' = \left|a + b\eta^2\right|$, $\eta(0) = 0$ is defined on $[0, L]$, then there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f > 0 \text{ in } [0, L],$$
$$g > 0 \text{ in } [0, L],$$
$$f' < 0 \text{ in } [0, L], \tag{26}$$
$$g' > 0 \text{ in } [0, L], \tag{27}$$
$$-f'g' > (af + bg)^2 \text{ in } [0, L]. \tag{28}$$

The point is that inequalities (26)-(27)-(28) hold in $[0, L]$ instead of a.e. in $[0, L]$ for inequalities (3)-(4)-(5). Now it is obvious that the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1)-(2)-(26)-(27)-(28) hold implies the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1)-(2)-(3)-(4)-(5) hold. Hence we have in fact established the following more general result.

Proposition 2. The three following statements are equivalent:

- There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1)-(2)-(3)-(4)-(5) hold.
- There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1)-(2)-(26)-(27)-(28) hold.
- The maximal solution η of the Cauchy problem $\eta' = \left|a + b\eta^2\right|$, $\eta(0) = 0$ is defined on $[0, L]$.

3. STABILISATION OF LINEAR SYSTEMS

We consider the linear 2×2 hyperbolic system in canonical form

$$\begin{aligned} \partial_t y_1 + \lambda_1(x)\partial_x y_1 + a_2(x)y_2 = 0, \\ \partial_t y_2 - \lambda_2(x)\partial_x y_2 + b_1(x)y_1 = 0, \end{aligned} \tag{29}$$

under the boundary conditions

$$\begin{aligned} y_1(t, 0) &= u_1(t), \\ y_2(t, L) &= u_2(t), \end{aligned} \tag{30}$$

where $t \in [0, +\infty)$ is the time variable, $x \in [0, L]$ is the space variable, the functions λ_1, λ_2 are in $C^1([0, L], ; \mathbb{R}_+)$ and the functions a_2, b_1 are in $C^1([0, L]; \mathbb{R})$.

This is a control system where, at time t, the state is $(y_1(t, \cdot), y_2(t, \cdot))^T \in L^2(0, L)^2$ and the control is $(u_1(t), u_2(t))^T \in \mathbb{R}^2$. Our concern is to analyze, by using a control Lyapunov function, the stabilisability of this system with linear decentralised boundary feedback control laws.

Remark 3. There is no loss of generality in considering systems in canonical form (29). Indeed, for any linear 2×2 hyperbolic system, there always exist canonical coordinates which allow to transform the system into canonical form. This will be illustrated in Section 4.

We consider the following control Lyapunov function candidate

$$V(y) := \int_0^L \left(q_1(x)y_1^2(t, x) + q_2(x)y_2^2(t, x)\right)dx, \tag{31}$$

where $q_1 \in C^1([0, L]; (0, +\infty))$ and $q_2 \in C^1([0, L]; (0, +\infty))$ have to be determined. The time derivative $\dot{V}$ of V along the trajectories of (29)-(30) is

$$\begin{aligned}\dot{V} &= \int_0^L \big(2q_1y_1\partial_t y_1 + 2q_2y_2\partial_t y_2\big)dx \\ &= -\int_0^L \big(2q_1y_1(\lambda_1\partial_x y_1 + a_2y_2) \\ &\qquad +2q_2y_2(-\lambda_2\partial_x y_2 + b_1y_1)\big)dx \\ &= -B - \int_0^L I dx,\end{aligned} \tag{32}$$

with

$$\begin{aligned}B &:= \lambda_1(L)q_1(L)y_1^2(t,L) - \lambda_2(L)q_2(L)u_2^2 \\ &\quad - \lambda_1(0)q_1(0)u_1^2 + \lambda_2(0)q_2(0)y_2^2(t,0),\end{aligned} \tag{33}$$

$$\begin{aligned}I &:= (-(\lambda_1q_1)_x)y_1^2 + 2(q_2b_1 + q_1a_2)y_1y_2 \\ &\quad + ((\lambda_2q_2)_x)y_2^2.\end{aligned} \tag{34}$$

A necessary condition for V to be a (strict) control Lyapunov is that I is a strictly positive quadratic form with respect to (y_1, y_2) for almost every x in $[0, L]$, i.e.

$$-(\lambda_1q_1)_x > 0 \text{ a.e. in } [0, L], \tag{35}$$

$$(\lambda_2q_2)_x > 0 \text{ a.e. in } [0, L], \tag{36}$$

$$-(\lambda_1q_1)_x(\lambda_2q_2)_x > (q_2b_1 + q_1a_2)^2 \text{ a.e. in } [0, L]. \tag{37}$$

We define the functions $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f(x) := \lambda_1(x)q_1(x), \forall x \in [0, L], \tag{38}$$

$$g(x) := \lambda_2(x)q_2(x), \forall x \in [0, L]. \tag{39}$$

The quadratic form V is coercive with respect to $(y_1, y_2) \in L^2(0, L)^2$ (i.e. $\exists\sigma > 0$ such that $V(y_1, y_2) \geqslant \sigma \int_0^L (y_1^2 + y_2^2)dx$) if and only if (1) and (2) hold. Note that (35) is equivalent to (3) and that (36) is equivalent to (4). Property (37) is equivalent to (5) with a and b defined by

$$a(x) := \frac{a_2(x)}{\lambda_1(x)}, \, b(x) := \frac{b_1(x)}{\lambda_2(x)}, \forall x \in [0, L]. \tag{40}$$

Following Proposition 1, we consider the maximal solution η of the Cauchy problem

$$\eta' = \left|\frac{a_2}{\lambda_1} + \frac{b_1}{\lambda_2}\eta^2\right|, \, \eta(0) = 0. \tag{41}$$

It follows from Proposition 1 that a necessary condition for the existence of a control Lyapunov function of the form (31) is that η is defined on $[0, L]$.

Let us now assume that η is indeed defined on $[0, L]$. We study the following two cases:

- (i) The control is on both sides: we can choose u_1 and u_2 for feedback stabilisation.
- (ii) The control u_2 is of the following form $u_2(t) = My_1(t, L)$, where M is a given constant. Only u_1 can be chosen freely.

(Note that the case where $u_1(t) = My_2(t, 0)$, where M is a given constant and u_2 is free follows from the case (ii) by replacing x by $L - x$.)

In case (i) there is a strict control Lyapunov V of the form (31). Indeed, by Proposition 2, there exist $q_1 \in C^1([0, L]; (0, +\infty))$ and $q_2 \in C^1([0, L]; (0, +\infty))$ such that (35), (36) and (37) hold everywhere in $[0, L]$ (instead a.e. in $[0, L]$). Then we consider the following decentralized feedback laws

$$u_1(t) := k_1y_2(t, 0), \, u_2(t) := k_2y_1(t, L). \tag{42}$$

If we take

$$k_2^2 \leqslant \frac{\lambda_1(L)q_1(L)}{\lambda_2(L)q_2(L)}, \, k_1^2 \leqslant \frac{\lambda_2(0)q_2(0)}{\lambda_1(0)q_1(0)}, \tag{43}$$

then

$$\dot{V} \leqslant -\delta(|y_1|^2_{L^2(0,L)} + |y_1|^2_{L^2(0,L)}), \tag{44}$$

for some $\delta > 0$ independent of (y_1, y_2).

We now turn to the case (ii). Note that in order to have $\dot{V} \leqslant 0$ we must have

$$M^2 \leqslant \frac{f(L)}{g(L)}. \tag{45}$$

However, it follows from our proof of Proposition 1 (and with the notations therein) that

$$\frac{g(L)}{f(L)} = g(L)h(L) = w^2(L) > \eta^2(L). \tag{46}$$

(See in particular (17).) Let us first treat the case where $a_2 \neq 0$. Then $\eta(L) > 0$ and it follows from (45) and (46), that a necessary condition for the existence of a control Lyapunov of the form (31) is that

$$|M| < \frac{1}{\eta(L)}. \tag{47}$$

Conversely, let us assume that (47) holds. Then, it follows from Proposition 1 that there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1) to (5) and (45) hold. Then it suffices to take the feedback law $u_1(t) := k_1y_2(t, 0)$ with

$$k_1^2 \leqslant \frac{\lambda_2(0)q_2(0)}{\lambda_1(0)q_1(0)} = \frac{g(0)}{f(0)}. \tag{48}$$

Finally let us deal with the case $a_2 = 0$. Then it follows from Proposition 2 that there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (1)-(2)-(26)-(27)-(28) and (45) hold (in fact in this case the existence of such f and g is straightforward). We then proceed as above: we take the feedback law $u_1(t) := k_1y_2(t, 0)$ with k_1 satisfying (48).

Remark 4. The proof of Proposition 1 provides a way to construct "good" coefficients q_1 and q_2 for the Lyapunov function: Take $\varepsilon > 0$ small enough and consider the solution of the Cauchy problem

$$\eta_\varepsilon' = \left|\frac{a_2}{\lambda_1} + \frac{b_1}{\lambda_2}\eta_\varepsilon^2\right| + \varepsilon, \, \eta_\varepsilon(0) = \varepsilon, \tag{49}$$

and then define q_1 and q_2 by

$$q_1(x) := \frac{1}{\lambda_1(x)\eta_\varepsilon(x)}, \forall x \in [0, L], \tag{50}$$

$$q_2(x) := \frac{\eta_\varepsilon(x)}{\lambda_2(x)}, \forall x \in [0, L]. \tag{51}$$

Of course (49) can be replaced by some similar Cauchy problem whose solution could be simpler to compute. For example, if $a_2 \geqslant 0$ and $b_1 \geqslant 0$, one can replace (49) by

$$\eta_\varepsilon' = (1 + \varepsilon)\left|\frac{a_2}{\lambda_1} + \frac{b_1}{\lambda_2}\eta_\varepsilon^2\right|, \, \eta_\varepsilon(0) = \varepsilon. \tag{52}$$

4. APPLICATION TO SAINT-VENANT EQUATIONS

We consider a pool of a prismatic horizontal open channel with a rectangular cross section and a unit width. The dynamics of the system are described by the Saint-Venant equations

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0 \\ &\partial_t V + \partial_x \left(\frac{V^2}{2} + gH \right) + gC\frac{V^2}{H} = 0 \end{aligned} \tag{53}$$

with the state variables $H(t,x)$ = water depth and $V(t,x)$ = water velocity. C is a friction coefficient and g the gravity acceleration.

The channel is provided with hydraulic control devices (pumps, valves, mobile spillways, sluice gates, ...) which are located at the two extremities and allow to assign the values of the flow-rate on both sides:

$$\begin{aligned} Q_1(t) = H(t,0)V(t,0) \\ Q_2(t) = H(t,L)V(t,L) \end{aligned} \tag{54}$$

The system (53)-(54) is a control system with state $H(t,x), V(t,x)$ and controls $Q_1(t), Q_2(t)$. This system is clearly open-loop unstable. The objective is to design decentralised control laws, with $Q_1(t)$ function of $H(t,0)$ and $Q_2(t)$ function of $H(t,L)$, in order to stabilise the system about a constant flow-rate set point Q^*.

A steady-state (or equilibrium profile), corresponding to the set-point Q^*, is a couple of time-invariant *non-uniform* (i.e. space-varying) state functions $H^*(x), V^*(x)$ such that $H^*(x)V^*(x) = Q^*$ which satisfy the differential equations

$$\begin{aligned} &\partial_x(H^*V^*) = 0, \\ &\partial_x\big(\frac{V^{*2}}{2} + gH^*\big) + gC\frac{V^{*2}}{H^*} = 0. \end{aligned}$$

These equations may also be written as

$$V^*\partial_x H^* = -H^*\partial_x V^* = -\frac{gCV^{*3}}{gH^* - V^{*2}}. \tag{55}$$

In this section, as a first stage towards a more comprehensive study of the problem, we shall focus on the stabilisability of the *linearised* system by using the analysis of the previous section.

In order to linearise the model, we define the deviations of the states $H(t,x)$ and $V(t,x)$ with respect to the steady-states $H^*(x)$ and $V^*(x)$:

$$h(t,x) \triangleq H(t,x) - H^*(x),\ v(t,x) \triangleq V(t,x) - V^*(x).$$

Then the linearised Saint-Venant equations around the steady-state are :

$$\begin{aligned} &\partial_t h + V^*\partial_x h + H^*\partial_x v + (\partial_x V^*)h + (\partial_x H^*)v = 0 \\ &\partial_t v + g\partial_x h + V^*\partial_x v - gC\frac{V^{*2}}{H^{*2}}h + \left[\partial_x V^* + 2gC\frac{V^*}{H^*}\right] v = 0. \end{aligned}$$

The characteristic (Riemann) coordinates are defined as follows:

$$\begin{aligned} z_1(t,x) = v(t,x) + h(t,x)\sqrt{\frac{g}{H^*(x)}} \\ z_2(t,x) = v(t,x) - h(t,x)\sqrt{\frac{g}{H^*(x)}} \end{aligned} \tag{56}$$

$$\iff \quad \begin{aligned} h(t,x) &= \frac{z_1(t,x) - z_2(t,x)}{2}\sqrt{\frac{H^*(x)}{g}} \\ v(t,x) &= \frac{z_1(t,x) + z_2(t,x)}{2} \end{aligned}$$

With these definitions and notations, the linearised Saint-Venant equations are written in characteristic form:

$$\begin{aligned} \partial_t z_1 + \lambda_1(x)\partial_x z_1 + \gamma_1(x)z_1 + \delta_1(x)z_2 = 0 \\ \partial_t z_2 - \lambda_2(x)\partial_x z_2 + \gamma_2(x)z_1 + \delta_2(x)z_2 = 0 \end{aligned} \tag{57}$$

with the characteristic velocities

$$\lambda_1(x) = V^*(x) + \sqrt{gH^*(x)},\ -\lambda_2(x) = V^*(x) - \sqrt{gH^*(x)}$$

and the coefficients

$$\begin{aligned} \gamma_1(x) &= g\frac{CV^{*2}}{H^*}\left[-\frac{3}{4(\sqrt{gH^*} + V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right] \\ \delta_1(x) &= g\frac{CV^{*2}}{H^*}\left[-\frac{1}{4(\sqrt{gH^*} + V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right] \\ \gamma_2(x) &= g\frac{CV^{*2}}{H^*}\left[\frac{1}{4(\sqrt{gH^*} - V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right] \\ \delta_2(x) &= g\frac{CV^{*2}}{H^*(x)}\left[\frac{3}{4(\sqrt{gH^*} - V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right] \end{aligned}$$

The steady-state flow is subcritical (or fluvial) if the following condition holds

$$gH^*(x) - V^{*2}(x) > 0 \ \ \forall x. \tag{58}$$

Under this condition, the system is strictly hyperbolic with

$$-\lambda_2(x) < 0 < \lambda_1(x) \ \ \forall x.$$

We now introduce the notations

$$\begin{aligned} \varphi_1(x) &= \exp\Big(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds\Big), \\ \varphi_2(x) &= \exp\Big(-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds\Big), \\ \varphi(x) &= \frac{\varphi_1(x)}{\varphi_2(x)}, \end{aligned}$$

and the canonical coordinates

$$y_1(t,x) = \varphi_1(x)z_1(t,x),\ \ y_2(t,x) = \varphi_2(x)z_2(t,x). \tag{59}$$

Then the model is written in canonical form

$$\begin{aligned} \partial_t y_1 + \lambda_1(x)\partial_x y_1 + \varphi(x)\delta_1(x)y_2 = 0, \\ \partial_t y_2 - \lambda_2(x)\partial_x y_2 + \varphi^{-1}(x)\gamma_2(x)y_1 = 0. \end{aligned} \tag{60}$$

According to Proposition 1 and our analysis in Section 3, in order to check the condition for the existence of the quadratic control Lyapunov function, we need to solve the following third-order differential system on $[0,L]$ (with $H^*(x) = Q^*/V^*(x)$):

$$\begin{aligned} &\frac{dV^*}{dx} = \frac{gC}{Q^*}\left(\frac{(V^*(x))^5}{gQ^* - (V^*(x))^3}\right) \qquad V^*(0) = V^*_o \\ &\frac{d\psi}{dx} = \frac{\gamma_1(x)}{\lambda_1(x)} + \frac{\delta_2(x)}{\lambda_2(x)} \qquad \psi(0) = 0 \\ &\frac{d\eta}{dx} = \frac{e^{\psi(x)}\delta_1(x)}{\lambda_1(x)} + \frac{\gamma_2(x)}{e^{\psi(x)}\lambda_2(x)}\eta^2(x) \qquad \eta(0) = 0 \end{aligned}$$

As a matter of illustration, we compute the solution of this system with the following parameter values : $L = 1000$ m, $g = 9.81$ m/s^2, $C = 0.001$ s^2/m, $Q^* = 5$ m^3/s and the initial condition $V^*_o = 1$ m/s. The function η exists over the interval $[0,L]$ and is shown in the following figure.

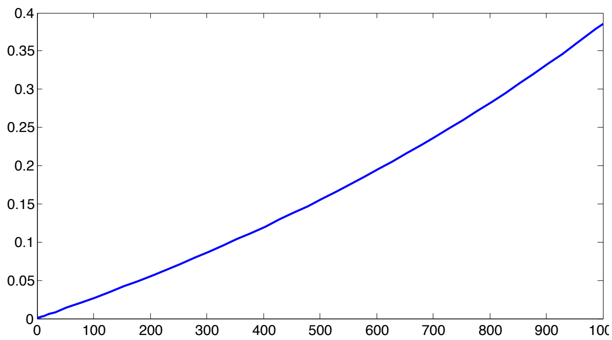
Let us now impose a boundary condition of the form

$$y_1(t,0) = k_1 y_2(t,0) \tag{61}$$

with

$$k_1^2 \leqslant \frac{\lambda_2(0)q_2(0)}{\lambda_1(0)q_1(0)}$$

to the canonical system (60). Then, using the definition (59) of the canonical coordinates, the definition (56) of the



Riemann coordinates and the physical boundary condition (54), it is a matter of few calculations to get the physical stabilising control law which implements the boundary condition (61)

$$Q_1(t) = \frac{H(t,0)}{H^*(0)} \times$$
$$\left[Q^* - \frac{\varphi_1(0) - k_1\varphi_2(0)}{\varphi_1(0) + k_1\varphi_2(0)} \sqrt{gH^*(0)}(H(t,0) - H^*(0)) \right]$$

for the open channel represented by the Saint-Venant equations. We remark that this control law is a non-linear feedback function of the water depth $H(t, 0)$ although it is derived on the basis of a linearised model. Obviously, a similar derivation leads to a control law for $Q_2(t)$ at the other side of the channel.

5. CONCLUSION AND FINAL REMARK

Conditions for boundary feedback stabilisability of linear hyperbolic systems in canonical form have been established. The main result was to show that the existence of a quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result has been used to give explicit conditions for the existence of linear boundary feedback control laws. The analysis is illustrated with an application to the boundary feedback stabilisation of open channels represented by Saint-Venant equations with non-uniorm steady-states.

An interesting final remark is that we could believe that more general stabilisability conditions could be obtained by considering a more general Lyapunov function candidate (with a cross term) of the form

$$V(y) := \int_0^L \big(q_1(x)y_1^2 + q_2(x)y_2^2 + q_3(x)y_1y_2\big)dx. \quad (62)$$

In fact this is not true because it can be shown that, for the canonical control system (29)-(30), there exist necessarily coefficients $\lambda_i(x)$, $a_2(x)$, $b_1(x)$ such that if (62) is a control Lyapunov function then $q_3(x)$ must be zero.

The results presented in this paper bring various extensions to our previous contributions to the same subject. The interested reader is referred to e.g. Bastin et al. [2008], Prieur et al. [2008], Coron et al. [2008], Bastin et al. [2009] and the references therein.

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REFERENCES

G. Bastin, J-M. Coron, and B. d'Andréa-Novel. Boundary feedback control and Lyapunov stability analysis for physical networks of 2x2 hyperbolic balance laws. In *47th IEEE Conference on Decision and Control*, pages 1454–1458, 2008.

G. Bastin, J-M. Coron, and B. d'Andréa-Novel. On lyapunov stability of linearised Saint-Venant equations for a sloping channel. *Networks and Heterogeneous Media*, 4(2):177–187, 2009.

J-M. Coron, G. Bastin, and B. d'Andréa-Novel. Dissipative boundary conditions for one dimensional nonlinear hyperbolic systems. *SIAM Journal of Control and Optimization*, 47(3):1460–1498, 2008.

C. Prieur, J. Winkin, and G. Bastin. Robust boundary control of systems of conservation laws. *Mathematics of Control, Signal and Systems (MCSS)*, 20:173–197, 2008.