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Boundary Control Design for Cascades of Hyperbolic 2×2 PDE systems via Graph Theory.

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***Abstract*— This article is concerned with the design of boundary control laws for stabilizing systems of 2×2 first order quasi-linear hyperbolic PDEs. A new graph representation of such systems represents the interactions between the characteristic curves and the boundary control laws, the *invariant graph*, is introduced. The structure of the *invariant graph* is used to design stabilizing control laws and an analytical stability condition is given.**

I. Introduction

Many distributed physical systems are described by systems of hyperbolic partial differential equations (e.g. [1]). Probably the most famous example is the shallow water equations that were introduced by Saint-Venant in 1871 (see [2]) and that describe the dynamics of open-channels. Typically, in such systems, the available control actions are hydraulic gates located at the boundaries of the channels. Moreover, the systems to be controlled are usually composed of several connected sub-systems, creating a network of channels and gates. This motivates the study of the design of stabilizing boundary control laws for networks of hyperbolic systems.

As far as the stability of systems of hyperbolic PDEs is concerned, Greenberg and Li have introduced in [3] a constructive proof for local asymptotic stability based on Riemann invariants. Later on, Li published in [4] a condition for the local asymptotic stability of hyperbolic systems. This result was generalized in [8]. Unfortunately, the stability condition, which is based on a spectral radius computation, becomes very complicated when the system dimension increases, which makes the controller design difficult.

In this paper, we provide a new control design technique that keeps the stability condition, given in [8], simple and practically usable. To do so, the constructive proof based on Riemann invariants is formalized by introducing the *invariant graph*, which is a weighted directed graph representation of an hyperbolic system. By taking advantage of the rich topological properties of the *invariant graph*, a stabilizing control design technique is built. Previous results relating graph theory to control design can be found in [5]

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In Section II, the notations are introduced and the concept of hyperbolic system is described. A brief overview of the Riemann invariant technique is given in Section III. Two graph representations of an hyperbolic system, namely the *invariant graph* and the *flux graph*, are given in Section IV. The control design is exposed in Section V. In Section VI, an application of this design to cascade of 2×2 systems is given.

II. Cascade of Hyperbolic Systems

Let us consider a cascade of $n \geq 2$ 2×2 first order quasi-linear hyperbolic PDE systems with linear boundary conditions.

The negative characteristic velocities are denoted with $\lambda_i (i = 1, \cdots, n)$ and the positive positive characteristic velocities are denoted with $\lambda_i (i = n+1, \cdots, 2n)$.

For each characteristic velocity λ_i, the associated *Riemann invariant* is denoted $\xi_i(x,t) \in C^1([0,1] \times [0,+\infty); \mathbb{R})$ where x is the space variable defined on the finite interval $[0,1]$ and $t \in [0,+\infty)$ is the time variable.

The first order quasi-linear hyperbolic PDE system is defined by the following equation :

$$\partial_t \boldsymbol{\xi} + \Lambda(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi} = 0 \ . \tag{1}$$

with the following linear boundary conditions

$$\begin{bmatrix} \boldsymbol{\xi}_-(1,t) \\ \boldsymbol{\xi}_+(0,t) \end{bmatrix} = K \begin{bmatrix} \boldsymbol{\xi}_-(0,t) \\ \boldsymbol{\xi}_+(1,t) \end{bmatrix} \tag{2}$$

with $K = (k_{ij}) \in M_{2n,2n}$ and the following compact vector notations

$$\begin{array}{rcll} \boldsymbol{\xi}_- & = & [\xi_1, \cdots, \xi_n]', & \\ \boldsymbol{\xi}_+ & = & [\xi_{n+1}, \cdots, \xi_{2n}]', & \\ \boldsymbol{\xi} & = & [\xi_1, \cdots, \xi_{2n}]', & \\ \lambda_i(\boldsymbol{\xi}) & < & 0, & i \in [1,n], \\ \lambda_i(\boldsymbol{\xi}) & > & 0, & i \in [n+1,2n], \\ \Lambda_-(\boldsymbol{\xi}) & = & \mathrm{diag}(\lambda_1, \cdots, \lambda_n), & \\ \Lambda_+(\boldsymbol{\xi}) & = & \mathrm{diag}(\lambda_{n+1}, \cdots, \lambda_{2n}), & \\ \Lambda(\boldsymbol{\xi}) & = & \mathrm{diag}(\lambda_1, \cdots, \lambda_{2n}). & \end{array} \tag{3}$$

Since we consider the particular case of cascade of systems, K has a special structure which we can describe. In fact, K can be divided into four sub-matrices as follows:

$$K = \begin{bmatrix} T & Q \\ R & S \end{bmatrix} \tag{4}$$

where $Q, R, S, T \in M_{n,n}$, Q and R are diagonal matrices, S is a lower diagonal matrix and T is an upper diagonal matrix. This characterisation, which is illustrated in Section

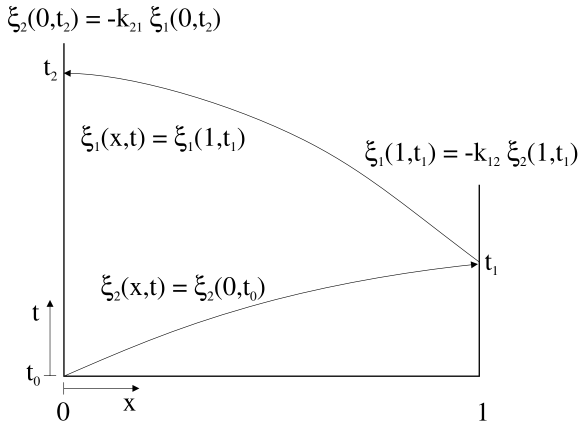
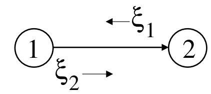


Fig. 1. Following invariants along their trajectories



$$
\begin{aligned}
V_F &= \{1,2\} \ , \\
E_F &= \{(1,2)\} \ .
\end{aligned}
$$

Fig. 2. *Flux graph* of a 2×2 system corresponding to the system in Figure 3.

III and IV, plays a crucial role in the control design process in Section VI.

Clearly the system (1)-(2) has an infinity of equilibria since any constant $\bar{\boldsymbol{\xi}}$ is a steady-state of the system. Our concern in this paper is to design a boundary control in order to stabilize the system at particular desirable steady-state, called the set point, which can be taken as the origin without loss of generality:

$$\bar{\boldsymbol{\xi}} = 0 . \tag{5}$$

An important property of *Riemann invariants* is that their value remains constant along the characteristic curves defined by the following differential equation

$$\frac{\partial x_i}{\partial t} = \lambda_i(x,t) \ . \tag{6}$$

When there is no ambiguity, the arguments (x,t) of $\xi_i(x,t)$ and $\boldsymbol{\xi}$ of $\lambda_i(\boldsymbol{\xi})$ will be omitted.

Moreover, the restriction to linear boundary conditions can be weakened by considering K as the gradient of non-linear boundary conditions linearised around the origin.

III. Riemann invariants and stability

This section gives a short overview of the constructive proof used to prove the stability of a hyperbolic PDE system using boundary control laws. This explanation is given in order to motivate the introduction of the *invariant graph* in Section IV. Note that this technique was first introduced by Greenberg and Li Ta-Tsien in [3], and was used in [4], [6], [7], [8].

We shall illustrate the issue with a single 2×2 system where the invariants can be intuitively understood as waves travelling along the space domain and bouncing at the boundaries. Figure 1 represents the travelling of the two Riemann invariants of a 2×2 system denoted $\xi_1(x,t)$ and $\xi_2(x,t)$. For this system, K is written:

$$K = \begin{bmatrix} 0 & -k_{12} \\ -k_{21} & 0 \end{bmatrix} \tag{7}$$

where k_{12} and k_{22} are positive constants.

Consider the positive Riemann invariant $\xi_2(x,t)$ along its characteristic curve starting from $(0,t_0)$ (recall that ξ_2 is constant along its characteristic curve).

For $|\xi_1(\ ,0)|C^0([0,1]) + |\xi_2(\ ,0)|C^0([0,1])|$ sufficiently small, there exists obviously a time instant $t_1 > t_0$ such that $\xi_2(1,t_1) = \xi_2(0,t_0)$. Suppose now that we are able to apply a boundary control at the right boundary ($x = 1$) such that $\xi_1(1,t_1) = -k_{12}\xi_2(1,t_1)$. Obviously, there exists a time instant $t_2 > t_1$ such that $\xi_1(0,t_2) = \xi_1(1,t_1)$. We now apply a boundary control at the left gate such that $\xi_2(0,t_2) = -k_{21}\xi_1(0,t_2)$. Chaining the results, we obtain

$$\xi_2(0,t_2) = k_{12}k_{21}\xi_2(0,t_0) \ .$$

Clearly, if k_{12} and k_{21} are chosen such that

$$k_{12}k_{21} < 1 \tag{8}$$

and if the bouncing process can be repeated sufficiently, ξ_i will tend asymptotically to zero. This result is in fact a direct application of [8, Theorem 6] which states, that for suitable initial conditions, the stability conditions is

$$\rho(\mathrm{abs}(K)) < 1$$

which is equivalent to (8) in the 2×2 example. The complete formulation of this Theorem has been added in Section A for convenience.

Assuming that the coefficients k_{12}, k_{21} are uniquely related to the control variables, the control design for the stability is trivial because of the simplicity of the stability condition. However, this is not true for less simple systems where the complexity of the stability condition rapidly increases.

IV. Graph representations

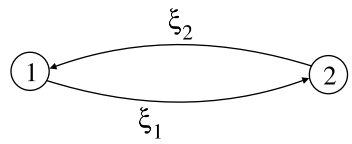
In this section, two graph representations are given.

The *flux graph* represents the network of 2×2 systems where each edge represents one system and the nodes represent the coupling between those systems through the boundary conditions. It is the natural representation of hyperbolic systems based on physical models.

The *invariant graph* represents the interaction of the invariants ξ_i through the boundary conditions.

We now give a formal definition of the *flux graph*:

Definition 1: The *flux graph* (G_F) is a directed graph defined by $G_F = (V_F, E_F)$ where V_F is the *flux vertex*-set and E_F is the *flux edge*-set. A *flux vertex* represents



$$
\begin{aligned}
V_I &= \{1,2\}\\
E_I &= \{(1,2),(2,1)\}\ ,\\
K &= \begin{bmatrix} 0 & -k_{12} \\ -k_{21} & 0 \end{bmatrix}
\end{aligned}
$$

Fig. 3. *Invariant graph* for $n = 2$. This is the G_I of the example system of Section III.

the coupling of systems through boundary conditions. A *flux edge* represents a 2×2 hyperbolic PDE system. The direction of the edge is chosen as the direction of the positive Riemann invariant.

The *flux graph* of the system depicted in Figure 3 is given in Figure 2.

The *invariant graph* is a weighted directed graph (e.g. [9, Chapter 1], [10, Chapter 2]) based on the K matrix.

Definition 2: The *invariant graph* (G_I) is a weighted directed graph defined by $(V_I, E_I, \text{abs}(K))$ where V_I is the *invariant vertex*-set , E_I the *invariant edge*-set associated to the K matrix and $\text{abs}(K)$ the *invariant edge* weights. An *invariant vertex* represents a boundary condition in (2), and an *invariant edge* (u, v) connects a vertex u to a vertex v if the corresponding entry k_{uv} in K is nonzero. Each edge is weighted by the absolute value of the corresponding entry in the K matrix.

The *invariant graph* of a simple system, corresponding to (7), is depicted in Figure 3.

Note that the *invariant graph* can be seen as the *associated directed graph* of the non-negative matrix $\text{abs}(K)$ (e.g. [10, Chapter 2]). Another way of interpreting the *invariant graph* is to consider it as a network flow graph (e.g. [9, Chapter 4]) where the edge weights are defined by the matrix $\text{abs}(K)$.

V. Control design for two cascaded systems

A. Model

In this section, we shall apply the control design to a simple network composed of two systems in cascade. The network is intentionally restricted to a cascade of two systems in order to simplify the notations and to give the reader a clear understanding of the design methodology. A general result for a cascade of n systems is given in SectionVI.

For this system, the boundary conditions (2) are written (the boundary conditions are illustrated in Figure 4):

$$
\begin{bmatrix} \xi_1(1,t) \\ \xi_2(1,t) \\ \xi_3(0,t) \\ \xi_4(0,t) \end{bmatrix} = \begin{bmatrix} 0 & -k_{12} & -k_{13} & 0 \\ 0 & 0 & 0 & k_{24} \\ k_{31} & 0 & 0 & 0 \\ 0 & -k_{42} & -k_{43} & 0 \end{bmatrix} \begin{bmatrix} \xi_1(0,t) \\ \xi_2(0,t) \\ \xi_3(1,t) \\ \xi_4(1,t) \end{bmatrix} \tag{9}
$$

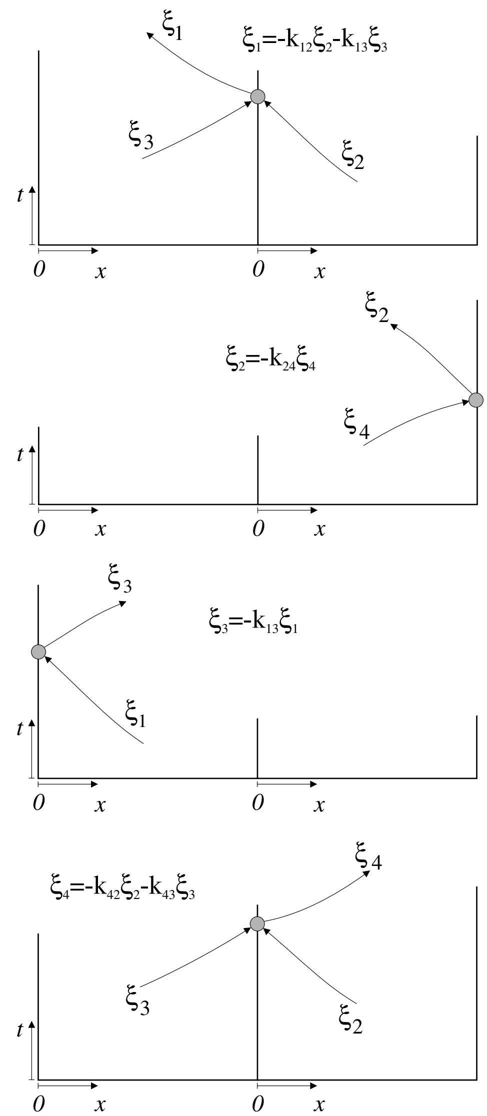


Fig. 4. Illustration of (9)

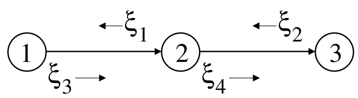
The *invariant graph* and the *flux graph* of this system are given in Figures 5 and 6.

B. Properties of the invariant graph

The *invariant graph* and $\text{abs}(K)$ of such systems have several interesting properties that come from the Non-negative Matrices framework (see [10, Chapter 2]). Some important Definitions and Theorems are given in Appendix B for convenience.

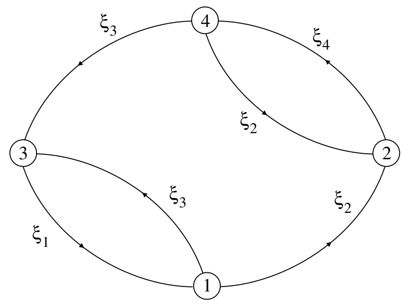
First of all, it is trivial to see that the *invariant graph* is *strongly connected* (see Definition 4). This property implies that $\text{abs}(K)$ is *irreducible* (see Definition 3 and Theorem 2).

The strong connectivity implies that there is always at least one circuit passing through any *invariant vertex*. Moreover, the circuit lengths are always a multiple of two since the "information" should always travel through two *invariant edges* before returning to the initial vertex. There-



$$\begin{aligned} V_F &= \{1,2,3\}\ , \\ E_F &= \{(1,2),(2,3)\}\ . \end{aligned}$$

Fig. 5. *Flux graph* of two 2×2 systems in cascade. The corresponding *invariant graph* is depicted in Figure 6.



$$\begin{aligned} V_I &= \{1,2,3,4\} \\ E_I &= \{(1,2),(1,3),(2,4),(3,1),(4,2),(4,3)\}\ , \\ K &= \begin{bmatrix} 0 & -k_{12} & -k_{13} & 0 \\ 0 & 0 & 0 & -k_{24} \\ -k_{31} & 0 & 0 & 0 \\ 0 & -k_{42} & -k_{43} & 0 \end{bmatrix} \end{aligned}$$

Fig. 6. The *invariant graph* of two 2×2 systems in cascade. The corresponding *flux graph* is depicted in Figure 5. The graph can be built from the Figure 4 and corresponds to (9).

fore, the index of cyclicity of abs(K) is 2 (see Definition 5 and Theorem 3).

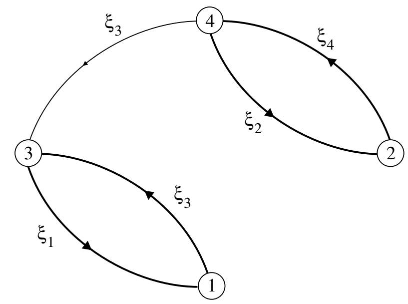
C. Control Design using the invariant graph

The control design goal is to simplify the control gains k_{ij} computation resulting from the [8, Theorem 6] (this result is added in Appendix A for convenience).

Recall that the method of proof relies on the "pursuit" of the Riemann invariants along their trajectories and the "rebounds" at the boundaries. The approach can be formalised with the *invariant graph* where following the trajectories is equivalent to following all the possible paths in the graph.

The idea of control design is to break the connectivity of the graph such that the maximum circuit length in the *invariant graph* is 2. Therefore, the *invariant graph* appears as a sequence of locally stabilizing loops similar to the *invariant graph* presented in Figure 3. For example, in Figure 6, removing edge $(1,2)$ breaks the circuit $(1,2,4,3)$ and the two remaining circuits have length two. Analytically, this connectivity breaking is obtained by setting $k_{12} = 0$ in (9). This is depicted in Figure 7.

It turns out in the general case that the connectivity breaking is also equivalent to setting T to zero in (4) which is a key property that shall be used to compute the stability condition.



$$\begin{aligned} V_I &= \{1,2,3,4\} \\ E_I &= \{(1,3),(2,4),(3,1),(4,2),(4,3)\}\ , \\ K &= \begin{bmatrix} 0 & 0 & -k_{13} & 0 \\ 0 & 0 & 0 & -k_{24} \\ -k_{31} & 0 & 0 & 0 \\ 0 & -k_{42} & -k_{43} & 0 \end{bmatrix} \end{aligned}$$

Fig. 7. The *invariant graph* of two 2×2 systems in cascade where the edge $(1,2)$ has been removed.

In the next section, we prove that breaking the connectivity in cascaded 2×2 systems leads to a global stability condition based on the stability of the local loops.

VI. Stability condition for cascades of 2×2 systems with degenerated boundary conditions

We shall need the following lemma to compute the eigenvalues of K:

Lemma 1: Let $n > 0$, $N \in M_{2n,2n}$, $Q, R, S \in M_{n,n}$ such that

$$N = \begin{bmatrix} 0 & Q \\ R & S \end{bmatrix}$$

where Q and R are diagonal matrices and S is a lower diagonal matrix. Then, the eigenvalues λ_i of N are

$$\lambda_i(N) = \pm\sqrt{q_{ii} r_{ii}} \quad \text{for } i = 1, ..., n.$$

Proof: We want to find the values λ such that

$$\det \begin{bmatrix} \lambda I & -Q \\ -R & \lambda I - S \end{bmatrix} = 0. \tag{10}$$

Taking the *Schur complement* of (10), we have

$$\begin{aligned} &\det \begin{bmatrix} \lambda I & -Q \\ -R & \lambda I - S \end{bmatrix} \\ &= \det(\lambda I) \det(\lambda I - S - R(\lambda I)^{-1} Q) \\ &= \det(\lambda^2 I - \lambda S - RQ) \\ &= \det \begin{bmatrix} \lambda_1^2 - q_{1,1} r_{1,1} & & & \\ -\lambda_2 s_{2,1} & \lambda_2^2 - q_{2,2} r_{2,2} & & \\ \ddots & \ddots & & \\ & -\lambda_n s_{n,n-1} & \lambda_n^2 - q_{n,n} r_{n,n} \end{bmatrix} \end{aligned}$$

Hence, this determinant is zero if

$$\lambda_i^2 - q_{i,i} r_{i,i} = 0 \qquad \text{for } i = 1, ...n.$$

This ends the proof.

□

The control design for n-dimensional systems is a straight generalization of the 2-dimensional system case. It consists of setting the T matrix in (4) to zero, by control design or due to a particular configuration of the physical devices represented by the boundary conditions as explained below. In terms of graph theory, this is equivalent to removing edges until the index of cyclicity of the *invariant graph* is equal to 2.

Of course, the assumption that T could be set to zero can seem to be restrictive but this is not always the case. In fact, there exists examples of physical systems that naturally have this property. For example, open channels separated by spillway gates, an hydraulic structure often used in open channel regulation, have this property (e.g. [11], [12]).

Using Lemma 1, the spectral radius of $\text{abs}(K))$ can be computed and is equivalent to

$$\max_{i\in[1,p]}\{k_{n+i,i}k_{i,n+i}\} < 1 \ . \tag{11}$$

The stability follows from a direct application of the [8, Theorem 6] given in Appendix A.

VII. Conclusion

In this paper, we have proposed a methodology for designing stabilizing controls for a cascade of n 2×2 hyperbolic systems based on the results of [8]. The design approach is based on breaking the connectivity of the *invariant graph* of the system in order to lower the index of cyclicity to 2. This step leads to a analytically simple stability condition.

It must be noted that methodology also applies to other network topologies such as stars (illustrated in Figure 9 and 8) and trees although this matter has not been investigated yet.

Appendix

A. Stability theorem

In this section, we state a result from [8, Theorem 6] that gives a condition for asymptotic stability.

Theorem 1: If

$$\rho(\text{abs}(K)) < 1 \ , \tag{12}$$

then there exists $\varepsilon > 0$, $\mu > 0$, $C > 0$ such that, for every $\boldsymbol{\xi}^\natural \in C^1([0,1];\Re^n)$ satisfying compatibility conditions with (1)-(2) and

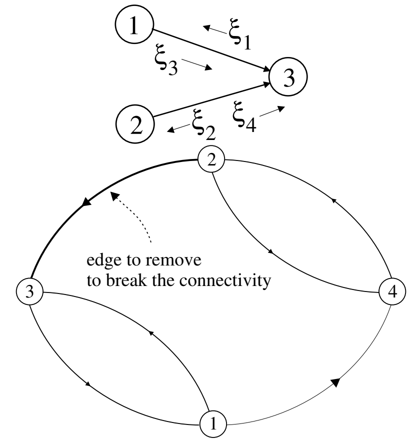
$$|\boldsymbol{\xi}^\natural|_{C^1([0,1])} \leq \varepsilon \ ,$$

there exists one and only one function $\boldsymbol{\xi} \in C^1([0,1]\times[0,+\infty);\Re^n)$ satisfying (1),(2) and

$$\boldsymbol{\xi}(x,0) = \boldsymbol{\xi}^\natural, \quad \forall x \in [0,1] \ .$$

Moreover, this function $\boldsymbol{\xi}$ satisfies

$$|\boldsymbol{\xi}(.,t)|_{C^1([0,1])} \leq C \exp^{-\mu t} |\boldsymbol{\xi}^\natural|_{C^1([0,1])}, \quad \forall t \geq 0 \ . \tag{13}$$

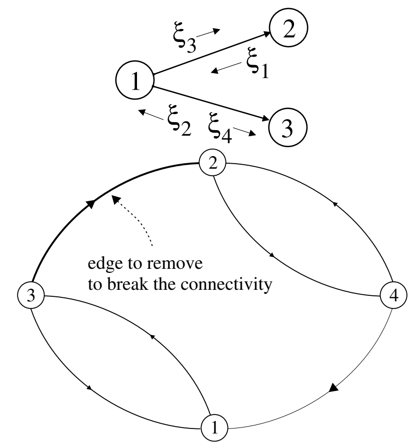


$$K = \begin{bmatrix} 0 & 0 & -k_{13} & -k_{14} \\ 0 & 0 & -k_{23} & -k_{24} \\ -k_{31} & 0 & 0 & 0 \\ 0 & -k_{42} & 0 & 0 \end{bmatrix} ,$$

By removing the edge $(2,3)$ (setting k_{23} to 0), we have

$$\rho(\text{abs}(K)) \leq \max\{k_{13}k_{31}, k_{24}k_{42}\}$$

Fig. 8. *Flux graph* and *invariant graph* of a tree.



$$K = \begin{bmatrix} 0 & 0 & -k_{13} & 0 \\ 0 & 0 & 0 & -k_{24} \\ -k_{31} & -k_{32} & 0 & 0 \\ -k_{41} & -k_{42} & 0 & 0 \end{bmatrix} ,$$

By removing the edge $(2,3)$ (setting k_{23} to 0), we have

$$\rho(\text{abs}(K)) \leq \max\{k_{13}k_{31}, k_{24}k_{42}\}$$

Fig. 9. *Flux graph* and *invariant graph* of a tree.

B. Nonnegative Matrices results

In this section, we state some important Definitions and Theorems from the Nonnegative Matrices framework, see [10, Chapter 2].

A *Nonnegative Matrix* is a matrix having nonnegative entries. It is denoted $A \geq 0$.

Definition 3: A $n \times n$ $A \geq 0$ matrix is *cogredient* to a matrix E if for some permutation matrix P, $PAP^T = E$. A is *reducible* if it is cogredient to

$$E = \left[\begin{array}{cc} B & 0 \\ C & D \end{array} \right] ,$$

where B and D are square matrices, or if $n = 1$ and $A = 0$. Otherwise, A is *irreducible*.

Definition 4: A directed graph G is *strongly connected* if for any ordered pair of vertices (u, v), there exists a sequence of edges (a path) which leads from u to v.

Theorem 2: A matrix $A \geq 0$ is irreducible if and only if $G(A)$ is strongly connected.

Definition 5: Let $A \geq 0$ be irreducible. The number h of eigenvalues of A of modulus $\rho(A)$ is called the *index of cyclicity* of A. If h is greater than one, A is said to be *cyclic of index* h.

Theorem 3: Let $A \geq 0$ be an irreducible matrix of order n. Let S_i be the set of all the lengths m_u of circuits in $G(A)$, through u. Let h_u the greatest common divisor of m_u in S_u,

$$h_u = \underset{m_u \in S_u}{\gcd} \{m_u\} .$$

Then $h_1 = h_2 = \cdots = h_n = h$ and h is the index of cyclicity of A.

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