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Abstract

In many process control applications, the system under consideration is *compartmental* and *positive*. This means that the system satisfies a mass conservation condition and that both the state variables and the control input are physically constrained to remain non-negative along the system trajectories. For such systems, the design of state feedback controllers makes sense only if the control function is guaranteed to provide a non-negative value at each time instant. The purpose of this paper is to present a positive control law for the feedback stabilisation of a class of positive compartmental systems which are dissipative but can nevertheless be globally unstable. The approach is illustrated with an application to the control of an industrial grinding circuit.

Keywords : positive control systems, stabilisation, dissipative, mass balance.

1. INTRODUCTION

In many practical applications of control engineering, the dynamical system under consideration is *compartmental* and *positive*. This means that the system is governed by a law of mass conservation and that both the state variables and the control input are physically constrained to remain non-negative along the system trajectories as stated in the following definition :

Definition 1. Positive System. A control system $\dot{x} = f(x,u) \;\; x \in I\!\!R^n, \;\; u \in I\!\!R$ is positive if

$$\left.\begin{array}{l} x(0) \in I\!\!R^n_+ \\ u(t) \in I\!\!R_+ \;\; \forall t \geq 0 \end{array}\right\} \Longrightarrow x(t) \in I\!\!R^n_+ \;\; \forall t \geq 0.$$

(**Notation.** The set of non-negative real numbers is denoted as usual $I\!\!R_+ = \{a \in I\!\!R, a \geq 0\}$. For any integer n, the set $I\!\!R^n_+$ is called the "non-negative orthant". Similarly the set of positive real numbers is denoted $I\!\!P = \{a \in I\!\!R, a > 0\}$ and $I\!\!P^n$ is called the "positive orthant".) ■

For such systems, it is an evidence that the design of state feedback controllers makes sense only if the control function is guaranteed to provide a non-negative value at each time instant.

The purpose of the present paper is to present a positive control law for the feedback stabilisation of compartmental systems which are described in Section 2. These systems have several interesting structural properties which are emphasized. In particular, they are dissipative but can nevertheless be globally unstable. In Section 3, a positive control law is proposed in order to achieve global output stabilization with state boudedness in the positive orthant. The controlled output has a clear physical meaning : it is the total mass contained in the system. The approach is illustrated with an application to a compartmental model of an industrial grinding circuit in Section 4. Some final comments are given in Section 5.

2. DISSIPATIVE COMPARTMENTAL SYSTEMS

We consider the general class of single-input compartmental systems described by a set of state equations of the form :

$$\dot{x}_i = \sum_{j \neq i} r_{ji}(x)x_j - \sum_{k \neq i} r_{ik}(x)x_i - q_i(x)x_i + b_i u \qquad i = 1, \dots, n \tag{1}$$

with:

- the state vector $x = (x_1, x_2, \dots, x_n) \in I\!R^n_+$ and the control input $u \in I\!R_+$
- the functions $r_{ij}(x)$ and $q_i(x)$: $I\!R^n_+ \rightarrow I\!R^n$ are differentiable
- the vector $b = (b_1, b_2, \dots, b_n)^T$ has non negative entries and at least one of them is positive : $b_i \geq 0 \quad \forall i$ and $b_i > 0$ for at least one i

System (1) is representative of a wide class of dynamical systems of interest in numerous engineering applications. Typical examples are chemical or biological industrial processes (see e.g. [1]), ecological systems or communication networks (see e.g. [2]).

In these systems :

1. Each state variable x_i is the amount of some material or immaterial "species" involved in the system.
2. The terms $r_{ik}(x)x_i$ represent various transport, transformation or interaction phenomena between the species inside the system
3. The terms $q_i(x)x_i$ represent an outflow of material leaving the system.
4. The terms $b_i u$ represent an inflow of material injected into the system from the outside.

Compartmental systems have several interesting structural properties which are now presented.

Property 1. The system (1) is *positive*. Indeed, if $x_i = 0$, then $\dot{x}_i = \sum_{j \neq i} r_{ji}(x)x_j + b_i u \geq 0$. This is sufficient to guarantee the forward invariance of the non negative orthant if the functions $r_{ij}(x)$ and $q_i(x)$ are differentiable. ■

The total mass contained in the system is

$$M(x) = \sum_{i=1}^{n} x_i$$

Property 2. A compartmental system is *mass conservative* in the sense that the mass balance is preserved inside the system. This is easily seen if we consider the special case of a closed system (1) without inflows ($u = 0$) and without outflows ($q_i(x) = 0, \forall i$). Then it is easy to check that $dM(x)/dt = 0$ which shows that the total mass is indeed conserved. ■

The model (1) is also written in matrix form as:

$$\dot{x} = G(x)x + bu \tag{2}$$

where $G(x)$ is a so-called *compartmental matrix* with the following properties:

1. $G(x)$ is a *Metzler* matrix, i.e. a matrix with non-negative off-diagonal entries:

   $$g_{ij}(x) = r_{ji}(x) \geq 0$$

   (note the inversion of indices !)

2. The diagonal entries of $G(x)$ are non-positive:

   $$g_{ii}(x) = -q_i(x) - \sum_{j \neq i} r_{ij}(x) \leq 0$$

3. The matrix $G(x)$ is diagonally dominant:

   $$|g_{ii}|(x) \geq \sum_{j \neq i} g_{ji}(x)$$

The term *compartmental* is motivated by the fact that such systems are usually represented by a network of conceptual reservoirs called compartments. Each quantity (state variable) x_i is supposed to be contained in a compartment which is represented by a node in the network. The flows between the compartments are represented by directed arcs $i \to j$ labeled with the so-called fractional rates r_{ij}. Additional arcs, labeled respectively with fractional outflow rates q_i and inflow rates $b_i u$ are used to represent inflows and outflows of the system. An illustrative example will be given in Fig. 2.

Definition 2. A compartment i is said to be *outflow connected at x* if there is a path $i \to j \to k \to \ldots \to \ell$ with positive fractional rates $r_{ij}(x) > 0, r_{jk}(x) > 0, \ldots$ from that compartment to a compartment ℓ from which there is a positive outflow $q_\ell(x) > 0$. The system is said to be *fully outflow connected at x* if all compartments are outflow connected at x. When the property holds for all x in the non-negative orthant, the system is simply said to be fully outflow connected. ■

Property 3. The compartmental matrix $G(x)$ is non singular if and only if the compartmental system is fully outflow connected at x. This shows that the non-singularity of a compartmental matrix can be directly checked by inspection of the associated compartmental network. ■

Property 4. If $u = 0$ (no inflow), the unforced system $\dot{x} = G(x)x$ is *dissipative* in the sense that the total mass $M(x)$ decreases along the system trajectories, because we have $dM(x)/dt = -\sum_{i=1}^{n} q_i(x)x_i \leq 0$. ■

Property 5. If $u = 0$ (no inflow) and if the system is fully outflow connected, the origin $x = 0$ is a globally asymptotically stable equilibrium of the unforced system $\dot{x} = G(x)x$ in the non-negative orthant, with the total mass $M(x) = \sum_{i=1}^{n} x_i$ as Lyapunov function. This property is equivalent to the zero state observability of the system $\dot{x} = G(x)x$ with output $y = M(x)$. (see [4] for a definition of the zero state observability) ■

3. CONTROL DESIGN FOR GLOBAL STABILISATION

Although the system is dissipative when the control input u is zero (no inflow), it can nevertheless be *globally unstable* when there is a non zero inflow $u(t) > 0$ which is the normal mode of operation in practical applications. The symptom of this instability is an unbounded accumulation of mass inside the system. An example will be given in the application section of the paper. This obviously makes the problem of feedback stabilisation of compartmental systems in the positive orthant relevant and sensible. One way of approaching the problem is to consider that the control objective is to globally stabilize the total mass $M(x)$ at a given positive set point $M^* > 0$ in order to prevent the unbounded mass accumulation. This control objective may be achieved with the following *positive* control law :

$$\begin{aligned} u(x) &= \max(0, \tilde{u}(x)) \\ \tilde{u}(x) &= \left(\sum_{i=1}^{n} b_i\right)^{-1} \left(\sum_{i=1}^{n} q_i(x)x_i + \lambda(M^* - M(x))\right) \end{aligned}$$

where λ is an arbitrary design parameter.

The set $\Omega = \{x : M(x) = M^* \text{ and } x \in \mathbb{R}_+^n\}$ is called an "iso-mass", because it is the set of all x corresponding to the same total mass M^*.

The stabilizing properties of this control law are given in the following theorem.

Theorem. For the closed loop system (2)-(3) with arbitrary initial conditions in the non-negative orthant $x(0) \in I\!\!R^n_+$:

(i) the iso-mass Ω is forward invariant

(ii) the state $x(t)$ is bounded for all $t > 0$ and converges to the iso-mass Ω.

Proof. Along the closed loop trajectories, we have :

$$\frac{dM(x)}{dt} = -\sum_{i=1}^{n} q_i(x)x_i + \left(\sum_{i=1}^{n} b_i\right) u(x)$$

(i) if $x \in \Omega$, then $M(x) = M^*$ and

$$u(x) = \tilde{u}(x) = \left(\sum_{i=1}^{n} b_i\right)^{-1} \left(\sum_{i=1}^{n} q_i(x)x_i\right)$$

hence $\dot{M}(x) = 0$ which proves that Ω is forward invariant.

(ii) if $x \neq \Omega$, then $u(x) = \begin{cases} 0 & \text{if } \tilde{u}(x) < 0 \\ \tilde{u}(x) & \text{if } \tilde{u}(x) \geq 0 \end{cases}$

Suppose that $\tilde{u}(x) < 0$ and $u(x) = 0$, then $M(x) > M^*$ necessarily.

Consider the Lyapunov function candidate $V = \frac{1}{2}(M^* - M(x))^2$. We have :

$$\dot{V} = (M^* - M(x))\frac{dM(x)}{dt} = (M^* - M(x)) \left(\sum_{i=1}^{n} q_i(x)x_i\right) \leq 0$$

Suppose that $u(x) = \tilde{u}(x) \geq 0$, then

$$\dot{V} = -\lambda(M^* - M(x))^2 \leq 0$$

If $\dot{V} = 0$ then either $x \in \Omega$ which is a forward invariant set of the closed loop (see above)

$$\text{or } x \in \{x : \sum_{i=1}^{n} q_i(x)x_i = 0 \text{ and } M(x) > M^*\}$$

which does not contain any invariant set from Property 5. The result then follows from Lasalle's theorem. ■

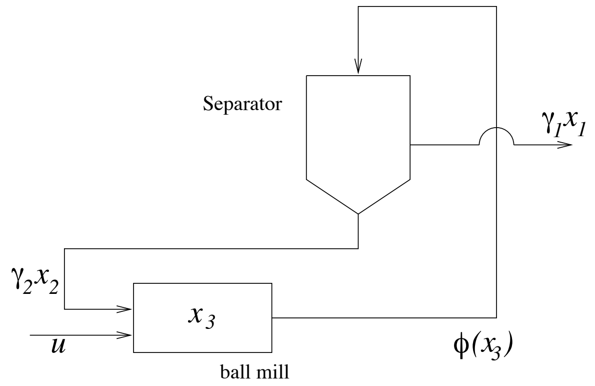

Figure 1: An industrial grinding circuit

4. APPLICATION TO AN INDUSTRIAL GRINDING CIRCUIT

A schematic lay-out of an industrial grinding circuit used in cement industries is depicted in Fig.1. It is made up of the interconnection of a ball mill and a separator as shown in the figure. The ball mill is fed with raw material. After grinding, the milled material is introduced in a separator where the finished product is separated from the oversize particles which are recycled to the ball mill. A simple dynamical model has been proposed (see [3]) for this system under the form:

$$\begin{aligned}\dot{x}_1 &= -\gamma_1 x_1 + (1-\alpha)\phi(x_3)\\ \dot{x}_2 &= -\gamma_2 x_2 + \alpha\phi(x_3)\\ \dot{x}_3 &= \gamma_2 x_2 - \phi(x_3) + u\end{aligned}$$

with the following notations and definitions :

x_1 = amount of finished product in the separator
x_2 = amount of oversize particles in the separator
x_3 = amount of material in the ball mill
u = feeding rate
$\gamma_1 x_1$ = outflow rate of finished product
$\gamma_2 x_2$ = flowrate of recycled product
$\phi(x_3)$ = grinding function

The parameter α is the separation constant of the separator ($0 < \alpha < 1$). The grinding function $\phi(x_3)$ is non monotonic as represented in Fig. 2. It can be written under the form $\phi(x_3) = x_3\varphi(x_3)$ with $\varphi(x_3)$ an appropriate monotonically decreasing function. This model is readily seen to be a special case of the general compartmental system (2) with the following

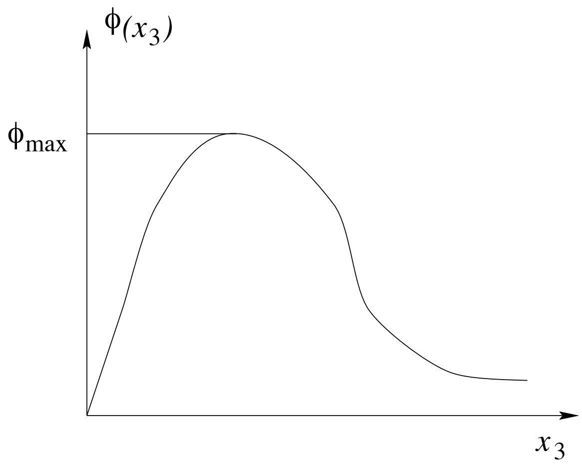

Figure 2: The grinding function

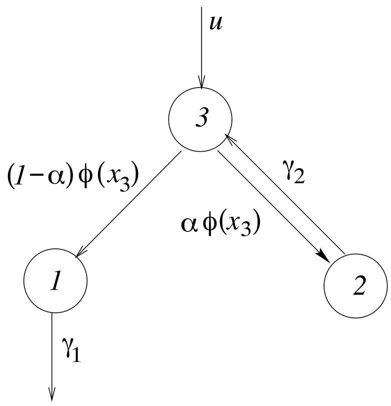

Figure 3: Compartmental network

definitions :

$$
\begin{array}{lll}
r_{12}(x) = 0 & r_{13}(x) = 0 & q_1(x) = \gamma_1 \\
r_{21}(x) = 0 & r_{23}(x) = \gamma_2 & q_2(x) = 0 \\
r_{31}(x) = (1-\alpha)\varphi(x_3) & r_{32}(x) = \alpha\varphi(x_3) & q_3(x) = 0
\end{array}
$$

The network associated to this model is represented in Fig. 3. Remark that only the links corresponding to physically existing transformation rates are represented in the network. In other terms, the fractional rates r_{ij} and q_i that are identically zero are not represented.

When the control input is constant $u = \bar{u}$ (constant) > 0, the global instability of the system appears if the state is initialised in the set D defined by the following inequalities:

$$
D \left\{ \begin{array}{c}
(1-\alpha)\phi(x_3) < \gamma_1 x_1 < \bar{u} \\
\alpha\phi(x_3) < \gamma_2 x_2 \\
\partial\phi/\partial x_3 < 0
\end{array} \right.
$$

Indeed, it can be shown that this set D is positively invariant and if $x(0) \in D$ then $x_1 \to 0$, $x_2 \to 0$, $x_3 \to \infty$. This means that there is an irreversible accumulation of material in the mill with a decrease of the production to zero. In the jargon of cement industries, this is called *mill plugging*. In practice, the state may be lead to the set D by intermittent disturbances like variations of hardness of the raw material.

The iso-mass is $M(x) = x_1 + x_2 + x_3$ and we have $\dot{M}(x) = -x_1 + u$. The control law is written :

$$u(x) = \max(0, \tilde{u}(x))$$

with :

$$\begin{aligned} \tilde{u}(x) &= \gamma_1 x_1 + \lambda(M^* - M(x)) \\ &= \lambda M^* + (\gamma_1 - \lambda)x_1 - \lambda x_2 - \lambda x_3 \end{aligned}$$

It is interesting to analyse the behaviour of the system in the invariant set Ω which is in fact the behaviour of the zero dynamics :

$$\begin{cases} \dot{x}_2 = & -\gamma_2 x_2 + (1-\alpha)\phi(x_3) \\ \dot{x}_3 = & \gamma_1(M^* - x_2 - x_3) + \gamma_2 x_2 - \phi(x_3) \end{cases}$$

The equilibria of the zero-dynamics must satisfy the following relation :

$$\underbrace{\left(\frac{(\gamma_1 - \gamma_2)}{\gamma_2}\alpha + 1\right)\phi(x_3) + \gamma_1 x_3}_{\psi(x_3)} = \gamma_1 M^*$$

The zero-dynamics have a unique equilbrium in Ω if the following inequality is satisfied :

$$\frac{\partial \psi}{\partial x_3} > 0 \Longrightarrow \frac{\partial \phi}{\partial x_3} > -\frac{\gamma_1 \gamma_2}{\alpha\gamma_1 + (1-\alpha)\gamma_2}$$

If $\gamma_2 \geq \gamma_1$, this unique equilibrium is easily shown to be globally asymptotically stable in Ω by using the Bendixsson theorem.

In this case, it follows that the feedback controller is able not only to prevent the mill from plugging by regulating the total mass at an arbitrary set point, but also to stabilise the system at a unique equilibrium which is globally asymptotically stable in its domain of physical existence (the positive orthant).

5. FINAL COMMENTS

1. The controller (3) proposed in this paper has an intereting robustness property. Indeed it is fully independent from the functions $r_{ij}(x)$. This means that the feedback

stabilisation is robust against a full modelling uncertainty regarding the internal transformations between the species inside the system. This is quite important because in many practical applications,the $r_{ij}(x)$ are precisely the most uncertain terms of the model.

2. The condition that the system is fully outflow connected which guarantees the dissipativity of the unforced system is critical for our result. It implies that, in absence of feeding ($u = 0$), there is a natural "wash-out" of the material contained in the system. Besides the fact that it is a common property in many practical applications, it must be emphasized that, without natural dissipativity, there is no hope to globally stabilise the total mass $M(x)$ at an *arbitrary* set point.

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