A system-optimal routing policy for road traffic networks

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A SYSTEM-OPTIMAL ROUTING POLICY FOR ROAD TRAFFIC NETWORKS

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Abstract: This article deals with the establishment of a new real-time routing strategy for road traffic networks. On the basis of the Lighthill-Whitham-Richards single order model, a "system-optimal" strategy is developed and validated with numerical simulations.

Keywords: Optimal routing, LWR, Road network

1. INTRODUCTION

In order to improve the actual efficiency of road networks, different control strategies may be envisaged. Besides ramp metering and speed limitations strategies a routing strategy may be used. Usually there are different paths available for a driver to reach a particular destination. One can think of the ring around a city where there are two possible paths between two particular ramps. It can be assumed that it is possible to influence the fraction of the drivers taking one of the two paths, for example by presenting some information about the time needed on each path.

The road network may be represented by a graph whose edges are roads and the nodes are the junction between the roads. The problem of "optimal routing" consists adjusting the "splitting rate" at some bifurcation node in order to optimize a cost criterion representing the efficiency of the network. This cost may vary depending on the objective : minimizing the pollution, maximizing the total outflow, avoiding to penalize too much some drivers... When real-time route guidance is considered a perfect solution is usually not achievable since the optimal splitting rates depend on the value of the inflows for future time which are not available in real time situations. It is this real-time case which is addressed by the new routing strategy presented in this article. After recalling the model used to describe the evolution of the traffic on a road, the strategy will be presented and validated using numerical simulations.

2. THE LWR MODEL

The routing strategy will be developed and validated using the classical LWR traffic model. In the LWR model (see Lighthill and Whitham (1955) and Richards (1956)), the traffic state is represented from a macroscopic point of view by the

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function $\rho_i(x,t)$ which represents the density of vehicles at position x and time t on road i. The dynamics of the traffic are represented by a conservation law expressed as

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial (\rho_i v_i)}{\partial x} = 0 \qquad (1)$$

where $v_i = v_i(x,t)$ is the velocity of cars at (x,t) on road i. The main assumption of the LWR model is that the drivers instantaneously adapt their speed in function of the surrounding density i.e. :

$$v(x,t) = V(\rho(x,t)). \qquad (2)$$

The function $f(\rho) = \rho V(\rho)$ is then the "flow rate" representing the number of vehicles per time unit passing through a particular position in function of the traffic state at this position. Inserting (2) in (1), the LWR model is :

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial f(\rho_i)}{\partial x} = 0. \qquad (3)$$

In accordance with the physical observations, it is usually assumed that the speed-density relation is a decreasing function ($\frac{\partial V}{\partial \rho} < 0$) defined on the interval $[0, \rho_{\max}]$ with :

$V(0) = V_{\mathbf{max}}$: the maximal velocity of the vehicles when the road is (almost) empty;

$V(\rho_{\mathbf{max}}) = 0$: the velocity drops to zero when the density is maximal and the traffic is totally congested.

Then the flow rate $f(\rho) = \rho V(\rho)$ is a non monotonic function with $f(0) = 0$ and $f(\rho_{\max}) = 0$ which is maximal at some critical value σ : the traffic is moving freely when $\rho < \sigma$ while the traffic is congested when $\rho > \sigma$ (see Fig. 1).

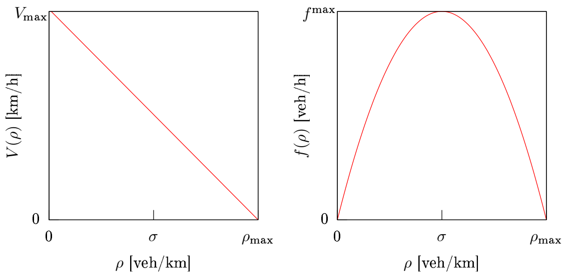

Fig. 1. The speed and the flow in function of the density.

This description of the evolution of the traffic on a infinite single road must be completed by the description of the behavior of the drivers at the junctions. Several models of junctions have been proposed in the literature (see Coclite et al. (2004), Herty and Klar (2003), Jin and Zhang (2003), Lebacque and Khoshyaran (2002) and Holden and Risebro (1995)). These models will not be detailed here since they do not influence the explanation of the new routing strategy.

3. NEW ROUTING CRITERION

Some routing strategy (see Wang et al. (2005)) tends to achieve user-equilibrium in order to improve the global efficiency of the network. This user-equilibrium situation occurs when *each driver* chooses the direction corresponding to his time-minimal path. For example we may consider a bifurcation node where two outgoing paths may be used to reach a final destination. The splitting rate α expresses the fraction of the flow leaving the node using the first outgoing path. If we denote by τ_i the time spent by the drivers to reach the final destination using the outgoing path i then the user-equilibrium condition is

$$\begin{aligned} \Delta\tau &\geq 0 \,\text{if}\, \alpha = 1 \\ \Delta\tau &= 0 \,\text{if}\, 0 < \alpha < 1 \\ \Delta\tau &\leq 0 \,\text{if}\, \alpha = 0 \end{aligned} \qquad (4)$$

where $\Delta\tau = \tau_2 - \tau_1$. This condition expresses the fact that, while the time used to travel on one of the two paths is lower than the time on the other path, the drivers will choose the first one.

The idea of the strategy presented in Wang et al. (2005) is to achieve the user-equilibrium (i.e. keep $\Delta\tau$ close to zero if α is between zero and one) by adapting the factor α with a feedback strategy :

$$\alpha(t) = K_p \Delta\tau(t) + \int K_i \Delta\tau(t) \qquad (5)$$

where $\alpha(t)$ is clipped between zero and one. In general, it is difficult to obtain the exact value of $\Delta\tau(t)$ since this value depends on the traffic state for time greater than t and thus the inputs of the network must be known in advance. One can use the *instantaneous travel time*, the time spent by the driver assuming that the traffic state will not change, to approximate $\Delta\tau(t)$.

An improvement of this strategy can be achieved by considering the fact that a user-equilibrium situation is usually not *system-optimum*. A system-optimum equilibrium corresponds to a particular choice of the splitting rate α which minimizes a cost function. A usual choice made for the cost is the *Total Time Spent* :

$$\text{TTS} = \iint \rho(x,t)\; dxdt \qquad (6)$$

where the integration is done over the whole network for all time. This cost is a "social cost" which represents the time spent by all the drivers on the network. With this definition, the cost rate $M(t)$ is

$$M(t) = \int \rho(x,t) dx$$

where the integration is done over each road. Let denote $M_i(t)$ the contribution of road i to the cost rate :

$$M_i(t) = \int_0^{L_i} \rho_i(x,t)dx$$

where L_i is the length of road i.

To illustrate the difference between the user-equilibrium and the system-optimality, the network represented in Fig 2 may be considerated. In this network all the drivers from the first (resp. second) input leave the network using the first (resp. second) output. The first drivers may choose to use two different paths (road 1 and 2) while the second drivers can only choose one path (road 1). The factor α represents the fraction of the first drivers using road 1. In this simple

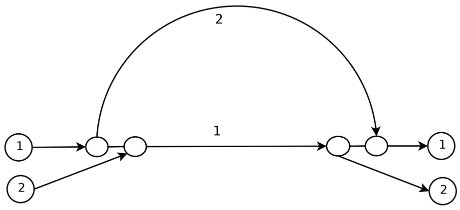

Fig. 2. A simple benchmark network

network if the length of road 2 is greater than the length of the 1 then the difference of cost between the user-optimum and the system-optimum may become important. For simplicity, consider that the length of road 1 is 1 km, the length of road 2 is l, the flow at the two inputs is 3000 [veh/h] and the function linking the density to the speed is

$$V(\rho) = 0.74(180 - \rho)[km/h].$$

The cost rates $M(t)$ and the splitting rates corresponding to the user-optimum and the system-optimum situations at the equilibrium are represented in Fig. 3. It can be seen that the difference between the two costs may be significant, especially for $l = 2$ where the difference is around 50 percent.

The strategy (5) may be improved by analyzing the influence of a change of α on the TTS. A change of the splitting rate α by $\Delta\alpha$ implies a change of the inflow of road i by Δq_i. If the density on the road i is strictly less than ρ_c then the density on road i will change by

$$\Delta\rho_i(x,t) \approx \frac{\Delta q_i}{f'(\rho_i(x,t))}.$$

This is an approximation not only because we take a first order approximation but also because we implicitly assume that the only change of flow on the path i comes from the change of the splitting rate α. The influence of Δq_i on M_i is then

$$\Delta M_i(t) \approx \int_0^{L_i} \frac{\Delta q_i}{f'(\rho_i(x,t))}dx.$$

Since $\Delta q_1 = -\Delta q_2$, the cost rate total change for the whole network is

$$\Delta M(t) \approx \Delta\alpha q_{in} \left(\int_0^{L_1} \frac{1}{f'(\rho_1(x,t))} - \int_0^{L_2} \frac{1}{f'(\rho_2(x,t))} \right) dx \quad (7)$$

where q_{in} is the inflow at the entrance of the bifurcation node. This equation is only valid if $\rho_i(x,t) < \sigma$ since, when this condition is not satisfied at a particular position x, an increase of the inflow of the road i does *not* lead to an increase of the density at x. In this case (7) tends to suggest that an increase of the inflow produces a decrease of the TTS (since $f'(\rho_i(x,t)) < 0$ when $\rho_i(x,t) > \sigma$) even it is clearly not the case. In order to alleviate this limitation and to penalize the inflow increase of a jammed road, (7) may be modified as

$$\Delta GM(t) \approx \Delta\alpha q_{in} \left(\int_0^{L_1} \frac{1}{g(f'(\rho_1(x,t)))} - \int_0^{L_2} \frac{1}{g(f'(\rho_2(x,t)))} \right) dx \quad (8)$$

where

$$g(x) = \begin{cases} x & \text{if } x \geq f'(\sigma-\epsilon) \\ f'(\sigma-\epsilon)\left(\dfrac{1}{2 - \frac{x}{f'(\sigma-\epsilon)}}\right) & \text{if } x \leq f'(\sigma-\epsilon) \end{cases} \quad (9)$$

where $\epsilon > 0$. With this choice of g, ΔGM is equal to ΔM for ρ less than $\sigma - \epsilon$ and an increase of flow on a jammed road is heavily penalized.

Based on this particular measure of the TTS evolution, some "system-optimality" conditions may be derived. These conditions are exact with respect the cost function (6) only at equilibrium and if $\rho_i < \sigma$ on all roads due to the various approximations made. Expressed in the same form as (4), the conditions are :

$$\begin{aligned} \Delta c &\geq 0 \,\text{if}\, \alpha = 1 \\ \Delta c &= 0 \,\text{if}\, 0 < \alpha < 1 \\ \Delta c &\leq 0 \,\text{if}\, \alpha = 0 \end{aligned} \quad (10)$$

where $\Delta c = c_2 - c_1$ and

$$c_i = \int_0^{L_1} \frac{1}{g(f'(\rho_1(x,t)))}dx.$$

These conditions express the fact that while the TTS can be decreased by modyfing α, the splitting rate must decrease or increase depending on the sign of Δc. When $\Delta c = 0$, the splitting rate α may be kept constant.

Based on these conditions, and by analogy with (5), a new strategy may be derived. The idea of

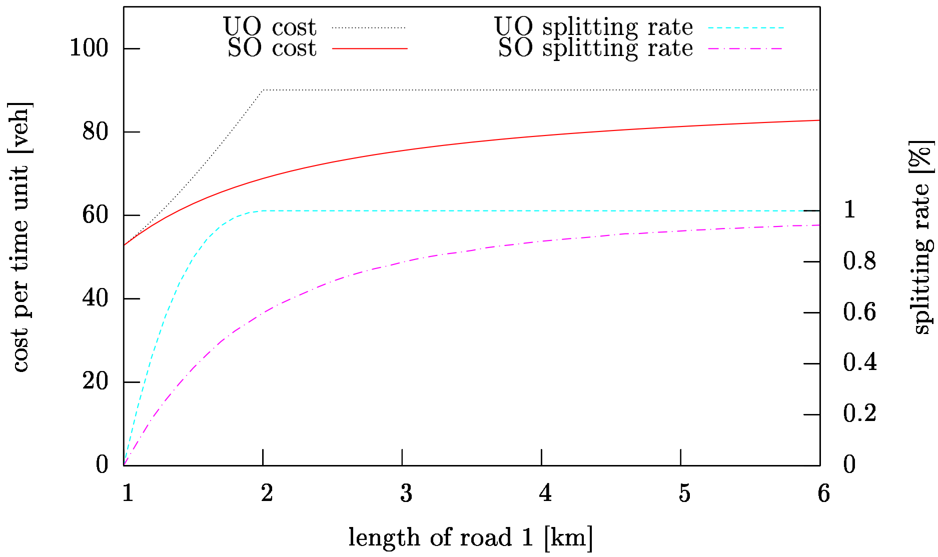

Fig. 3. The evolution of the user-optimum (UO) and system-optimum (SO) costs and the splitting rates α in function of the length of road 1.

this new strategy is to keep Δc close to zero if α is between zero and one by adapting α using a PI controller :

$$\alpha(t) = K_p \Delta c(t) + \int K_i \Delta c(t) \tag{11}$$

where $\alpha(t)$ is clipped between zero and one.

4. NUMERICAL SIMULATIONS

A set of numerical simulations were performed to compare this new routing strategy (11) with the user-equilibrium strategy (5). The network considered is represented in Fig. 2. The results of four scenarii with different inflows are represented in Table 1. The parameters of the controller were the same for the two strategies and are not particularly tuned in order to achieve the best TTS for this particular network. In all cases, the simulation is two hours long. The length of road 1 is 5 km and the length of road 2 is 7 km. The function $V(\rho)$ used in the simulation is a simple linear relation such that $f^{\max} = 6000$ veh/h.

The four scenarii are :

(1) At the two inputs, the inflows are constant (3000 veh/h). The two strategies lead to a constant splitting rate ($\alpha = 0.47$ for the system-optimum and $\alpha = 1$ for the user-optimum).
(2) At the two inputs, the inflows are constant (3200 veh/h for input 1 and 3000 veh/h for input 2). In this case, the two strategies produced two strongly different TTS. The reason is that the user-optimum strategy leads to a splitting rate of one : all drivers try to pass through road 1. But since the traffic on road 1 is limited to 6000 veh/h and 6200 veh/h are trying to pass, a traffic jam is growing before the bifurcation node.
(3) The inflows of the two inputs vary with time (see Fig. 4). Sometimes the total inflow is higher than the capacity of road 1, sometimes it is lower.
(4) The inflows are the same as in scenario 3 with the artificial appearance of a traffic jam to validate the use of the function g (9) when the density is higher than σ. This traffic jam appears by limiting the outflow of the output 1 to 300 veh/h during 6 minutes after 30 minutes of simulation.

As represented in Table 1, for the four scenarii, using the new routing strategy (11) allows a better TTS on the network at the end of the simulation.

5. CONCLUSIONS

As illustrated in the numerical simulations, this new strategy is able to achieve better performance. In this example the benefit on the TTS is important (around 15%) since the difference between the user-equilibrium and the system-equilibrium cost on this network is significant.

The main difference between the two strategies is that (5) assumes that increasing the flow on a road will not change the time spent by the users on this road while (11) takes into account the change of speed on the road (via f') due to an increase of the inflow.

This strategy was developed and numerically validated with the LWR model. It would be interesting to validate it in real conditions using an experimental function $f(\rho)$ or at least using a more precise traffic model using a function $V(\rho)$

	Scenario 1	Scenario 2	Scenario 3	Scenario 4
user-optimum strategy	847	1246	754	864
system-optimum strategy	697	731	644	738
benefit	17.8 %	41.3 %	14,6 %	14.6 %

Table 1. The Total Time Spent (veh·h) on the network for the four scenarii.

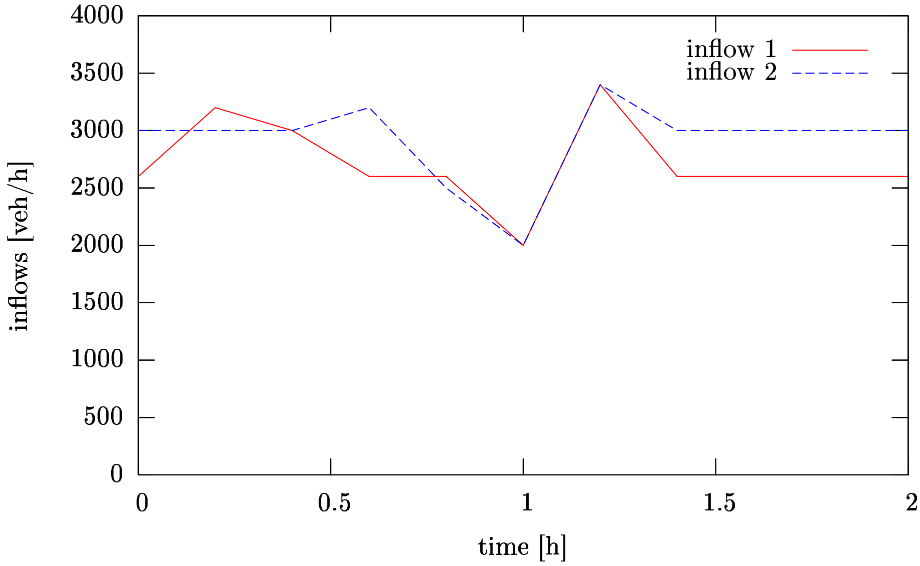

Fig. 4. The inflows of the two inputs for scenarii 3 and 4.

describing the preferential speed of the drivers such that $f(\rho) = \rho V(\rho)$. For example, it may be interesting to evaluate the performance of this new strategy using a second order model (like Aw and Rascle (2000)) where a relaxation term was added.

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