SIAM J. CONTROL OPTIM.
Vol. 53, No. 3, pp. 1464–1483

DISSIPATIVE BOUNDARY CONDITIONS FOR ONE-DIMENSIONAL QUASI-LINEAR HYPERBOLIC SYSTEMS: LYAPUNOV STABILITY FOR THE C^1-NORM*

JEAN-MICHEL CORON† AND GEORGES BASTIN‡

Abstract. This paper is concerned with boundary dissipative conditions that guarantee the exponential stability of classical solutions of one-dimensional quasi-linear hyperbolic systems. We present a comprehensive review of the results that are available in the literature. The main result of the paper is then to supplement these previous results by showing how a new Lyapunov stability approach can be used for the analysis of boundary conditions that are known to be dissipative for the C^1-norm.

Key words. hyperbolic systems, Lyapunov stability

AMS subject classifications. 35F30, 35F25, 93D20, 93D30

DOI. 10.1137/14097080X

1. Introduction.

We are concerned with one-dimensional quasi-linear strictly hyperbolic systems of the form

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \quad x \in [0,1],\, t \in [0,+\infty), \tag{1.1}$$

where $\mathbf{u} : [0,\infty) \times [0,1] \to \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathcal{M}_{n,n}(\mathbb{R})$, $\mathcal{M}_{n,n}(\mathbb{R})$ denoting, as usual, the set of $n \times n$ real matrices.

Since the system is strictly hyperbolic, the matrix $A(\mathbf{0})$ has, by definition, n real eigenvalues denoted $\Lambda_1, \dots, \Lambda_n$ satisfying

$$\Lambda_i \neq \Lambda_j \quad \forall i,j \in \{1,\dots,n\} \text{ such that } i \neq j. \tag{1.2}$$

We assume that none of these eigenvalues is zero. Then, possibly after a suitable linear change of variables, we may assume that there exists $m \in \{0,1,\dots,n\}$ such that

$$\Lambda_i > 0 \quad \forall i \in \{1,\dots,m\}, \qquad \Lambda_i < 0 \quad \forall i \in \{m+1,\dots,n\},$$

and that $A(\mathbf{0})$ is a diagonal matrix:

$$A(\mathbf{0}) \triangleq \operatorname{diag}\{\Lambda_1, \Lambda_2, \dots, \Lambda_n\}.$$

Our concern is to analyze the asymptotic convergence of the classical solutions of system (1.1) under a boundary condition of the form

$$\begin{pmatrix}\mathbf{u}_+(t,0)\\ \mathbf{u}_-(t,1)\end{pmatrix} = G\begin{pmatrix}\mathbf{u}_+(t,1)\\ \mathbf{u}_-(t,0)\end{pmatrix}, \qquad t \in [0,+\infty), \tag{1.3}$$

*Received by the editors May 29, 2014; accepted for publication (in revised form) March 2, 2015; published electronically June 9, 2015. This work was supported by the ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7) and by the Belgian Programme on Interuniversity Attraction Poles (IAP VII/19).
http://www.siam.org/journals/sicon/53-3/97080.html
†Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 4 place Jussieu, 75252 Paris cedex 05, France (coron@ann.jussieu.fr).
‡Corresponding author. Department of Mathematical Engineering, ICTEAM, Université Catholique de Louvain, 4 avenue G. Lemaitre, 1348 Louvain-la-Neuve, Belgium (georges.bastin@uclouvain.be).

where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at $\mathbf{0}$ while $\mathbf{u}_+ \in \mathbb{R}^m$, $\mathbf{u}_- \in \mathbb{R}^{n-m}$ are defined by requiring that $\mathbf{u} \triangleq (\mathbf{u}_+^\mathsf{T}, \mathbf{u}_-^\mathsf{T})^\mathsf{T}$. The challenge is to give explicit conditions on the map G such that the boundary condition (1.3) is exponentially dissipative, i.e., implies that the equilibrium solution $\mathbf{u} \equiv \mathbf{0}$ of system (1.1) with the boundary condition (1.3) is exponentially stable.

In order to state our results, we first introduce the functions $\rho_p : \mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\rho_p(M) \triangleq \inf \left\{ \|\Delta M \Delta^{-1}\|_p, \ \ \Delta \in \mathcal{D}_n^+ \right\}, \qquad 1 \leqslant p \leqslant \infty, \tag{1.4}$$

where $\mathcal{D}_n^+$ denotes the set of diagonal $n \times n$ real matrices with strictly positive diagonal entries and with

$$\|\boldsymbol{\xi}\|_p \triangleq \left[\sum_{i=1}^{n} |\xi_i|^p \right]^{\frac{1}{p}}, \ \ \|\boldsymbol{\xi}\|_\infty \triangleq \max\{|\xi_i|; i \in \{1, \ldots, \}\} \qquad \forall \boldsymbol{\xi} \triangleq (\xi_1, \ldots, \xi_n)^\mathsf{T} \in \mathbb{R}^n,$$

$$\text{and} \ \ \|M\|_p \triangleq \max_{\|\boldsymbol{\xi}\|_p = 1} \|M\boldsymbol{\xi}\|_p \qquad \forall M \in \mathcal{M}_{n,n}(\mathbb{R}).$$

In this paper, our main result (section 3) is a new proof, achieved using a Lyapunov function approach, that a sufficient condition for the exponential stability of the steady state $\mathbf{u} \equiv \mathbf{0}$ for the C^1-norm is to have a map G such that

$$\rho_\infty(G'(\mathbf{0})) < 1, \tag{1.5}$$

where G' denotes the Jacobian matrix of the map G.

This new proof supplements the previous results which are already available in the literature and are reviewed in section 2.

2. Literature review. The well-posedness of the Cauchy problem associated with nonlinear hyperbolic systems of the kind (1.1) and (1.3) was studied in 1985 by Li and Yu [26] in the framework of C^1-solutions. For such systems, the issue of finding sufficient dissipative boundary conditions has been addressed in the literature for more than thirty years. To our knowledge, the first results were published by Slemrod [41] in 1983 and by Greenberg and Li [17] in 1984 for the special case of systems of size $n = 2$. A generalization to systems of size n was then progressively elaborated by the students of Ta-Tsien Li, in particular by Qin [36] in 1985 and by Zhao [46] in 1986. All these contributions deal with the particular case of "local" boundary conditions having the specific form

$$\mathbf{u}_+(t, 0) = G_0(\mathbf{u}_-(t, 0)), \qquad \mathbf{u}_-(t, 1) = G_1(\mathbf{u}_+(t, 1)). \tag{2.1}$$

With these boundary conditions, the analysis can be based on the method of characteristics, which can exploit an explicit computation of the "reflection" of the solutions at the boundaries along the characteristic curves. This has given rise to the sufficient condition

$$\rho_\infty \left(\begin{pmatrix} \mathbf{0} & G_0'(\mathbf{0}) \\ G_1'(\mathbf{0}) & \mathbf{0} \end{pmatrix} \right) < 1 \tag{2.2}$$

for the dissipativity of the boundary conditions (2.1) for the C^1-norm. This result is given, for instance, by Li [24, Theorem 1.3, p. 173] in his seminal book of 1994 on the stability of the classical solutions of quasi-linear hyperbolic systems. Finally, by using an appropriate dummy doubling of the system size, de Halleux et al. [13, Theorem 4] showed in 2003 how the general dissipative boundary condition $\rho_\infty(G'(\mathbf{0})) < 1$ can be established for systems with the general "nonlocal" boundary condition (1.3). This dummy doubling has also been used by Li, Rao, and Wang [25] to prove the well-posedness of the Cauchy problem associated with (1.1) and (1.3) still in the framework of C^1-solutions.

Another approach to the analysis of dissipative boundary conditions is based on the use of Lyapunov functions. The first attempts were in using entropies as Lyapunov functions, as done, for instance, by Coron, Andréa-Novel, and Bastin [9] in 1999 or by Leugering and Schmidt [23] in 2002. The drawback of this approach was, however, that the time derivatives of such entropy-based Lyapunov functions are necessarily only *semidefinite negative*. Hence one would like to conclude with the LaSalle invariant set principle. However, this principle requires the precompactness of the trajectories, a property which is difficult to get in the case of nonlinear partial differential equations.

In order to overcome this difficulty, Coron, Andréa-Novel, and Bastin [10] in 2007 proposed, for systems of size $n = 2$, a strict Lyapunov function whose time derivative is *strictly negative definite* when $\rho_\infty(G'(\mathbf{0})) < 1$. The advantage of this is that the proof is less elaborate than that using the method of characteristics because it uses more direct computations. Furthermore, another advantage of a Lyapunov analysis is to directly induce robustness properties with respect to small uncertainties and disturbances. Then, in the paper [8], Coron, Bastin, and Andréa-Novel generalized this Lyapunov approach to general nonlinear hyperbolic systems of the kind (1.1) and (1.3). In particular, they emphasize a new weaker dissipative boundary condition which is formulated as follows.

A sufficient condition for the exponential stability of the steady state $\mathbf{u} \equiv \mathbf{0}$ for the H^2-norm is to have a map G such that

$$\rho_2(G'(\mathbf{0})) < 1. \tag{2.3}$$

Moreover, it is also shown in the same paper that $\rho_2(G'(\mathbf{0})) \leqslant \rho_\infty(G'(\mathbf{0}))\ \ \forall n \geqslant 1$ and that this inequality can be strict if $n \geqslant 2$. However, it has been proved in [11] that $\rho_2(G'(\mathbf{0})) < 1$ is not a sufficient condition for the exponential stability of the steady state $\mathbf{u} \equiv \mathbf{0}$ for the C^1-norm.

Various recent contributions and extensions of the previous results are also worth mentioning:

- Dos Santos and Maschke [39] in 2009 gave a Hamiltonian perspective to the stabilization of systems of two conservation laws.
- Using the Lyapunov approach, Castillo et al. [7] in 2013 gave sufficient conditions for the existence of exponentially stable observers in the case where all eigenvalues Λ_i are positive.
- Perrollaz and Rosier [31] showed in 2013 that there exist boundary dissipative conditions for systems of size $n = 2$ that achieve finite-time stabilization.
- Using an approach via time delay equations, Coron and Nguyen [11] showed in 2014 that $\rho_p(G'(\mathbf{0})) < 1$ implies the exponential stability in the Sobolev norm $W^{2,p}$.

For so-called inhomogeneous quasi-linear hyperbolic systems (i.e., with additional zero-order terms),

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u}) = \mathbf{0},$$

the analysis of dissipative boundary conditions is much more intricate and only very partial results are known.

- Using the method of characteristics, Prieur, Winkin, and Bastin [35] in 2008 showed that the stability condition (2.2) holds for inhomogeneous systems when $\|B(\mathbf{u})\|$ is small enough. Prieur [32] also showed in 2009 how this result can be extended to deal with differential or integral boundary errors.
- Using the Lyapunov approach, Gugat and Herty [19] in 2009 and Gugat, Dick, and Leugering [18] in 2011 analyzed the boundary feedback stabilization of gas flow in fan-shaped networks described by isentropic Euler equations.
- Using the Lyapunov approach, Pavel and Chang [30] in 2012 analyzed the boundary control of hyperbolic Lotka–Volterra systems, with application to pumped Raman amplifiers on optical fibres (see also Dower and Farrell [16] from 2006).
- For inhomogeneous systems of size $n = 2$ with $m = 1$, Coron et al. [12] showed in 2013 the existence of the full-state feedback law, which achieves exponential stability for the H^2-norm. The proof uses a backstepping transformation (see [22]) to find new variables for which a strict Lyapunov function can be constructed.

Linear hyperbolic systems. A linear hyperbolic system,

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{0}, \quad x \in [0, 1], \, t \in [0, +\infty), \tag{2.4}$$

with linear boundary conditions

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,1) \end{pmatrix} = K \begin{pmatrix} \mathbf{u}_+(t,1) \\ \mathbf{u}_-(t,0) \end{pmatrix}, \qquad t \in [0, +\infty), \tag{2.5}$$

is a special case of (1.1) and (1.3), where the matrix $A \in \mathcal{D}_n(\mathbb{R})$ with nonzero distinct diagonal entries, $\mathcal{D}_n(\mathbb{R})$ denotes the set of $n \times n$ diagonal real matrices, and the matrix $K \in \mathcal{M}_{n,n}(\mathbb{R})$. As can be expected, the analysis of dissipative boundary conditions is both simpler and more comprehensive for linear than for quasi-linear systems.

The linear hyperbolic system (2.4)–(2.5) is equivalent to a system of interconnected scalar time-delays $\tau_i \triangleq \Lambda_i^{-1}$. For such systems, it follows from a theorem by Silkowski [40] (quoted in [20, Chapter 9, Theorem 6.1]; see also [29]), which relies on the *Kronecker density theorem* (e.g., [6]), that the boundary conditions (2.5) are, for any L^p-norm, *robustly* dissipative with respect to arbitrary small perturbations on the Λ_i's if and only if

$$\bar{\rho}(K) \triangleq \max\{\rho(\mathrm{diag}(e^{\iota\theta_1}, \ldots, e^{\iota\theta_n})K; (\theta_1, \ldots, \theta_n)^\mathsf{T} \in \mathbb{R}^n\} < 1, \tag{2.6}$$

where $\iota \in \mathbb{C}$ is such that $\iota^2 = -1$ and $\rho(M)$ is the spectral radius of $M \in \mathcal{M}_{n,n}(\mathbb{C})$. It is shown in [4, 8] that $\bar{\rho}(K) = \rho_2(K)$ if there exists a permutation matrix P such that the matrix $\tilde{K} = PKP^{-1}$ is a block diagonal matrix $\tilde{K} = \mathrm{diag}\{\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_p\}$, where each block $\tilde{K}_i$ is a real $n_i \times n_i$ matrix with $n_i \in \{1, 2, 3, 4, 5\}$. Moreover, when $n > 5$, there always exists K such that $\bar{\rho}(K) < \rho_2(K)$.

The exponential stability of nonuniform inhomogeneous linear hyperbolic systems has also received a lot of attention in the literature for a long time and can be traced back to Rauch and Taylor [37, section 4] in 1974 or Russell [38, section 3] in 1978. These systems are written as

$$\mathbf{u}_t + A(x)\mathbf{u}_x + B(x)\mathbf{u} = 0, \tag{2.7}$$

where $A(x) \in \mathcal{D}_n(\mathbb{R})$ and $B(x) \in \mathcal{M}_{n,n}(\mathbb{R})$ $\forall x \in [0,1]$. Using the Lyapunov approach (as do, for instance, Xu and Sallet [45] in 2002 or Diagne, Bastin, and Coron [14] in 2012 for the uniform case), it can be shown that the equilibrium $\mathbf{u} \equiv \mathbf{0}$ is exponentially stable for the L^2-norm if there exists a positive diagonal matrix P such that the following hold:

(a) $P\left(\begin{smallmatrix} |A_+(1)| & 0 \\ 0 & |A_-(0)| \end{smallmatrix}\right) - K^{\mathsf{T}} P \left(\begin{smallmatrix} |A_+(0)| & 0 \\ 0 & |A_-(1)| \end{smallmatrix}\right) K$ is positive definite;

(b) $PB(x) + B^{\mathsf{T}}(x)P - P\partial_x|A(x)|$ is semi–positive definite for all $x \in [0,1]$,

where, for every $x \in [0,1]$, $A_+(x) \in \mathcal{D}_m(\mathbb{R})$ and $A_-(x) \in \mathcal{D}_{n-m}(\mathbb{R})$ are defined by requiring

$$A(x) = \begin{pmatrix} A_+(x) & 0 \\ 0 & A_-(x) \end{pmatrix}.$$

Obviously conditions (a) and (b) are rather conservative. Bastin and Coron [3] in 2011 gave a less restrictive sufficient condition for inhomogeneous linear systems of size $n = 2$ with $m = 1$ under the form of the existence of the solution of an associated ordinary differential equation.

Other recent significant contributions are worth mentioning:

- Tchousso, Besson, and Xu [43] showed in 2009 how the Lyapunov approach can be extended to inhomogeneous linear hyperbolic systems of higher spatial dimension.
- Litrico and Fromion [27] in 2009 used a frequency domain method for the boundary control of homogeneous systems of size $n = 2$. They showed that the transfer functions belong to the Callier–Desoer algebra, which opens the way to necessary and sufficient conditions for closed-loop stability and the use of a Nyquist-type test.
- Amin, Hante, and Bayen [2] and Prieur, Girard, and Witrant [33] showed in 2012 how the Lyapunov approach can be extended to switched linear hyperbolic systems.
- Prieur and Mazenc [34] in 2012 showed how time-varying strict Lyapunov functions can be defined to get input-to-state stability for time-varying inhomogeneous linear hyperbolic systems.
- Dick, Gugat, and Leugering [15] in 2012 addressed the feedback stabilization of quasi-linear hyperbolic systems with varying delays.
- The use of a backstepping method for boundary stabilization by full state feedback and observer design for nonuniform inhomogeneous linear systems has also been extensively studied recently: Krstic et al. [21] in 2008 and Smyshlyaev, Cerpa, and Krstic [42] in 2010 for unstable wave equations; Vazquez, Krstic, and Coron [44] in 2011 for general systems of size $n = 2$ with $m = 1$; Di Meglio, Vazquez, and Krstic [28] in 2012 for systems with $m = n - 1$ and a single controller at the right boundary; and Aamo [1] in 2013 for the rejection of differential boundary disturbances.
- Using a frequency domain approach, Bastin, Coron, and Tamasoiu [5] in 2015 gave necessary and sufficient conditions for the stability of linear density-flow systems under proportional-integral control.

Our contribution in this paper fills a small gap in this panorama of analysis of boundary dissipative conditions for one-dimensional linear and quasi-linear hyperbolic systems. We show how the Lyapunov approach can be used to establish the dissipativity of the boundary condition $\rho_\infty(G'(\mathbf{0})) < 1$.

3. Main result. We consider hyperbolic systems of the form

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \quad x \in [0,1],\, t \in [0,+\infty), \tag{3.1}$$

under a boundary condition of the form

$$\begin{pmatrix} \mathbf{u}_+(t,0) \\ \mathbf{u}_-(t,1) \end{pmatrix} = G\begin{pmatrix} \mathbf{u}_+(t,1) \\ \mathbf{u}_-(t,0) \end{pmatrix}, \qquad t \in [0,+\infty). \tag{3.2}$$

For σ, let $\mathcal{B}_\sigma$ be the open ball with radius σ in $\mathbb{R}^n$ for the norm

$$|\mathbf{u}|_0 \triangleq \max\{|u_j|;\, j \in \{1,\dots,n\}\}, \qquad \mathbf{u} = (u_1,\dots,u_n)^\mathsf{T} \in \mathbb{R}^n. \tag{3.3}$$

We assume that, for some $\sigma \in (0,+\infty)$, $A : \mathcal{B}_\sigma \to \mathcal{M}_{n,n}(\mathbb{R})$ and $G : \mathcal{B}_\sigma \to \mathbb{R}^n$ are of class C^1. We define $A_+(\mathbf{u}) \in \mathcal{M}_{m,n}(\mathbb{R})$, $A_-(\mathbf{u}) \in \mathcal{M}_{(n-m),n}(\mathbb{R})$, $G_+(\mathbf{u}) \in \mathbb{R}^m$, and $G_-(\mathbf{u}) \in \mathbb{R}^{n-m}$ by requiring

$$A(\mathbf{u}) = \begin{pmatrix} A_+(\mathbf{u}) \\ A_-(\mathbf{u}) \end{pmatrix}, \qquad G(\mathbf{u}) = \begin{pmatrix} G_+(\mathbf{u}) \\ G_-(\mathbf{u}) \end{pmatrix}.$$

Our concern is to analyze the exponential stability of this system for the C^1-norm under an initial condition

$$\mathbf{u}(0,x) = \mathbf{u}^o(x), \quad x \in (0,1), \tag{3.4}$$

which satisfies the compatibility conditions

$$\begin{pmatrix} \mathbf{u}^o_+(0) \\ \mathbf{u}^o_-(1) \end{pmatrix} = G\begin{pmatrix} \mathbf{u}^o_+(1) \\ \mathbf{u}^o_-(0) \end{pmatrix}, \tag{3.5}$$

$$\begin{aligned} A_+(\mathbf{u}^o(0))\mathbf{u}^o_x(0) = {} & \left[G'_{+\mathbf{u}_+}\begin{pmatrix} \mathbf{u}^o_+(1) \\ \mathbf{u}^o_-(0) \end{pmatrix}\right] A_+(\mathbf{u}^o(1))\mathbf{u}^o_x(1) \\ & + \left[G'_{+\mathbf{u}_-}\begin{pmatrix} \mathbf{u}^o_+(1) \\ \mathbf{u}^o_-(0) \end{pmatrix}\right] A_-(\mathbf{u}^o(0))\mathbf{u}^o_x(0), \end{aligned} \tag{3.6}$$

$$\begin{aligned} A_-(\mathbf{u}^o(1))\mathbf{u}^o_x(1) = {} & \left[G'_{-\mathbf{u}_+}\begin{pmatrix} \mathbf{u}^o_+(1) \\ \mathbf{u}^o_-(0) \end{pmatrix}\right] A_+(\mathbf{u}^o(1))u^o_x(1) \\ & + \left[G'_{-\mathbf{u}_-}\begin{pmatrix} \mathbf{u}^o_+(1) \\ \mathbf{u}^o_-(0) \end{pmatrix}\right] A_-(\mathbf{u}^o(0))\mathbf{u}^o_x(0). \end{aligned} \tag{3.7}$$

Let us now define some norms which will be useful in stating and proving our results. For $\mathbf{u} \in C^0([0,1];\mathbb{R}^n)$ (resp., in $C^0([0,1]\times[T_1,T_2];\mathbb{R}^n)$ one defines

$$|\mathbf{u}|_0 \triangleq \max\{|\mathbf{u}(x)|_0;\, x \in [0,1]\} \tag{3.8}$$

$$(\text{resp., } |\mathbf{u}|_0 \triangleq \max\{|\mathbf{u}(t,x)|_0;\, t \in [T_1,T_2],\, x \in [0,1]\}). \tag{3.9}$$

For $\mathbf{u} \in C^1([0,1];\mathbb{R}^n)$ (resp., in $C^1([T_1,T_2]\times[0,1];\mathbb{R}^n)$) we define

$$|\mathbf{u}|_1 \triangleq |\mathbf{u}|_0 + |\mathbf{u}'|_0 \tag{3.10}$$

$$(\text{resp., } |\mathbf{u}|_1 \triangleq |\mathbf{u}|_0 + |\partial_t\mathbf{u}|_0 + |\partial_x\mathbf{u}|_0). \tag{3.11}$$

Let us first recall the following theorem, due to Li and Yu [26], on the well-posedness of the Cauchy problem (3.1)–(3.2)–(3.4).

THEOREM 3.1. *Let $T > 0$. There exist $C_1 > 0$ and $\varepsilon_1 > 0$ such that, for every $\mathbf{u}^o \in C^1([0,1];\mathbb{R}^n)$ satisfying the compatibility conditions* (3.5) *to* (3.7) *and such that*

$$|\mathbf{u}^o|_1 \leqslant \varepsilon_1, \tag{3.12}$$

there exists one and only one solution on $[0,T]\times[0,1]$ to the Cauchy problem (3.1)–(3.2)–(3.4)*. Moreover, this solutions satisfies*

$$|\mathbf{u}|_1 \leqslant C_1|\mathbf{u}^0|_1. \tag{3.13}$$

(In fact, [26] deals with the case of local boundary conditions (2.1); however, the general case follows from this particular case by using the dummy doubling of the system size introduced in de Halleux et al. [13], already mentioned in section 2; see [25].)

Our definition of exponential stability for the C^1-norm is as follows.

DEFINITION 3.2. *The steady state $x \in [0,1] \mapsto \mathbf{0} \in \mathbb{R}^n$ of the system* (3.1)–(3.2) *is exponentially stable for the C^1-norm if there exist $\varepsilon > 0$, $\nu > 0$, and $C > 0$ such that, for every $\mathbf{u}^o$ such that $|\mathbf{u}^o|_1 < \varepsilon$ and satisfying the compatibility conditions* (3.5) *to* (3.7)*, the Cauchy problem* (3.1)–(3.2)–(3.4) *has a unique C^1-solution which satisfies*

$$|\mathbf{u}(t,.)|_1 \leqslant Ce^{-\nu t}|\mathbf{u}^o|_1 \quad \forall t \in [0,+\infty).$$

The main goal of this article is to give a new proof of the following theorem due to Qin [36] and Zhao [46] (see also Li [24, Theorem 1.3, p. 173] and de Halleux et al. [13, Theorem 4]).

THEOREM 3.3. *If*

$$\rho_\infty(G'(\mathbf{0})) < 1, \tag{3.14}$$

the steady state $x \in [0,1] \mapsto \mathbf{0} \in \mathbb{R}^n$ of the system (3.1)–(3.2) *is exponentially stable for the C^1-norm.*

In the remaining part of this section we give our proof of Theorem 3.3. This proof is divided in two parts:

- In section 3.1, we study the case where all the Λ_i are positive, i.e., the case where $m = n$.
- In section 3.2, we explain how to modify the Lyapunov function introduced in section 3.1 in order to treat the general case.

3.1. Proof of Theorem 3.3 in the special case where $m = n$. For the clarity of the demonstration, we shall first prove the theorem in the special case where $m = n$, which means the matrix $A(\mathbf{0})$ is the positive diagonal matrix $\text{diag}\{\Lambda_1,\ldots,\Lambda_n\}$ with $\Lambda_i > 0$ $\forall i = 1,\ldots,n$. In that case, the boundary condition (3.2) and the compatibility conditions (3.5) to (3.7) are simply rewritten as

$$\mathbf{u}(t,0) = G(\mathbf{u}(t,1)), \tag{3.15}$$

$$\mathbf{u}^o(0) = G(\mathbf{u}^o(1)), \quad A(\mathbf{u}^o(0))\mathbf{u}^o_x(0) = G'(\mathbf{u}^o(1))A(\mathbf{u}^o(1))\mathbf{u}^o_x(1). \tag{3.16}$$

As already mentioned above, our proof is based on a Lyapunov approach. In order to define an appropriate Lyapunov function for the analysis, we need the following technical classical lemma, for which our assumption (1.2) is crucial and which also holds if $m \neq n$.

LEMMA 3.4. *Let $D(\mathbf{u})$ be the diagonal matrix whose diagonal entries are the eigenvalues $\lambda_i(\mathbf{u})$, $i = 1, \ldots, n$, of the matrix $A(\mathbf{u})$. There exist a positive real number $\eta < \sigma$ and a map $M : \mathcal{B}_\eta \to \mathcal{M}_{n,n}(\mathbb{R})$ of class C^1 such that*

$$M(\mathbf{u})A(\mathbf{u}) = D(\mathbf{u})M(\mathbf{u}) \quad \forall \mathbf{u} \in \mathcal{B}_\eta, \tag{3.17}$$

$$M(\mathbf{0}) = Id_n, \tag{3.18}$$

where Id_n is the identity matrix of $\mathcal{M}_{n,n}(\mathbb{R})$.

Let $\mathbf{W}_1 : C^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ and $\mathbf{W}_2 : C^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\mathbf{W}_1(\mathbf{u}) \triangleq \left(\int_0^1 \sum_{i=1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j \right)^{2p} e^{-2p\mu x} dx \right)^{\frac{1}{2p}}, \tag{3.19}$$

$$\mathbf{W}_2(\mathbf{u}) \triangleq \left(\int_0^1 \sum_{i=1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_t u_j \right)^{2p} e^{-2p\mu x} dx \right)^{\frac{1}{2p}}, \tag{3.20}$$

with $p \in \mathbb{N}_+ \triangleq \{1, 2, 3, \ldots\}$ and $p_i > 0$ $\forall i \in \{1, \ldots, n\}$. In (3.19) and (3.20), $m_{ij}(\mathbf{u})$ denotes the (i,j)th entry of the matrix $M(\mathbf{u})$, and in (3.20), $\partial_t u_j$ is defined by $(\partial_t u_1, \ldots, \partial_t u_n)^{\mathsf{T}} = \partial_t \mathbf{u}$, with

$$\partial_t \mathbf{u} \triangleq -A(\mathbf{u})\partial_x \mathbf{u} \quad \forall \mathbf{u} \in C^1([0,1]; \mathbb{R}^n) \text{ with } |\mathbf{u}|_0 \text{ small enough.} \tag{3.21}$$

We use this slight abuse of (useful) notation (3.21) elsewhere in this article.

Throughout this section, $\mathbf{u} : [0,T] \to \mathbb{R}^n$ denotes a C^1-solution of (3.1) and (3.15). We define $W_1 : [0,T] \to \mathbb{R}$ and $W_2 : [0,T] \to \mathbb{R}$ by

$$W_1(t) \triangleq \mathbf{W}_1(\mathbf{u}(t)), \quad W_2(t) \triangleq \mathbf{W}_2(\mathbf{u}(t)) \quad \forall t \in [0,T]. \tag{3.22}$$

In (3.22) and in the following, for $t \in [0,T]$, $\mathbf{u}(t) : [0,1] \to \mathbb{R}^n$ is defined by

$$\mathbf{u}(t)(x) \triangleq \mathbf{u}(t,x) \quad \forall x \in [0,1]. \tag{3.23}$$

Of course $W_1(t)$ and $W_2(t)$ depend also on $\mathbf{u}$. So we should in fact write $W_1^{\mathbf{u}}(t)$ and $W_2^{\mathbf{u}}(t)$. But, to simplify the notation, we omit this dependence on $\mathbf{u}$ in the notation $W_1(t)$ and $W_2(t)$.

The proof of the theorem will be based on two preliminary lemmas. These lemmas provide estimates on dW_1/dt and dW_2/dt.

LEMMA 3.5. *If $\rho_\infty(\mathbf{G}'(\mathbf{0})) < 1$, there exist $p_i > 0$ $\forall i \in \{1, \ldots, n\}$ and positive real constants α, β_1, and δ_1 such that, for every $\mu \in (0, \delta_1)$, for every $p \in (1/\delta_1, +\infty)$, and for every C^1-solution $\mathbf{u} : [0,T] \times [0,1] \to \mathbb{R}^n$ of (3.1) and (3.15) satisfying $|\mathbf{u}|_0 < \delta_1$, we have*

$$\frac{dW_1}{dt}(t) \leqslant \Big(-\mu\alpha + \beta_1 |\mathbf{u}_x(t)|_0 \Big) W_1(t) \qquad \forall t \in [0,T]. \tag{3.24}$$

Proof. Let $\mathbf{u} : [0,T] \times [0,1] \to \mathbb{R}^n$ be a C^1-solution of (3.1) and (3.15). The time derivative of W_1 is

$$\frac{dW_1}{dt} = \frac{1}{2p} W_1^{1-2p} \int_0^1 \sum_{i=1}^n 2p\, p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j \right)^{2p-1} \cdot \left(\left[\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_t u_j \right] + \sum_{j=1}^n \big(\partial_t m_{ij}(\mathbf{u})\big) u_j \right) e^{-2p\mu x} dx. \tag{3.25}$$

Using (3.1), the term between brackets can be written as

$$\begin{aligned} \sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_t u_j &= - \sum_{j=1}^n m_{ij}(\mathbf{u}) \left(\sum_{k=1}^n a_{jk}(\mathbf{u}) \partial_x u_k \right) \\ &= - \sum_{k=1}^n \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) a_{jk}(\mathbf{u}) \partial_x u_k \right), \end{aligned} \tag{3.26}$$

where $a_{jk}(\mathbf{u})$ is the (j,k)th entry of the matrix $A(\mathbf{u})$. Now, from (3.17), we have

$$\sum_{j=1}^n m_{ij}(\mathbf{u}) a_{jk}(\mathbf{u}) \partial_x u_k = \sum_{j=1}^n d_{ij}(\mathbf{u}) m_{jk}(\mathbf{u}) \partial_x u_k = \lambda_i(\mathbf{u}) m_{ik}(\mathbf{u}) \partial_x u_k, \tag{3.27}$$

where $d_{ij}(\mathbf{u})$ is the (i,j)th entry of the matrix $D(\mathbf{u})$. From (3.26) and (3.27), we have

$$\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_t u_j = -\lambda_i(\mathbf{u}) \sum_{k=1}^n m_{ik}(\mathbf{u}) \partial_x u_k = -\lambda_i(\mathbf{u}) \sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_x u_j. \tag{3.28}$$

By substituting this expression for the term between brackets in (3.25), we get

$$\begin{aligned} \frac{dW_1}{dt} = \frac{1}{2p} W_1^{1-2p} \int_0^1 \sum_{i=1}^n 2p\, p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j \right)^{2p-1} \\ \cdot \left(\left(-\lambda_i(\mathbf{u}) \sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_x u_j \right) + \left(\sum_{j=1}^n \big(\partial_t m_{ij}(\mathbf{u})\big) u_j \right) \right) e^{-2p\mu x} dx, \end{aligned}$$

which leads to

$$\begin{aligned} \frac{dW_1}{dt} = \frac{1}{2p} W_1^{1-2p} \Bigg[\int_0^1 - \sum_{i=1}^n p_i^p \lambda_i(\mathbf{u}) \left(\left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j \right)^{2p} \right)_x e^{-2p\mu x} dx \\ + \int_0^1 2p \sum_{i=1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j \right)^{2p-1} \Bigg(\lambda_i(\mathbf{u}) \left(\sum_{j=1}^n (\partial_x m_{ij}(\mathbf{u})) u_j \right) \\ + \left(\sum_{j=1}^n \big(\partial_t m_{ij}(\mathbf{u})\big) u_j \right) \Bigg) e^{-2p\mu x} dx \Bigg]. \end{aligned} \tag{3.29}$$

Using integrations by parts, we now get

$$\frac{dW_1}{dt} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3, \tag{3.30}$$

with

$$\mathcal{T}_1 \triangleq \frac{W_1^{1-2p}}{2p}\left[-\sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u})\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j e^{-\mu x}\right)^{2p}\right]_0^1, \tag{3.31}$$

$$\mathcal{T}_2 \triangleq -\mu W_1^{1-2p}\int_0^1 \sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u})\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j\right)^{2p} e^{-2p\mu x}dx, \tag{3.32}$$

$$\begin{aligned}\mathcal{T}_3 \triangleq W_1^{1-2p}\int_0^1 \sum_{i=1}^{n} p_i^p \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j\right)^{2p-1}&\left(\left(\lambda_i(\mathbf{u})\sum_{j=1}^{n}(\partial_x m_{ij}(\mathbf{u}))u_j\right)\right.\\ &\left.+\left(\sum_{j=1}^{n}(\partial_t m_{ij}(\mathbf{u}))u_j\right)+\frac{1}{2p}\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j\right)\frac{\partial\lambda_i}{\partial\mathbf{u}}(\mathbf{u})\partial_x\mathbf{u}\right)e^{-2p\mu x}dx.\end{aligned} \tag{3.33}$$

Analysis of the first term $\mathcal{T}_1$. From (3.31), we have

$$\begin{aligned}\mathcal{T}_1 = -\frac{W_1^{1-2p}}{2p}&\left[\sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u}(t,1))\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,1))u_j(t,1)e^{-\mu}\right)^{2p}\right.\\ &\left.-\sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u}(t,0))\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,0))u_j(t,0)\right)^{2p}\right].\end{aligned} \tag{3.34}$$

Let $K \triangleq G'(\mathbf{0})$. By the definition of ρ_∞ (see (1.4)) and (3.14), there exist $\Delta_i > 0$, $i \in \{1,\dots,n\}$, such that

$$\theta \triangleq \sum_{j=1}^{n} |K_{ij}|\frac{\Delta_i}{\Delta_j} < 1. \tag{3.35}$$

The parameters p_i are selected such that

$$p_i^p \Lambda_i = \Delta_i^{2p}, \qquad i = 1,\dots,n. \tag{3.36}$$

We define $\xi_i : [0,T] \to \mathbb{R}$, $i = 1,\dots,n$, by

$$\xi_i(t) \triangleq \Delta_i u_i(t,1) \quad \forall t \in [0,T]. \tag{3.37}$$

From (3.34), (3.36), and (3.37), we have

$$\begin{aligned}\mathcal{T}_1 = -\frac{W_1^{1-2p}}{2p}&\left[\sum_{i=1}^{n} \frac{\lambda_i(\mathbf{u}(t,1))}{\Lambda_i}\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,1))\frac{\Delta_i}{\Delta_j}\xi_j(t)e^{-\mu}\right)^{2p}\right.\\ &\left.-\sum_{i=1}^{n} \frac{\lambda_i(\mathbf{u}(t,0))}{\Lambda_i}\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,0))\Delta_i u_j(t,0)\right)^{2p}\right].\end{aligned} \tag{3.38}$$

Let $t \in [0,T]$. Without loss of generality, we may assume that

$$\xi_1^2(t) = \max\{\xi_i^2(t),\, i = 1,\dots,n\}. \tag{3.39}$$

Let us denote by δ and C various positive constants which may vary from place to place but are independent of $t \in [0,T]$, $\mathbf{u}$, and $p \in \mathbb{N}_+$. From (3.18) and (3.39), we have, for $|\mathbf{u}(t,1)|_0 \leqslant \delta$,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\lambda_i(\mathbf{u}(t,1))}{\Lambda_i}\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,1))\frac{\Delta_i}{\Delta_j}\xi_j(t)e^{-\mu}\right)^{2p}\\
&\quad \geqslant \frac{\lambda_1(\mathbf{u}(t,1))}{\Lambda_1}\left(\sum_{j=1}^{n} m_{1j}(\mathbf{u}(t,1))\frac{\Delta_1}{\Delta_j}\xi_j(t)e^{-\mu}\right)^{2p}\\
&\quad \geqslant e^{-2p\mu}\left(1 - C|\xi_1(t)|\right)\left(|\xi_1(t)| - C|\xi_1(t)|^2\right)^{2p}\\
&\quad = e^{-2p\mu}\left(1 - C|\xi_1(t)|\right)^{2p+1}\left(\xi_1(t)\right)^{2p}.
\end{aligned} \tag{3.40}$$

From (3.15), (3.35), (3.37), and (3.39), we have, for $|\mathbf{u}(t,0)|_0 \leqslant \delta$,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\lambda_i(\mathbf{u}(t,0))}{\Lambda_i}\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,0))\Delta_i u_j(t,0)\right)^{2p}\\
&\quad \leqslant \left(1 + C|\xi_1(t)|\right)\left(\sum_{i=1}^{n}\left(C|\xi_1(t)|^2 + \sum_{j=1}^{n}|K_{ij}|\frac{\Delta_i}{\Delta_j}|\xi_j(t)|\right)^{2p}\right)\\
&\quad \leqslant n\left(1 + C|\xi_1(t)|\right)\left(\theta|\xi_1(t)| + C|\xi_1(t)|^2\right)^{2p}.
\end{aligned} \tag{3.41}$$

From (3.35), (3.38), (3.39), and (3.41), there exists $\delta_{11} \in (0,1)$, independent of $\mathbf{u}$, such that for every $\mu \in (0,\delta_{11})$, for every $p \in (1/\delta_{11}, +\infty) \cap \mathbb{N}_+$, and for every $\mathbf{u}$ we have

$$\mathcal{T}_1(t) \leqslant 0 \quad \text{if } |\mathbf{u}(t)|_0 < \delta_{11}. \tag{3.42}$$

Analysis of the second term $\mathcal{T}_2$. Let

$$\alpha \triangleq \min(\Lambda_1,\dots,\Lambda_n)/2. \tag{3.43}$$

From (3.19) and (3.32) there is a $\delta_{12} \in (0,\eta]$ such that, for every $\mu \in (0,+\infty)$, for every $p \in \mathbb{N}_+$, and for every $\mathbf{u}$,

$$\mathcal{T}_2 \leqslant -\mu\alpha W_1 \quad \text{if } |\mathbf{u}|_0 < \delta_{12}. \tag{3.44}$$

Analysis of the third term $\mathcal{T}_3$. Using (3.1) and (3.33), we have

$$\begin{aligned}
\mathcal{T}_3 = W_1^{1-2p}\int_0^1 \sum_{i=1}^{n} p_i^p &\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j\right)^{2p-1}\\
\cdot\left(\frac{1}{2p}\frac{\partial\lambda_i}{\partial\mathbf{u}}\mathbf{u}_x\left(\sum_{j=1}^{n} m_{ij}(\mathbf{u})u_j\right)\right. &+ \left.\sum_{j=1}^{n}\left(\frac{\partial m_{ij}}{\partial\mathbf{u}}\big(-A(\mathbf{u}) + \lambda_i(\mathbf{u})\big)\mathbf{u}_x\right)u_j\right)e^{-2p\mu x}dx.
\end{aligned} \tag{3.45}$$

From (3.18), (3.19), and (3.45) one gets the existence of $\beta_1 > 0$ and $\delta_{13} > 0$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$, and for every $\mathbf{u}$,

$$\mathcal{T}_3 \leqslant \beta_1 |\mathbf{u}_x|_0 W_1 \quad \text{if } |\mathbf{u}|_0 < \delta_{13}. \tag{3.46}$$

Let $\delta_1 \triangleq \min\{\delta_{11}, \delta_{12}, \delta_{13}\}$. From (3.30), (3.42), (3.44), and (3.46), we conclude that

$$\frac{dW_1}{dt} \leqslant -\mu\alpha_1 W_1 + \beta_1 |\mathbf{u}_x|_0 W_1,$$

provided that $\mathbf{u}$ is such that $|\mathbf{u}|_0 < \delta_1$, $p \in (1/\delta_1, +\infty) \cap \mathbb{N}_+$, and $\mu \in (0, \delta_1)$. This completes the proof of Lemma 3.5. ◻

LEMMA 3.6. *Let p_i $(i = 1, \dots, n)$ be given by* (3.36). *If $\rho_\infty(G'(\mathbf{0})) < 1$, there exist β_2 and δ_2 such that, for every $\mu \in (0, \delta_2)$, for every $p \in (1/\delta_2, +\infty) \cap \mathbb{N}_+$, and for every C^1-solution $\mathbf{u} : [0, T] \times [0, 1] \to \mathbb{R}^n$ of* (3.1) *and* (3.15) *such that $|\mathbf{u}|_0 < \delta_2$, we have, in the sense of distributions in $(0, T)$,*

$$\frac{dW_2}{dt} \leqslant \Big(-\mu\alpha + \beta_2 |\mathbf{u}_x(t)|_0 \Big) W_2(t) \quad \forall t \in [0, T],$$

with α defined by (3.43).

Proof. We first deal with the case where A, G, and u are of class C^2. By time differentiation of (3.1) and (3.15), we see that $\mathbf{u}_t$ satisfy the following hyperbolic dynamics for $t \in [0, T]$ and $x \in [0, 1]$:

$$(\mathbf{u}_t)_t + A(\mathbf{u})(\mathbf{u}_t)_x + A'(\mathbf{u}, \mathbf{u}_t)\mathbf{u}_x = 0, \tag{3.47}$$

$$\partial_t \mathbf{u}(t, 0) = \frac{\partial G(\mathbf{u}(t, 1))}{\partial \mathbf{u}(t, 1)} \partial_t \mathbf{u}(t, 1), \tag{3.48}$$

where $A'(\mathbf{u}, \mathbf{u}_t)$ is a compact notation for the matrix whose entries are

$$A'(\mathbf{u}, \mathbf{u}_t)_{i,j} \triangleq \frac{\partial a_{ij}(\mathbf{u})}{\partial \mathbf{u}} \mathbf{u}_t, \quad i \in \{1, \dots, n\}, j \in \{1, \dots, n\}.$$

Using (3.47)–(3.48), we see that the time derivative of W_2 is

$$\begin{aligned} \frac{dW_2}{dt} = \frac{1}{2p} W_2^{1-2p} \int_0^1 \sum_{i=1}^n 2p\, p_i^p &\left(\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_t u_j \right)^{2p-1} \\ &\left(\left[\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_{tt} u_j \right] + \sum_{j=1}^n \big(\partial_t m_{ij}(\mathbf{u}) \big) \partial_t u_j \right) e^{-2p\mu x} dx. \end{aligned} \tag{3.49}$$

From (3.17), similarly as for (3.28), it can be shown that

$$\sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_{tt} u_j = -\lambda_i(\mathbf{u}) \sum_{j=1}^n m_{ij}(\mathbf{u}) \partial_x \big(\partial_t u_j \big) + \sum_{j=1}^n m_{ij}(\mathbf{u}) \left(\sum_{k=1}^n \tilde{a}_{jk}(\mathbf{u}, \mathbf{u}_t) \partial_t u_k \right),$$

where $\tilde{a}_{ij}(\mathbf{u}, \mathbf{u}_t)$ is the (i, j)th entry of the matrix $\tilde{A}(\mathbf{u}, \mathbf{u}_t) \triangleq A'(\mathbf{u}, \mathbf{u}_t) A^{-1}(\mathbf{u})$. Then, by substituting this expression for the term between brackets in (3.49), we get

$$\frac{dW_2}{dt} = \frac{1}{2p} W_2^{1-2p} \int_0^1 \sum_{i=1}^{n} 2p\, p_i^p \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) \partial_t u_j \right)^{2p-1}$$
$$\cdot \Bigg[-\lambda_i(\mathbf{u}) \sum_{j=1}^{n} m_{ij}(\mathbf{u}) \partial_x \big(\partial_t u_j\big)$$
$$+ \sum_{j=1}^{n} \left(\left(\sum_{k=1}^{n} m_{ik}(\mathbf{u}) \tilde{a}_{kj}(\mathbf{u}, \mathbf{u}_t) \right) + \partial_t m_{ij}(\mathbf{u}) \right) \partial_t u_j \Bigg] e^{-2p\mu x} dx.$$

Using integration by parts as in the proof of Lemma 3.5, we get

$$\frac{dW_2}{dt} = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3,$$

with

$$\mathcal{U}_1 \triangleq \frac{W_2^{1-2p}}{2p} \left[- \sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u}) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) \big(\partial_t u_j\big) e^{-\mu x} \right)^{2p} \right]_0^1,$$

$$\mathcal{U}_2 \triangleq -\mu W_2^{1-2p} \int_0^1 \sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u}) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) \big(\partial_t u_j\big) \right)^{2p} e^{-2p\mu x} dx,$$

$$\mathcal{U}_3 \triangleq W_2^{1-2p} \int_0^1 \sum_{i=1}^{n} p_i^p \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) \big(\partial_t u_j\big) \right)^{2p-1} \Bigg[\sum_{j=1}^{n} \Bigg(\lambda_i(\mathbf{u}) (\partial_x m_{ij}(\mathbf{u}))$$
$$+ \left(\sum_{k=1}^{n} m_{ik}(\mathbf{u}) \tilde{a}_{kj}(\mathbf{u}, \mathbf{u}_t) \right) + \partial_t m_{ij}(\mathbf{u}) \Bigg) \big(\partial_t u_j\big)$$
$$+ \frac{1}{2p} \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) \partial_t u_j \right) \frac{\partial \lambda_i}{\partial \mathbf{u}}(\mathbf{u}) \mathbf{u}_x \Bigg] e^{-2p\mu x} dx.$$

Analysis of the first term $\mathcal{U}_1$. Using the boundary conditions (3.15) and (3.48), we have

$$\mathcal{U}_1 = -\frac{W_2^{1-2p}}{2p} \Bigg[\sum_{i=1}^{n} p_i^p \lambda_i(\mathbf{u}(t,1)) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}(t,1)) \big(\partial_t u_j(t,1)\big) \right)^{2p} e^{-2p\mu}$$
$$- \sum_{i=1}^{n} p_i^p \lambda_i(G(\mathbf{u}(t,1))) \left(\sum_{j=1}^{n} m_{ij}(G(\mathbf{u}(t,1))) \frac{\partial G_j(\mathbf{u}(t,1))}{\partial \mathbf{u}(t,1)} \partial_t \mathbf{u}(t,1) \right)^{2p} \Bigg].$$

Then, in a way similar to the analysis of $\mathcal{T}_1$ in the proof of Lemma 3.5, we can show that, since $\rho_\infty(K) < 1$, there exists $\delta_{21} \in (0,1)$, such that, for every $\mu \in (0, \delta_{21})$, for every $p \in (1/\delta_{21}, +\infty) \cap \mathbb{N}_+$, and for every $\mathbf{u}$, we have $\mathcal{U}_1 \leqslant 0$, provided that $|\mathbf{u}|_0 < \delta_{21}$.

Analysis of the second term $\mathcal{U}_2$. Proceeding as in the proof of (3.44), we get the existence of $\delta_{22} \in (0, \eta]$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$, and for every $\mathbf{u}$,

$$\mathcal{U}_2 \leqslant -\mu\alpha W_2 \quad \text{if } |\mathbf{u}|_0 < \delta_{22}, \tag{3.50}$$

with α defined as in (3.43).

Analysis of the third term $\mathcal{U}_3$. Proceeding as in the proof of (3.46), we get the existence of $\beta_2 > 0$ and $\delta_{23} > 0$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$, and for every $\mathbf{u}$,

$$\mathcal{U}_3 \leqslant \beta_2 |\mathbf{u}_x|_0 W_2 \quad \text{if } |\mathbf{u}|_0 < \delta_{23}.$$

From the analysis of $\mathcal{U}_1, \mathcal{U}_2$, and $\mathcal{U}_3$, we conclude that, with $\delta_2 \triangleq \min\{\delta_{21}, \delta_{22}, \delta_{23}\}$,

$$\frac{dW_2}{dt} \leqslant -\mu\alpha W_2 + \beta_2 |\mathbf{u}_x|_0 W_2$$

for all $\mathbf{u}$ such that $|\mathbf{u}|_0 < \delta_2$, provided that $\mu \in (0, \delta_2)$ and that $p \in (1/\delta_2, +\infty) \cap \mathbb{N}_+$.

The above estimates were obtained for A, G, and u of class C^2. But their proofs show that they do not depend on the C^2-norm of A, G, and $\mathbf{u}$. Hence, by density arguments, they remain valid with A, G, and $\mathbf{u}$ only of class C^1. This completes the proof of Lemma 3.6. $\square$

Proof. We now prove Theorem 3.3. Let us choose $\mu \in \mathbb{R}$ such that

$$0 < \mu < \min\{\delta_1, \delta_2\}, \tag{3.51}$$

where δ_1 and δ_2 are as in Lemmas 3.5 and 3.6, respectively. Let us define two functionals $\mathbf{V}_1 : C^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ and $\mathbf{V}_2 : C^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ by

$$\mathbf{V}_1(\mathbf{u}) \triangleq \left| \left(\Delta_1^2 \sum_{j=1}^n m_{1j}(\mathbf{u}) u_j e^{-\mu x}, \dots, \Delta_n^2 \sum_{j=1}^n m_{nj}(\mathbf{u}) u_j e^{-\mu x} \right)^{\mathsf{T}} \right|_0, \tag{3.52}$$

$$\mathbf{V}_2(\mathbf{u}) \triangleq \left| \left(\Delta_1^2 \sum_{j=1}^n m_{1j}(\mathbf{u}) (\partial_t u_j) e^{-\mu x}, \dots, \Delta_n^2 \sum_{j=1}^n m_{nj}(\mathbf{u}) (\partial_t u_j) e^{-\mu x} \right)^{\mathsf{T}} \right|_0 \tag{3.53}$$

for every $\mathbf{u} \in C^1([0,1]; \mathbb{R}^n)$ with $|\mathbf{u}|_0 < \varepsilon_0$, where $\varepsilon_0 > 0$ is chosen small enough. In (3.53), we again use (3.21) for every $\mathbf{u} \in C^1([0,1]; \mathbb{R}^n)$ with $|\mathbf{u}|_0 < \varepsilon_0$. This is a valid definition if, again, $\varepsilon_0 > 0$ is chosen small enough. We consider the Lyapunov function candidate $\mathbf{V} : C^1([0,1]; \mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathbf{V}(\mathbf{u}) \triangleq \mathbf{V}_1(\mathbf{u}) + \mathbf{V}_2(\mathbf{u}) \quad \forall \mathbf{u} \in C^1([0,1]; \mathbb{R}^n) \text{ such that } |\mathbf{u}|_0 < \varepsilon_0. \tag{3.54}$$

Still, if $\varepsilon_0 > 0$ is small enough, which will be always assumed, we can select $\gamma \in (1, +\infty)$ such that

$$\frac{1}{\gamma} \mathbf{V}(\mathbf{u}) \leqslant |\mathbf{u}|_1 \leqslant \gamma \mathbf{V}(\mathbf{u}) \quad \text{for every } \mathbf{u} \in C^1([0,1]; \mathbb{R}^n) \text{ such that } |\mathbf{u}|_0 < \varepsilon_0. \tag{3.55}$$

Let us now choose $T > 0$ large enough so that

$$\gamma^2 e^{-\mu\alpha T/2} \leqslant \frac{1}{2}, \tag{3.56}$$

with α defined by (3.43). Let $\varepsilon_2 \in (0, +\infty)$ be such that

$$\varepsilon_2 < \min\left\{\frac{\delta_1}{C_1}, \frac{\delta_2}{C_1}, \frac{\varepsilon_0}{C_1}, \varepsilon_1\right\}, \tag{3.57}$$

where, again, δ_1 and δ_2 are as in Lemmas 3.5 and 3.6, respectively. Let $\mathbf{u}^o$ be in $C^1([0,1];\mathbb{R}^n)$, satisfy the compatibility conditions (3.16), and be such that

$$|\mathbf{u}^o|_1 \leqslant \varepsilon_2. \tag{3.58}$$

Let $\mathbf{u} : [0,T] \times [0,1] \to \mathbb{R}^n$ be the solution of class C^1 of (3.1) and (3.15) satisfying the initial condition $\mathbf{u}(0,\cdot) = \mathbf{u}^o(\cdot)$. By Theorem 3.1, (3.57), and (3.58), $\mathbf{u}$ is well defined and satisfies

$$|\mathbf{u}|_0 < \min\{\delta_1, \delta_2, \varepsilon_0\}. \tag{3.59}$$

In order to emphasize the dependence of $\mathbf{W}_1$ and $\mathbf{W}_2$ on $p \in \mathbb{N}_+$, we now write them as $\mathbf{W}_{1,p}$ and $\mathbf{W}_{2,p}$, respectively. For $t \in [0,T]$, let us define

$$V(t) \triangleq \mathbf{V}(\mathbf{u}(t)), \quad V_1(t) \triangleq \mathbf{V}_1(\mathbf{u}(t)), \quad V_2(t) \triangleq \mathbf{V}_2(\mathbf{u}(t)), \tag{3.60}$$

$$W_{1,p}(t) \triangleq \mathbf{W}_{1,p}(\mathbf{u}(t)), \quad W_{2,p}(t) \triangleq \mathbf{W}_{2,p}(\mathbf{u}(t)). \tag{3.61}$$

Let us point out that, for every continuous function $\mathbf{f} = (f_1, \dots, f_n)^{\mathsf{T}} : [0,1] \to \mathbb{R}^n$, we have

$$|\mathbf{f}|_0 = \lim_{p\to\infty} \left(\int_0^1 \sum_{i=1}^n |f_i(x)|^{2p} dx\right)^{1/(2p)}, \tag{3.62}$$

$$\left(\int_0^1 \sum_{i=1}^n |f_i(x)|^{2p} dx\right)^{1/(2p)} \leqslant n^{1/(2p)} |\mathbf{f}|_0 \qquad \forall p \in [1, +\infty). \tag{3.63}$$

By the definition of $\mathbf{W}_1$ and $\mathbf{W}_2$, (3.60), (3.61), (3.62), and (3.63), we have

$$V_1(t) = \lim_{p\to\infty} W_{1,p}(t) \quad \text{and} \quad V_2(t) = \lim_{p\to\infty} W_{2,p}(t) \quad \forall t \in [0,T], \tag{3.64}$$

$$\exists M > 0 \text{ such that } W_{1,p}(t) + W_{2,p}(t) \leqslant M|u|_1 \quad \forall p \in [1, +\infty), \forall t \in [0,T]. \tag{3.65}$$

In particular,

$$W_{1,p} \overset{*}{\rightharpoonup} V_1 \text{ in the weak}^* \text{ topology } \sigma(L^\infty(0,T), L^1(0,T)) \text{ as } p \to +\infty, \tag{3.66}$$

$$W_{2,p} \overset{*}{\rightharpoonup} V_2 \text{ in the weak}^* \text{ topology } \sigma(L^\infty(0,T), L^1(0,T)) \text{ as } p \to +\infty. \tag{3.67}$$

(Let us recall that, for a bounded sequence $(\varphi_p)_{p\in\mathbb{N}_+}$ of elements of $L^\infty(0,T)$ and $\varphi \in L^\infty(0,T)$, one says that $\varphi_p \overset{*}{\rightharpoonup} \varphi$ in the weak* topology $\sigma(L^\infty(0,T), L^1(0,T))$ if, for every $\psi \in L^1(0,T)$,

$$\int_0^{\mathsf{T}} \varphi_p(t)\psi(t)dt \to \int_0^{\mathsf{T}} \varphi(t)\psi(t)dt \text{ as } p \to +\infty.)$$

From Lemmas 3.5 and 3.6, (3.59), (3.66), and (3.67), we have, in the distribution sense in $(0,T)$,

$$\frac{dV_1}{dt} \leqslant -\mu\alpha V_1 + \beta_1 |\mathbf{u}_x|_0 V_1, \tag{3.68}$$

$$\frac{dV_2}{dt} \leqslant -\mu\alpha V_2 + \beta_2 |\mathbf{u}_x|_0 V_2. \tag{3.69}$$

Summing (3.68) and (3.69), we get, in the distribution sense in $(0, T)$,

$$\frac{dV}{dt} \leqslant -\mu\alpha V + \beta|\mathbf{u}|_1 V, \tag{3.70}$$

with $\beta \triangleq \max\{\beta_1, \beta_2\}$. Let us impose on ε_2, in addition to (3.57), that

$$\varepsilon_2 \leqslant \frac{\mu\alpha}{2\beta C_1}. \tag{3.71}$$

From (3.13), (3.58), and (3.71), we get that

$$\beta|\mathbf{u}|_1 \leqslant \frac{\mu\alpha}{2}. \tag{3.72}$$

From (3.70) and (3.72), we have, in the distribution sense in $(0, T)$,

$$\frac{dV}{dt} \leqslant -\frac{\mu}{2}\alpha V, \tag{3.73}$$

which implies that

$$V(T) \leqslant e^{-\alpha\mu T/2} V(0). \tag{3.74}$$

From (3.55) and (3.74), we obtain that

$$|\mathbf{u}(T)|_1 \leqslant \gamma^2 e^{-\alpha\mu T/2} |\mathbf{u}(0)|_1, \tag{3.75}$$

which, together with (3.56), implies that

$$|\mathbf{u}(T)|_1 \leqslant \frac{1}{2}|\mathbf{u}(0)|_1. \tag{3.76}$$

This completes the proof of Theorem 3.3. ∎

Remark 3.7. Instead of using time derivatives in the distribution sense as, for instance, in (3.70), we could have directly derived inequality (3.74) on $V(t)$ by establishing the same inequality for $W(t) \triangleq W_1(t) + W_2(t)$ and letting $p \to +\infty$.

3.2. Proof of Theorem 3.3 in the general case where $0 < m < n$. In this section, we explain the modifications of the proof that must be used to deal with the case $0 < m < n$. (Of course, the case $m = 0$ is equivalent to the case $m = n$ by considering $\mathbf{u}(t, 1 - x)$ instead of $\mathbf{u}(t, x)$.)

The functionals $\mathbf{W}_1$ and $\mathbf{W}_2$ are now defined as follows:

$$\mathbf{W}_1(\mathbf{u}) \triangleq \left(\int_0^1 \left[\sum_{i=1}^m p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j\right)^{2p} e^{-2p\mu x} + \sum_{i=m+1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) u_j\right)^{2p} e^{2p\mu x}\right] dx\right)^{\frac{1}{2p}},$$

$$\mathbf{W}_2(\mathbf{u}) \triangleq \left(\int_0^1 \left[\sum_{i=1}^m p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) \big(\partial_t u_j\big)\right)^{2p} e^{-2p\mu x} + \sum_{i=m+1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u}) \big(\partial_t u_j\big)\right)^{2p} e^{2p\mu x}\right] dx\right)^{\frac{1}{2p}}.$$

Let $\mathbf{u} : [0, T] \times [0, 1] \to \mathbb{R}^n$ be a solution of (3.1)–(3.2) of class C^1. With computations similar to those of Lemma 3.5, it is readily seen that the time derivative of $W_1(t) \triangleq \mathbf{W}_1(\mathbf{u}(t))$ is given by

$$\frac{dW_1}{dt} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$$

with

$$\mathcal{T}_1 \triangleq \frac{W_1^{1-2p}}{2p} \left[-\sum_{i=1}^{m} p_i^p \lambda_i(\mathbf{u}) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j e^{-\mu x} \right)^{2p} \right.$$
$$\left. - \sum_{i=m+1}^{n} p_i^p \lambda_i(\mathbf{u}) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j e^{\mu x} \right)^{2p} \right]_0^1,$$

$$\mathcal{T}_2 \triangleq -\mu W_1^{1-2p} \int_0^1 \left[\sum_{i=1}^{m} p_i^p \lambda_i(\mathbf{u}) \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right)^{2p} e^{-2p\mu x} \right.$$
$$\left. + \sum_{i=m+1}^{m} p_i^p |\lambda_i(\mathbf{u})| \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right)^{2p} e^{2p\mu x} \right] dx,$$

$$\mathcal{T}_3 \triangleq W_1^{1-2p} \int_0^1 \left[\sum_{i=1}^{m} p_i^p \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right)^{2p-1} \left(\lambda_i(\mathbf{u}) \sum_{j=1}^{n} (\partial_x m_{ij}(\mathbf{u})) u_j \right. \right.$$
$$\left. + \left(\sum_{j=1}^{n} (\partial_t m_{ij}(\mathbf{u})) u_j \right) + \frac{1}{2p} \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right) \frac{\partial \lambda_i}{\partial \mathbf{u}}(\mathbf{u}) \partial_x \mathbf{u} \right) e^{-2p\mu x}$$
$$+ \sum_{i=m+1}^{n} p_i^p \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right)^{2p-1} \left(\lambda_i(\mathbf{u}) \sum_{j=1}^{n} (\partial_x m_{ij}(\mathbf{u})) u_j \right.$$
$$\left. \left. + \left(\sum_{j=1}^{n} (\partial_t m_{ij}(\mathbf{u})) u_j \right) + \frac{1}{2p} \left(\sum_{j=1}^{n} m_{ij}(\mathbf{u}) u_j \right) \frac{\partial \lambda_i}{\partial \mathbf{u}}(\mathbf{u}) \partial_x \mathbf{u} \right) e^{2p\mu x} \right] dx.$$

The parameters $p_i > 0$ are now selected such that (compare with (3.36))

$$p_i^p |\Lambda_i| = \Delta_i^{2p}, \quad i = 1, \ldots, n. \tag{3.77}$$

Concerning the ξ_i, $i = 1, \ldots, n$, they are now defined by (compare with (3.37))

$$\xi_i(t) \triangleq \Delta_i u_i(t, 1) \text{ for } i = 1, \ldots, m \quad \text{and} \quad \xi_i(t) \triangleq \Delta_i u_i(t, 0) \text{ for } i = m+1, \ldots, n.$$

It is then a straightforward exercise to verify that Theorem 3.3 can be established for the case $0 < m < n$ in a manner completely parallel to the one we have followed in the case $m = n$.

Remark 3.8. Looking at our proof of Theorem 3.3, we see that for every

$$\nu \in (0, -\min(|\Lambda_1|, \ldots, |\Lambda_n|) \ln (\rho (|G'(\mathbf{0})|))) \tag{3.78}$$

there exist $\varepsilon > 0$ and $C > 0$ such that, for every $\mathbf{u}^o$ such that $|\mathbf{u}^o|_1 < \varepsilon$ and satisfying the compatibility conditions (3.5) to (3.7), the Cauchy problem (3.1)–(3.2)–(3.4) has a unique C^1-solution which satisfies

$$|\mathbf{u}(t,.)|_1 \leqslant Ce^{-\nu t}|\mathbf{u}^o|_1 \quad \forall t \in [0,+\infty).$$

See in particular (3.40) and (3.41), replace the definition of α given in (3.43) by the condition

$$\alpha \in \big(0, \min(|\Lambda_1|, \dots, |\Lambda_n|)\big), \tag{3.79}$$

and slightly modify some arguments in order to deal with (3.79) instead of (3.43).

4. Conclusion and final remark. In this article, we have addressed the issue of exponential stability for the C^1-norm of quasi-linear hyperbolic systems of the form (3.1), (3.2). Our main result (section 3) has been to give a new proof, using a strict Lyapunov function, that a sufficient condition for the exponential stability of the steady state is to have a map G such that $\rho_\infty(G'(\mathbf{0})) < 1$ for the boundary conditions (3.2).

The approach followed in this paper can be easily adapted to the exponential stability for the C^ℓ-norm with $\ell \in \mathbb{N}_+$, provided that the compatibility conditions (3.5), (3.6), (3.7) are adapted accordingly (see [11] for details) and that A and G are of class C^ℓ. The analysis requires the following extensions of formulas (3.19) or (3.20) and (3.52) or (3.53):

$$\mathbf{W}_k(\mathbf{u}) \triangleq \left(\int_0^1 \sum_{i=1}^n p_i^p \left(\sum_{j=1}^n m_{ij}(\mathbf{u})\partial_t^{k-1}u_j\right)^{2p} e^{-2p\mu x}dx\right)^{\frac{1}{2p}}, \tag{4.1}$$

(4.2)
$$\mathbf{V}_k(\mathbf{u}) \triangleq \left|\left(\Delta_1^2 \sum_{j=1}^n m_{1j}(\mathbf{u})(\partial_t^{k-1}u_j)e^{-\mu x}, \dots, \Delta_n^2 \sum_{j=1}^n m_{nj}(\mathbf{u})(\partial_t^{k-1}u_j)e^{-\mu x}\right)^{\mathsf{T}}\right|_0,$$

where $\partial_t^{k-1}\mathbf{u}$ is defined by differentiating (1.1) formally with respect to time. For example,

$$\partial_t^2\mathbf{u} \triangleq (A'(\mathbf{u})\,(A(\mathbf{u})\partial_x\mathbf{u}))\,\partial_x\mathbf{u} + A(\mathbf{u})\,((A'(\mathbf{u})\partial_x\mathbf{u})\,\partial_x\mathbf{u}) \\ + A(\mathbf{u})\,\big(A(\mathbf{u})\partial_{xx}^2\mathbf{u}\big) \qquad \forall \mathbf{u} \in C^2([0,1];\mathbb{R}^n) \text{ with } |\mathbf{u}|_0 \text{ small enough.}$$

Then, using the Lyapunov function

$$\mathbf{V}(\mathbf{u}) = \sum_{k=1}^{\ell+1} \mathbf{V}_k(\mathbf{u}), \tag{4.3}$$

the following theorem holds.

THEOREM 4.1. *Assume that A and G are of class C^ℓ. Then, if*

$$\rho_\infty(G'(\mathbf{0})) < 1, \tag{4.4}$$

the steady state $x \in [0,1] \mapsto \mathbf{0} \in \mathbb{R}^n$ of the system (3.1)–(3.2) *is exponentially stable for the C^ℓ-norm.*

REFERENCES

[1] O.-M. Aamo, *Disturbance rejection in 2×2 linear hyperbolic systems*, IEEE Trans. Automat. Control, 58 (2013), pp. 1095–1106.

[2] S. Amin, F. Hante, and A. Bayen, *Exponential stability of switched linear hyperbolic initial-boundary value problems*, IEEE Trans. Automat. Control, 56 (2012), pp. 291–301.

[3] G. Bastin and J.-M. Coron, *On boundary feedback stabilization of non-uniform 2×2 hyperbolic systems over a bounded interval*, Systems Control Lett., 60 (2011), pp. 900–906.

[4] G. Bastin and J.-M. Coron, *Stability and Boundary Stabilisation of One-Dimensional Hyperbolic Systems*, book in preparation, 2015.

[5] G. Bastin, J.-M. Coron, and S.O. Tamasoiu, *Stability of linear density-flow hyperbolic systems under PI boundary control*, Automatica, 53 (2015), pp. 37–42.

[6] D. Bridges and P. Schuster, *A simple constructive proof of Kronecker's density theorem*, Elem. Math., 61 (2006), pp. 152–154.

[7] F. Castillo, E. Witrant, C. Prieur, and L. Dugard, *Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control*, Automatica, 49 (2013), pp. 3180–3188.

[8] J.-M. Coron, G. Bastin, and B. d'Andréa-Novel, *Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems*, SIAM J. Control Optim., 47 (2008), pp. 1460–1498.

[9] J.-M. Coron, B. d'Andréa-Novel, and G. Bastin, *A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations*, in Proceedings of the European Control Conference (ECC'99 CD-ROM), Karlsruhe, Germany, 1999, F1008-5.

[10] J.-M. Coron, B. d'Andréa-Novel, and G. Bastin, *A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Trans. Automat. Control, 52 (2007), pp. 2–11.

[11] J.-M. Coron and H.-M. Nguyen, *Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: Sharp conditions through an approach via time-delay systems*, SIAM J. Math. Anal., to appear.

[12] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, *Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping*, SIAM J. Control Optim., 51 (2013), pp. 2005–2035.

[13] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa-Novel, and G. Bastin, *Boundary feedback control in networks of open-channels*, Automatica, 39 (2003), pp. 1365–1376.

[14] A. Diagne, G. Bastin, and J.-M. Coron, *Lyapunov exponential stability of linear hyperbolic systems of balance laws*, Automatica, 48 (2012), pp. 109–114.

[15] M. Dick, M. Gugat, and G. Leugering, *Feedback stabilization of quasilinear hyperbolic systems with varying delays*, in Proceedings of the 17th International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland, 2012, pp. 125–130.

[16] P.M. Dower and P.M. Farrel, *On linear control of backward pumped Raman amplifiers*, in Proceedings of the IFAC Symposium on System Identification, Newcastle, Australia, 2006, pp. 547–552.

[17] J.M. Greenberg and T. Li, *The effect of boundary damping for the quasilinear wave equations*, J. Differential Equations, 52 (1984), pp. 66–75.

[18] M. Gugat, M. Dick, and G. Leugering, *Gas flow in fan-shaped networks: Classical solutions and feedback stabilization*, SIAM J. Control Optim., 49 (2011), pp. 2101–2117.

[19] M. Gugat and M. Herty, *Existence of classical solutions and feedback stabilisation for the flow in gas networks*, ESAIM Control Optim. Calc. Var., 17 (2009), pp. 28–51.

[20] J.K. Hale and S.M. Verduyn-Lunel, *Introduction to Functional-Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, 1993.

[21] M. Krstic, B.-Z. Guo, A. Balogh, and A. Smyshlyaev, *Output-feedback stabilization of an unstable wave equation*, Automatica, 44 (2008), pp. 63–74.

[22] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, Adv. Des. Control 16, SIAM, Philadelphia, 2008.

[23] G. Leugering and E.J.P.G. Schmidt, *On the modelling and stabilization of flows in networks of open canals*, SIAM J. Control Optim., 41 (2002), pp. 164–180.

[24] T.-T. Li, *Global Classical Solutions for Quasi-linear Hyperbolic Systems*, Res. Appl. Math., Masson, Paris, and Wiley, New York, 1994.

[25] T.-T. Li, B. Rao, and Z. Wang, *Exact boundary controllability and observability for first order quasilinear hyperbolic systems with a kind of nonlocal boundary conditions*, Discrete Contin. Dyn. Syst. Ser. A, 28 (2010), pp. 243–257.

[26] T.-T. Li and W.-C. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke Univ. Math. Ser.V, Duke University Mathematics Department, Durham, NC, 1985.
[27] X. Litrico and V. Fromion, *Boundary control of hyperbolic conservation laws using a frequency domain approach*, Automatica, 45 (2009), pp. 647–656.
[28] F. Di Meglio, R. Vazquez, and M. Krstic, *Stabilization of a linear hyperbolic system with one boundary controlled transport PDE coupled with n counterconvecting PDEs*, in Proceedings of the 51st IEEE Conference on Decision and Control, 2012, pp. 2914–2919.
[29] W. Michiels and T. Vyhlidal, *An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type*, Automatica, 41 (2005), pp. 991–998.
[30] L. Pavel and L. Chang, *Lyapunov-based boundary control for a class of hyperbolic Lotka–Volterra systems*, IEEE Trans. Automat. Control, 57 (2012), pp. 701–714.
[31] V. Perrollaz and L. Rosier, *Finite-time stabilization of* 2×2 *hyperbolic systems on tree-shaped networks*, SIAM J. Control Optim., 52 (2014), pp. 143–163.
[32] C. Prieur, *Control of systems of conservation laws with boundary errors*, Netw. Heterog. Media, 4 (2009), pp. 393–407.
[33] C. Prieur, A. Girard, and E. Witrant, *Lyapunov functions for switched linear hyperbolic systems*, in Proceedings of the 4th IFAC Conference on Analysis and Design of Hybrid Systems, M. Lazar, M. Heemels, and B. De Schutter, eds., Eindhoven, The Netherlands, 2012, pp. 382–387.
[34] C. Prieur and F. Mazenc, *ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws*, Math. Control Signal Systems, 24 (2012), pp. 111–134.
[35] C. Prieur, J. Winkin, and G. Bastin, *Robust boundary control of systems of conservation laws*, Math. Control Signal Systems , 20 (2008), pp. 173–197.
[36] T.-H. Qin, *Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 6 (1985), pp. 289–298.
[37] J. Rauch and M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J., 24 (1974), pp. 79–86.
[38] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.
[39] V. Dos Santos and B. Maschke, *A Hamiltonian perspective to the stabilization of systems of two conservations laws*, Netw. Heterog. Media, 4 (2009), pp. 249–266.
[40] R.A. Silkowski, *Star Shaped Regions of Stability in Hereditary Systems*, Ph.D. thesis, Brown University, Providence, RI, 1976.
[41] M. Slemrod, *Boundary feedback stabilization for a quasilinear wave equation*, in Control Theory for Distributed Parameter Systems, Lecture Notes in Control and Inform. Sci. 54, Springer-Verlag, Berlin, 1983, pp. 221–237.
[42] A. Smyshlyaev, E. Cerpa, and M. Krstic, *Boundary stabilization of a 1-D wave equation with in-domain antidamping*, SIAM J. Control Optim., 48 (2010), pp. 4014–4031.
[43] A. Tchousso, T. Besson, and C.-Z. Xu, *Exponential stability of distributed parameter systems governed by symmetric hyperbolic partial differential equations using Lyapunov's second method*, ESAIM Control Optim. Calc. Var., 15 (2009), pp. 403–425.
[44] R. Vazquez, M. Krstic, and J.-M. Coron, *Backstepping boundary stabilization and state estimation of a* 2×2 *linear hyperbolic system*, in Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, 2011, pp. 4937–4942.
[45] C.-Z. Xu and G. Sallet, *Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems*, ESAIM Control Optim. Calc. Var., 7 (2002), pp. 421–442.
[46] Y.-C. Zhao, *Classical Solutions for Quasilinear Hyperbolic Systems*, Thesis, School of Mathematical Sciences, Fudan University, Shanghai, China, 1986 (in Chinese).