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Sampled-data adaptive control of a class of continuous nonlinear systems

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The problem is addressed of deriving sampled-data control laws for a class of nonlinear continuous systems. The concept of approximate state-feedback linearizing control is defined for sampled-data models arising from the sampling of continuous-time systems. Such control laws are then designed for both non-adaptive and adaptive cases and a detailed performance analysis is performed.

1. Introduction

Adaptive control of nonlinear continuous time systems has been, in the recent years, a subject of growing interest. Many direct and indirect adaptive schemes have been discussed and analysed for nonlinear systems which are linearly parameterized. Typical recent references are Campion and Bastin (1990), Bastin and Campion (1989), Teel *et al.* (1991), Kanellakopoulos *et al.* (1991), Marino and Tomei (1991), Praly *et al.* (1991), Pomet and Praly (1992), Mareels *et al.* (1993). An overview of the state-of-the-art can also be found in a recently published special issue of the *International Journal of Adaptive Control and Signal Processing* (Praly *et al.* 1992).

The interest in a continuous time design is, however, restricted by the fact that controllers are most often implemented digitally. The issue therefore arises of designing discrete time controllers based on the sampled-data models of continuous time systems. Our objective, in this paper, is to derive and analyse a sampled-data counterpart of an adaptive continuous-time linearizing controller proposed by Bastin and Campion (1989) and further analysed by Teel *et al.* (1991). As we shall emphasize, this derivation is not immediate, mainly because both state feedback linear stabilizability and linear parametrization can be destroyed by the sampling process. For simplicity, we shall limit ourselves to adaptive regulation of nonlinear systems which are full-state linearly stabilizable without a parametrized diffeomorphism. This restriction should be interpreted as a first attempt towards a more general theory of adaptive control of sampled-data nonlinear systems.

The paper is organized as follows. The class of nonlinear systems under consideration is described in § 2, characterized by four basic assumptions which guarantee discretizability, feedback linear stabilizability, linear parametrization and strict model matching. The exact sampled-data model of these continuous systems is stated in § 3. The discrete time adaptive control problem we address in this paper is formulated in § 4, where the difficulties of the transposition from

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continuous time to discrete time are also emphasized. The limitations owing to sampling are analysed in § 4.3 (namely the risk of finite escape time between the sampling instants). The control law is based on the 'certainty equivalence principle' and is obtained by combining the 'approximate' linearizing control law presented in § 4.4 and a suitable parameter estimator, which is presented and analysed in § 6. In § 5, we discuss the non-adaptive case and in § 7, we will finally combine all the previous results to build up and analyse an adaptive approximate linearizing control law for the considered class of systems.

2. System description

We consider a class of nonlinear continuous time systems, with parameter uncertainty, which are linear in the control input, linearly parameterized and state feedback linearly stabilizable (see the definition in § 2.1).

These systems are described by the following state-space model

$$\dot{x} = f(x, \theta) + G(x, \theta)u \qquad (2.1)$$

where $x \in \mathbb{R}^n$ is the state of the system which is fully measured
$u \in \mathbb{R}^m$ is the control input
$\theta \in \mathbb{R}^p$ is the parameter vector
$\forall \theta$, $f(x, \theta)$ is a function of $\mathbb{R}^n$ and $G(x, \theta)$ is a $n \times m$ full rank matrix whose columns are the functions $g_i(x, \theta)$ in $\mathbb{R}^n$ $(i = 1, \dots m)$.

We assume that the structure of the functions $f(x, \theta)$ and $g_i(x, \theta)$ is exactly known and linearly parameterized in θ.

Hence, (2.1) describes a family of models parameterized by θ. The true system belongs to this family and is characterized by $\theta = \theta^*$. This true value θ^* of the parameter vector is unknown but an estimate $\hat{\theta}$ is available.

In this paper, we address the problem of the digital adaptive regulation of the state $x(t)$ at the origin $x = 0$.

2.1. *Assumptions*

This subsection gathers together all the assumptions on the family of systems (2.1) that will be used in this paper.

We first assume the existence of two compact sets $B_x \in \mathbb{R}^n$ and $B_\theta \in \mathbb{R}^p$ containing respectively the desired state $x = 0$ and the true parameter vector $\theta = \theta^*$ as interior points. We then make the following assumptions.

Assumption A1—Discretizability:

(i) *For all $\theta \in B_\theta$, the functions $f(., \theta)$ and $g_i(., \theta)$ are analytic on B_x for $i = 1, \dots m$.*

(ii) *There exist a continuous positive definite and integrable function $\rho: \mathbb{R} \to \mathbb{R}$ and two positive constants M and $\bar{M}$ with $M < \bar{M}$ and $\bar{M} > \max_{x \in B_x} \|x\|$ such that for all $\theta \in B_\theta$*

$$\text{if } \|x\| \leqslant \bar{M} \text{ then } f(x, \theta) \text{ and } g_i(x, \theta) \text{ are continuous}$$

$$\text{if } M \leqslant \|x\| \leqslant \bar{M} \text{ then } \|f(x, \theta)\| \leqslant \rho(\|x\|)$$
$$\|g_i(x, \theta)\| \leqslant \rho(\|x\|), \quad \text{for } i = 1, \dots m \qquad (2.2)$$

Assumption A1(i) is needed to guarantee the consistency of the sampled data model of system (2.1) which will be introduced in the next section. Assumption A1(ii) will be used in § 4.3 to ensure the boundedness of the solution of (2.1) between the sampling instants.

Definition: A system of the form $\dot{x} = f(x) + G(x)u$ with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is said to be *state feedback linearly stabilizable on a domain D* iff for all x in D there exists a feedback control $u(x) = \alpha(x) + \beta(x)v$ with $\alpha(x) \in \mathbb{R}^m$, $\beta(x) \in \mathbb{R}^{m \times m}$ non-singular and $v \in \mathbb{R}^m$ an external reference, such that, after feedback, the system is written as

$$\dot{x} = [f(x) + G(x)\alpha(x)] + G(x)\beta(x)v = Ax + Bv \qquad (2.3)$$

with (A, B) a stabilizable pair. □

Assumption A2—Linear stabilizability: *The true system $\dot{x} = f(x, \theta^*) + G(x, \theta^*)u$ is state feedback linearly stabilizable on B_x.*

Assumption A2 will be used in § 5 and § 7 for the design and the analysis of the non-adaptive and adaptive control algorithm respectively. Under Assumption A2, there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$A + BK = \Lambda \quad \text{with } \Lambda \text{ Hurwitz stable}$$

Then there exists a continuous feedback control law $u(x) = \alpha(x) + \beta(x)Kx$ which, applied to the true system, provides the following linear stable closed loop

$$\dot{x} = \Lambda x \qquad (2.4)$$

It also follows from Assumption A2 that $\Lambda x - f(x, \theta^*) \in \operatorname{span} G(x, \theta^*)$ and hence, it can be easily shown that this feedback control can be written in terms of $f(x, \theta^*)$ and $G(x, \theta^*)$ as follows

$$u_{\text{cont}}(x(t), \theta^*) = [G^{\mathrm{T}}(x, \theta^*)G(x, \theta^*)]^{-1}G^{\mathrm{T}}(x, \theta^*)(\Lambda x - f(x, \theta^*)) \qquad (2.5)$$

Assumption A3—Linear parameterization: *For all θ in B_θ, the functions $f(x, \theta)$ and $g_i(x, \theta)$ are linearly parameterized as follows*

$$f(x, \theta) \triangleq f_0(x) + \sum_{k=1}^{p} f_k(x)\theta_k = f(x, \theta^*) + \sum_{k=1}^{p} f_k(x)(\theta_k - \theta_k^*) \qquad (2.6)$$

$$g_i(x, \theta) \triangleq g_{i0}(x) + \sum_{k=1}^{p} g_{ik}(x)\theta_k = g(x, \theta^*) + \sum_{k=1}^{p} g_{ik}(x)(\theta_k - \theta_k^*) \qquad (2.7)$$

where f_0, g_{i0}, f_k and g_{ik} are functions $\in \mathbb{R}^n$ and $F(x)$ and $G_i(x)$ are $n \times p$ matrices.

This assumption will be used in § 6 for the discrete parameter estimation.

It states that the parameter vector enters the system (2.1) linearly. This allows us to write the model in regressor form as follows:

$$\dot{x} = \Phi_0(x, u) + \Phi^{\mathrm{T}}(x, u)\theta \qquad (2.8)$$

where

$$\Phi_0(x, u) \triangleq f_0(x) + \sum_{i=1}^{m} g_{i0}(x)u_i \qquad (2.9)$$

$$\Phi^T(x, u) \triangleq (\phi_1^T, \dots \phi_p^T) \quad \text{with } \phi_k^T = f_k + \sum_{i=1}^{m} g_{ik} u_i \tag{2.10}$$

A standard recursive linear estimation scheme can then be used to implement continuous adaptive control laws.

Assumption A4—Strict matching condition: *For all $x \in B_x$, $f_k(x)$ and $g_{ik}(x) \in$ span $G(x, \theta^*)$, for $i = 1, \dots m$ and $k = 1, \dots p$.*

This assumption will be used in § 7 to design and analyse the adaptive control algorithm.

It can be shown that, under Assumptions A2 and A3, A4 is equivalent to the state feedback linear stabilizability of the system (2.1) on B_x, for all θ in B_θ, with the same pair (A, B).

We can then define, for all θ in B_θ, a continuous feedback control of the form

$$u_{\text{cont}}(x(t), \theta) = [G^T(x, \theta)G(x, \theta)]^{-1} G^T(x, \theta)(Ax - f(x, \theta)) \tag{2.11}$$

Applied to (2.1), it will provide the same linear stable closed loop (2.4) for all θ in B_θ, i.e. for the whole family of systems defined by (2.1).

In the following, we will refer to the control (2.11) as to the continuous linearizing control of (2.1).

2.2. *Comments*

(1) A model of the system in the form (2.1), fulfilling Assumptions A1 to A4, might as well be obtained after application of a suitable change of coordinates to a given original 'physical' model.

As a matter of illustration, let us consider one of the simplest significant examples

$$\dot{z}_1 = f(z_1, z_2, \theta) + g(z_1, z_2, \theta)u$$
$$\dot{z}_2 = -\tan z_2$$

with $z_i \in \mathbb{R}$, $g(z_1, z_2, \theta) \neq 0$ on the domain of interest, f and g linearly parametrized according to Assumption A3, such that Assumption A1 is satisfied. Assumption A4 is then automatically fulfilled and A2 is the only missing assumption.

The change of coordinates $x_1 = z_1$, $x_2 = \sin z_2$ leads, however, to the model

$$\dot{x}_1 = f(x_1, \arcsin x_2) + g(x_1, \arcsin x_2)u$$
$$\dot{x}_2 = -x_2$$

which clearly fulfils all Assumptions A1 to A4.

It can also be shown that, under Assumption A4, if there exists a suitable change of coordinates for the true system, this same change of coordinates will transform, in the desired framework, the whole family of systems, i.e. it will be valid for all θ in B_θ. As a consequence, the change of coordinates will never destroy Assumptions A1, A3 or A4.

(2) The class of systems that are state feedback linearizable (see e.g. Isidori 1989 § 5.2) is, after the change of coordinates is applied, a subclass of the state feedback linearly stabilizable systems.

The model (2.12) gives a precise example of a state feedback linearly stabilizable system which is not state feedback linearizable.

(3) The Strict Matching Condition is identical to that of Taylor *et al.* (1989) (see also Kanellakopoulos *et al.* 1991 and Kokotovic *et al.* 1991) where it is used in the context of continuous time adaptive control of feedback linearizable systems.

2.3. *Notation*

In the following we will need the following bounds

$$k_1 \triangleq \max_{x \in B_x} \|x\| \tag{2.12}$$

$$k_2 \triangleq \max_{\theta \in B_\theta} \|\theta - \theta^*\| \tag{2.13}$$

$$k^* \triangleq \max\{k_1, M\} \text{ with } M \text{ from Assumption A1} \tag{2.14}$$

3. Exact sampling of nonlinear continuous systems

We are concerned with computer control of nonlinear continuous time systems of the form (2.1), implemented using sampling of the state and a zero-order hold control action as depicted in the Figure. The state $x(t)$ and the control $u(t)$ are supposed to be sampled at the same rate, with a sampling period denoted δ which is > 0.

The sampled state is defined at the sampling instants as follows

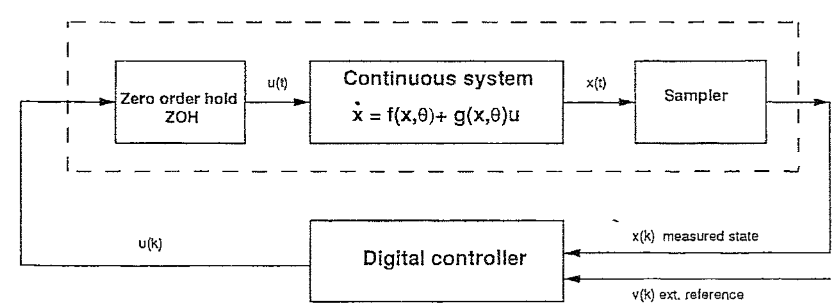
$$x(k) \triangleq x(t = k\delta) \tag{3.1}$$

The ZOH control action is denoted

$$u(t) = u(k\delta) \triangleq u(k) \quad \forall t: k\delta \le t < (k+1)\delta \tag{3.2}$$

Note that the argument 'δ' is omitted in $x(k)$ and $u(k)$ without risk of confusion. With the notations (3.1) and (3.2), it can be shown (see e.g. Monaco and Normand-Cyrot 1985) that the sampled-data version of (2.1) is written as follows

$$Dx(k) = \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \left[L_f + \sum_{i=1}^{m} u_i(k) L_{g_i} \right]^j (x(k)) \tag{3.3}$$



Structure of closed loop.

reminding that, for a function $\lambda: \mathbb{R}^n \to \mathbb{R}$, $L_f\lambda$ denotes the Lie derivative of λ along the vector field $f: \mathbb{R}^n \to \mathbb{R}^n$ and, for $h: \mathbb{R}^n \to \mathbb{R}^n$

$$L_f h \text{ denotes} \begin{bmatrix} L_f h_1 \\ \vdots \\ L_f h_n \end{bmatrix}$$

$Dx(k)$ denotes the finite difference operator

$$Dx(k) \triangleq \frac{x(k+1) - x(k)}{\delta} \tag{3.4}$$

In the following, the discrete model (3.3) will be termed the 'exact' sampled data model because its state exactly coincides with the state of the continuous system (2.1) at the sampling instants. It will be used as basis for the analysis.

4. Statement of the adaptive control problem

Adaptive linearizing control of nonlinear continuous systems has been extensively studied in the literature. In particular, under Assumption A1 to A3 (and even under less restrictive conditions) several continuous time indirect adaptive linearizing control techniques have been proposed by Bastin and Campion (1989), Campion and Bastin (1990), Pomet and Praly (1992) and Teel *et al.* (1991). One of these algorithms, taken from Bastin and Campion (1989), will be the continuous time reference for our discrete time design.

Our objective in this paper is to design a discrete time adaptive linearizing control (i.e. a piecewise constant function $u(x(k), \hat{\theta}(k))$ for the continuous system (2.1)). The design and the analysis of this controller will be based on the 'exact' sampled data model (3.3) which coincides with (2.1) at the sampling instants.

A discrete time transposition of continuous time design should consist of: first finding a linearizing control law for the discrete time model (3.3) and second, using it according to the certainty equivalence principle, in combination with a discrete time least squares recursive estimation.

Such a direct transposition is actually impossible because the discrete time approach introduces new difficulties. Hence, the design of an adaptive discrete time controller requires a specific study, which will be carried out in this paper.

In this section, we will first (§ 4.1) briefly review the main features of the reference continuous time algorithm and will then (§ 4.2) examine the specific difficulties which occur in the discrete time case. The subsequent sections (§ 4.3 to 4.5) will then develop how to tackle these difficulties.

4.1. *Review of a continuous time adaptive control algorithm from Bastin and Campion (1989)*

The continuous control law (2.11) performs state feedback linear stabilization of (2.1) and provides the stable closed-loop behaviour (2.4).

In Bastin and Campion (1989), two continuous indirect adaptive control algorithms are proposed. We focus on the second one and give here its main lines in the framework described in § 2:

Step 1

A 'filtered regressor form' model is first obtained from the regressor form equation (2.8)

$$x = \Psi_0 + \Psi^{\mathrm{T}}\theta \tag{4.1}$$

where the filtered regressor Ψ and the auxiliary quantity Ψ_0 are outputs of a (stable) linear filter ($\omega \in \mathbb{R}^+$)

$$\dot{\Psi}^{\mathrm{T}} = -\omega\Psi^{\mathrm{T}} + \Phi^{\mathrm{T}}$$
$$\dot{\Psi}_0 = -\omega\Psi_0 + \omega x + \Phi_0$$

where, without any risk of confusion, we drop the arguments (x, u) in the expressions of Φ, Ψ and Ψ_0.

Step 2

The state x is fully measured and a (unnormalized) recursive least squares estimation algorithm is applied to (4.1)

$$\begin{aligned} \hat{x} &= \Psi_0 + \Psi^{\mathrm{T}}\hat{\theta} \\ \dot{\hat{\theta}} &= \alpha P\Psi(x - \hat{x}) \\ \dot{P} &= -\alpha P\Psi\Psi^{\mathrm{T}}P \end{aligned} \tag{4.2}$$

where $\alpha \in \mathbb{R}^+$ and P is a $p \times p$ matrix with $P(0) > 0$.

Step 3

The adaptive control law is then obtained by combining (2.11) and (4.2) through a certainty equivalence principle

$$u(t) = u_{\mathrm{cont}}(x(t), \hat{\theta}(t)) \tag{4.3}$$

The closed-loop stability analysis is given in Bastin and Campion (1989) and ensures that there exists a domain $B_0 \subset B_x \times B_\theta$ such that if $(x(0), \hat{\theta}(0)) \in B_0$, then $x(t)$, $\hat{\theta}(t)$ and $(x(t) - \hat{x}(t))$ are bounded for every t, and the asymptotic closed-loop behaviour is given by (2.4) and $\lim_{t\to\infty} x(t) = 0$.

4.2. *Difficulties of the discrete time approach*

The transposition from continuous time to discrete time comes up against the following problems.

Difficulty D1

The exact discrete model makes sense only if the series in the RHS of (3.3) does converge. We must thus ensure that, for any k, when a constant bounded input $u(k)$ is applied between two sampling instants, the series keeps converging. Finite escape time problems might indeed appear through sampling since the continuous system (2.1) is evolving in open loop with constant input between the sampling instants.

Difficulty D2

The RHS of (3.3) is an infinite series with respect to the sampling period δ. This means that the model is not immediately tractable for control design purpose and that truncation and approximation might be necessary.

Difficulty D3

Although system (2.1) is linear with respect to the control $u(t)$ and is state feedback linearizable, its discretized counterpart (3.3) is *not* linear in the control input. This implies that state feedback linear stabilizability of (3.3) is not guaranteed *a priori* despite Assumption A2.

Difficulty D4

Although system (2.1) is linearly parameterized, its discretized counterpart (3.3) is *not* linearly parameterized any more. This will obviously introduce difficulties in the parameter adaptation design.

4.3. *Tackling D1: limitation of admissible sampling periods* δ

The aim is here to derive a bound on the sampling period δ which will guarantee, for constant bounded input $u(k)$ and for any k, the boundedness of the state $x(k)$ between the sampling instants, that is for $t \in [k\delta, (k+1)\delta]$.

An existence result will be given in Theorem 1 under Assumption A1. It is obtained via continuous time analysis of the solution of (2.1) using the following theorem from Hartman (1964).

Theorem of Wintner:

(i) *Let* $Y(t, y)$ *be a continuous scalar function for* $t_0 \leq t \leq t_0 + a$ *and* $y_0 \geq 0$ *such that the solution of the differential equation* $\dot{y} = Y(t, y)$ *with* $y(t_0) = y_0 \geq 0$ *exists on* $[t_0, t_0 + a]$.

(ii) *Let* $f(t, x)$ *be a continuous function on the strip* $t_0 \leq t \leq t_0 + a$ *and* x *be arbitrary in* $\mathbb{R}^n$ *and satisfy*

$$\|f(t, x)\| \leq Y(t, \|x\|) \tag{4.4}$$

Then the solution of the vector differential equation

$$\dot{x} = f(t, x) \quad x(t_0) = x_0 \text{ with } \|x_0\| = y_0$$

exists for every $t \in [t_0, t_0 + a]$ *with* $\|x(t)\| \leq y(t)$.

Proof: For the proof see Hartman (1964, p. 30). □

Application of this theorem leads to the determination of an upper bound δ^* of the maximum allowable sampling period that will ensure the boundedness of the state $x(t)$ of the continuous differential equation (2.1) $\forall t \in [k\delta, (k+1)\delta]$ and $\forall \delta < \delta^*$, when $u(k)$ is constant.

More precisely, we will ensure that $\|x(t)\| < \bar{M}$ with $\bar{M}$ from A1.

Theorem 1: *Under Assumption A1, considering a scalar, monotonically increasing function* $U(\delta)$ *such that the control law* $u(x(k), \delta)$ *applied to* (2.1) *satisfies*

$$\forall x(k) \in B_x, \ \|u(x(k), \delta)\| \leq U(\delta) \tag{4.5}$$

then, there exists a sampling period δ^* *(function of* $\bar{M}$, $\rho(x)$, $U(\delta)$ *and* k^**) ensuring that, for all* $\delta \leq \delta^*$, *if* $x(k) \in B_x$ *then the solution of* (2.1) *exists on* $[t_k, t_{k+1}]$ *and is strictly bounded by* $\bar{M}$.

Proof:

(1) $\rho(y)$ is integrable and definite positive, we can therefore define $P(y)$ such that $(d/dy)P(y) = \rho(y)^{-1} > 0$. $P(y)$ is thus monotonically increasing.

We then define δ^* as the maximum value such that

$$(1 + mU(\delta))\delta < P(\bar{M}) - P(k^*) \quad \forall 0 < \delta < \delta^* \tag{4.6}$$

Since $U(\delta)$ is positive, monotically increasing, this definition makes sense and δ^* is strictly positive.

(2) Let us now prove *ab absurdo* that $\forall t \in [t_k, t_{k+1}]$, $\|x(t)\| < \bar{M}$.

Consider that $\exists t^* \in [t_k, t_{k+1}]$ such that $\|x(t^*)\| = \bar{M}$.

Then, since $x(t_k) \in B_x$ and by continuity, $\exists t_1$ such that $\|x(t_1)\| = k^*$ and such that $\forall t \in [t_1, t^*]$, $M \leq \|x(t)\| \leq \bar{M}$.

From A1 and (4.5) we have in that interval that

$$\|f(x) + g(x)u(x)\| \leq (1 + mU(\delta))\rho(\|x\|) \tag{4.7}$$

Consider the associated scalar differential equation as in the Theorem of Wintner

$$\dot{y} = (1 + mU(\delta))\rho(y) \tag{4.8}$$

From (4.8), we then have that

$$\begin{aligned} \forall t \in [t_1, t^*], \ P(y(t)) &\leq P(y(t_1)) + (1 + mU(\delta))(t - t_1) \\ &\leq P(y(t_1)) + (1 + mU(\delta))\delta \\ &< \bar{M} \end{aligned}$$

Since $P(y)$ is monotonically increasing, it follows that $\forall t \in [t_1, t^*]$, $y(t) < \bar{M}$ and by application of the theorem of Wintner, $\forall t \in [t_1, t^*]$, $\|x(t)\| < \bar{M}$ and hence $\|x(t^*)\| < \bar{M}$ which raises the contradiction.

Therefore $\nexists t^* \in [t_k, t_{k+1}]$ such that $\|x(t^*)\| = \bar{M}$ and hence, $\forall t \in [t_k, t_{k+1}]$, $\|x(t)\| < \bar{M}$. □

Comments

(1) This theorem gives an existence result under the weak Assumption A1 and gives a rather conservative bound δ^* of the maximum admissible sampling period. However, the function $\rho(y)$ should be chosen to bound the functions $\|f(x)\|$ and $\|g_i(x)\|$ as closely as possible for δ^* not to be uselessly conservative. $\bar{M}$ gives a measure of the tolerated intersample excursion of $x(t)$. More practical or refined results can be derived, for example for a specified function $\rho(.)$ or when different bounding functions are used for $f(x)$ and $g_i(x)$. Explicit bounds can be derived for various choices of bounding functions but this is not the scope of this paper.

(2) If $\rho(y)$ is linear, application of Theorem 1 shows, as could be expected, that no finite escape time can occur whatever the sampling period.

(3) Equation (4.5) in Theorem 1 concerning the existence of $U(\delta)$ is not an assumption, it is only the anticipation of a straightforward result of § 5 and 7 where the control action will be bounded by a linear function $U(\delta)$ with non-negative coefficients.

(4) Assumption A1(ii) does not imply a global Lipschitz condition on f and g_i. This appears particularly when Assumption A1(ii) is satisfied $\forall \bar{M} > k^*$.

Then f and g_i are, of course, Lipschitz on every closed set $\{x: M \leqslant \|x\| \leqslant \bar{M}\}$ but are not globally Lipschitz. □

4.4. *Tackling D2 and D3: approximate state feedback linearization*

As stated in § 4.2, the discrete system (3.3) representing the sampled behaviour of the continuous system (2.1) is not necessarily state feedback linearly stabilizable.

Grizzle (1986), Jakubczyk (1987) and Arapostathis *et al.* (1989) have studied linearizability conditions for discrete systems and have shown that sampling can destroy the state feedback linearizability of a system. See also Grizzle and Kokotovic (1988), Nam (1989), Lee and Marcus (1986), Barbot *et al.* (1992) and Castillo *et al.* (1993) for results about linearization of discrete time nonlinear systems.

On the other hand, the exact discrete model (3.3) is given through an infinite series development and a compact, summed-up form of this can only rarely be obtained. It is thus usually very difficult to take into account all the terms of this expression.

To cope with these difficulties, we will not focus in this paper on exact linearization of the discrete time system (3.3) but we will derive a more pragmatic approach that will be successful: the design of a controller performing approximate state feedback linearization of (3.3).

What is meant by 'approximate state feedback linearization' is stated next. In continuous time, the linearizing control law (2.11) provides the desired linear closed-loop behaviour (2.4). In discrete time, exact linearization should consist of finding a controller providing a closed loop behaviour that coincides with the sampled version of the linear system (2.4) which is obtained by applying the exact discretization technique (3.3) to (2.4) and gives

$$Dx(k) = \left(\sum_{i=1}^{\infty} \frac{\delta^{i-1}}{i!} \Lambda^i\right) x(k) \triangleq \Lambda_d x(k) \tag{4.9}$$

Exact matching between this form and the discrete model (3.3) via feedback is generally not feasible.

Approximate linearization will consist of matching only a limited number of terms of both series expansions, as stated in the following definition.

Definition: The discretized model (3.3) is said to be rth-*degree state feedback linearizable* if there exists a control law $u(x(k), \delta)$ such that the closed-loop dynamics coincide with the linear reference model (4.9) up to the degree r, i.e. it can be written

$$Dx(k) = \left(\sum_{i=1}^{r} \frac{\delta^{i-1}}{i!} \Lambda^i\right) x(k) + R(\delta^r) \tag{4.10}$$

where r is called the linearization degree and $R(\delta^r)$ is a function of x and δ whose norm is of the order of δ^r. □

Notation: Throughout this paper, we use the notation $R(\delta^\alpha)$ to denote a residual function of x and δ such that, $\forall x \in B_x$

$$\|R(\delta^\alpha)\| \leqslant O(\delta^\alpha) \text{ i.e. is such that } \lim_{\delta \to 0} \frac{\|R(\delta^\alpha)\|}{\delta^\alpha} < \infty$$

The definition clearly shows that the performances of the approximate linearizing controllers are highly dependent on the sampling period δ and on the linearization degree r.

Qualitatively, the performances will improve with increasing r, even though it is difficult to quantify the gains in the general case.

Several values of r are worth special interest.

(i) $r = \infty$ corresponds to the exact state linear stabilizability of (3.3) but, as stated before, it does not always exist.

(ii) $r = 1$ will give the ZOH of the continuous control law (2.11). This is, in practice, the way which is most often followed for a practical implementation of the continuous control law. The analysis of 1-linearization is therefore a useful tool to get insight into the consequences of this kind of practical implementation.

(iii) $r = 2$: our aim in this paper is to develop more efficient control laws and therefore we go one step further in the series development and focus on 2-linearization of (3.3). Moreover, we show in the next section that 2-linearization is always possible. The analysis of the next sections could, however, easily be transposed to infer similar but weaker results for 1-linearization control law. □

It should be noticed that this approach differs radically from that of Krener (1984), Kang and Krener (1992) and Nam *et al.* (1993) where approximate linearization is also considered, but the remainder terms are higher order terms in x and u, while we consider remainder terms of higher order in the sampling period δ.

Prior to the design of the controllers, we have still to ensure that we fit to a stable discrete linear model. This can lead to a new limitation of the admissible sampling periods as stated in the following lemma.

Lemma 1: *If Λ is Hurwitz stable, there exists $\delta_\Lambda > 0$ such that $(I + \delta\Lambda + (\delta^2/2)\Lambda^2)$ is Schur-stable for every δ such that $0 < \delta < \delta_\Lambda$.*

Proof: Let λ_i denote the eigenvalues of Λ $(i = 1, \dots n)$. It is easily shown that the maximum value of δ such that $|1 + \delta\lambda_i + (\delta^2/2)\lambda_i^2| < 1$ $(i = 1, \dots n)$ can be used as bound δ_Λ.

Further calculations show that

(i) since $\mathrm{Re}(\lambda_i) < 0$, this value is always strictly positive.

(ii) $\delta_\Lambda \geqslant \min_{i=1,\dots,n} \frac{-2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$ □

Definition:

$$\delta_M \triangleq \min\{\delta_\Lambda, \delta^*\} \tag{4.11}$$

with δ_Λ from Lemma 1 and δ^* from Theorem 1. Choosing $\delta < \delta_M$ ensures that the truncated linear closed-loop is stable and that no finite escape time effects can occur between sampling instants. □

4.5. *Tackling D4: (over-)reparametrization of the discrete model*

Although (3.3) is not linearly parameterized, it is however easily seen that it takes the form of a polynomial in the different components of the parameter vector θ.

We can therefore define an infinite series of new parameter vectors β_j ($j = 1, \dots \infty$) as follows:

$$\left.\begin{aligned} \beta_1 &= \theta \\ \beta_2 &= \text{vector of the components of the tensor product } \theta \star \theta \\ &\vdots \\ \beta_j &= \text{vector of the components of the tensor product } \beta_{j-1} \star \theta \end{aligned}\right\} \quad (4.12)$$

We denote also p_j the dimension of the vector β_j.

Thanks to this reparametrization, the discrete model is linear with respect to the new parameter vector set and can be written in linear regressor form as follows

$$Dx(k) = \sum_{i=1}^{\infty} \frac{\delta^{i-1}}{i!}\left[\varphi_{i0}(x(k), u(k)) + \sum_{j=1}^{i} \varphi_{ij}^{\mathrm{T}}(x(k), u(k))\beta_j\right] \quad (4.13)$$

where $\varphi_{ij}(k), u(k)$ are the appropriate regressors derived by introducing (2.6)–(2.7) into (3.3). $\varphi_{i0}(x(k), u(k))$ are functions in $\mathbb{R}^n$ and $\varphi_{ij}(x(k), u(k))$ are matrices of $\mathbb{R}^{n\times p_j}$. These functions are nonlinear functions of $x(k)$ and polynomial functions of the components of $u(k)$.

Note in particular that the term which does not depend on δ is written

$$\varphi_{10}(x(k), u(k)) + \varphi_{11}^{\mathrm{T}}(x(k), u(k))\beta_1 = f(x(k), \beta_1) + G(x(k), \beta_1)u(k) \quad (4.14)$$

and coincides precisely with the continuous time regressor form (2.9) i.e. $\varphi_{10} = \Phi_0$ and $\varphi_{11}^{\mathrm{T}} = \Phi^{\mathrm{T}}$.

Note also that the new parameter set is infinite dimensional but, in § 6, the estimation algorithm will aim at obtaining an estimate of a finite subset of these new parameter vectors: $\{\beta_j \text{ with } j = 1, \dots q\}$.

4.6. *Guidelines*

To close this section, let us sum up the guidelines that will drive us in the design of a discrete non-adaptive 2-linearizing control $u(x(k))$ for (3.3) in § 5; of a discrete parameter estimation algorithm of order q, i.e. estimating $[\hat{\beta}_1(k), \dots \hat{\beta}_q(k)]$ in § 6 and of a discrete adaptive 2-linearizing control law $u(x(k), \hat{\beta}_1(k), \hat{\beta}_2(k))$ in § 7.

(1) The control design will be based on the exact discrete model (3.3) but must be applied to the real continuous system (2.1). We therefore require that no finite escape time effects occur between two sampling instants.

(2) We aim at designing discrete time counterparts of continuous results. We therefore want that, when $\delta \to 0$, the discrete results tend to the continuous ones so that both approaches are consistent.

(3) Regarding control purposes, we design 2-linearizing control. As a by-

product, the design will provide the 1-linearizing control which will coincide with the ZOH of the continuous control (2.5) or (2.11).

(4) In the analysis of the control and the parameter estimation algorithms, we want to exhibit clearly the influence of the choice of δ, q and r so that consistent choices can be made.

(5) Parallel to the continuous time approach, we want to exhibit a domain of initial conditions of the state and (in the adaptive version) of the parameter error, which will ensure that the algorithms can be applied successfully. This domain will not be of infinitesimal nature and therefore, the results should not be considered as local results. They can be considered as semi-global results (see for example Sussmann and Kokotovic 1991 and Teel 1992 for this notion) if the choice of the sampling period is considered as a design parameter.

5. Non-adaptive 2-linearizing control

In this section, we assume that the value of the parameter $\theta = \theta^*$ is perfectly known and we design an approximate linearizing controller $u(x(k), \delta)$ for (3.3) as follows

$$u(x(k), \delta, \theta^*) \triangleq u_0(x(k), \theta^*) + \frac{\delta}{2}u_1(x(k), \theta^*) \quad (5.1)$$

with

$$u_0(x(k), \theta^*) \triangleq [G^{\mathrm{T}}(x, \theta^*)G(x), \theta^*)]^{-1}G^{\mathrm{T}}(x, \theta^*)[\Lambda x - f(x, \theta^*)] \quad (5.2)$$

$$u_1(x(k), \theta^*) \triangleq \frac{\partial u_0(x(k), \theta^*)}{\partial x}\Lambda x \quad (5.3)$$

Lemma 2: *For every $x(k)$ in B_x, the control law (5.1)–(5.3) is well defined and there exists a uniform bounding function $\bar{U}(\delta) \triangleq \bar{U}_0 + (\delta/2)\bar{U}_1$ (with $\bar{U}_0$ and $\bar{U}_1$ two positive constants) such that*

$$\|u(x(k), \delta)\| \leqslant \bar{U}(\delta) = \bar{U}_0 + \frac{\delta}{2}\bar{U}_1 \quad (5.4)$$

Proof: $[G^{\mathrm{T}}(x, \theta^*)G(x, \theta^*)]$ is invertible $\forall x \in B_x$ and the functions involved in (5.1)–(5.3) are continuous on a compact set and therefore uniformly bounded. □

$\bar{U}(\delta)$ satisfies assumption (4.5) of Theorem 1. This theorem is thus applicable. This, together with Lemma 1, enables us to define the value of δ_M in expression (4.11).

The following theorem establishes that the control law (5.1)–(5.3) exists for all $x(k)$ in B_x and is the 2-linearizing control of (3.3).

Theorem 3: *Under Assumptions A1 and A2, for every $\delta < \delta_M$, if $x(k) \in B_x$, then the control law (5.1)–(5.3) realizes the 2-linearization of the system (3.3) i.e. provides closed-loop dynamics that satisfy the following relationship*

$$Dx(k) = \left(\Lambda + \frac{\delta}{2}\Lambda^2\right)x(k) + R_1(\delta^2) \quad (5.5)$$

$$\text{where } \|R_1(\delta^2)\| \leq \frac{\delta^2}{3!} M_1(\delta) < \infty \tag{5.6}$$

with $M_1(\delta)$ polynomial in δ

Proof:

(i) Introducing expressions (5.1) in the discrete model (3.3) and identifying this new series in δ with the desired closed-loop behaviour (4.9) we obtain the following system of equations in u_0, u_1 (where the arguments (x, θ^*) are dropped for simplicity)

$$Gu_0 = \Lambda x - f \tag{5.7}$$

$$Gu_1 = \Lambda^2 x - \left(\frac{\partial f}{\partial x} + \sum_{i=1}^{m} \frac{\partial g_i}{\partial x} u_{0i}\right)(f + Gu_0) \tag{5.8}$$

The solution of (5.7) is clearly (5.2) as in continuous time. Moreover, (5.8) is solvable iff the RHS belongs to the span of G.

By differentiating (5.7) with respect to x, we obtain

$$\frac{\partial f}{\partial x} + \sum_{i=1}^{m} \frac{\partial g_i}{\partial x} u_{0i} + G\frac{\partial u_0}{\partial x} \equiv \Lambda \tag{5.9}$$

and hence, (5.8) is equivalent to $Gu_1 = G(\partial u_0/\partial x)\Lambda x$. Consequently, with u_1 defined as in (5.3), (5.5) is verified.

(ii) The boundedness of $\|R_1(\delta^2)\|$ follows from Theorem 1. Furthermore, replacing $u(k)$ by its expression, $R_1(\delta^2) = (\delta^2/3!)\sum_{i=0}^{\infty}\delta^i r_i(x(k))$ where $r_i(x(k))$ are nonlinear continuous functions on a compact set and, therefore, (5.6) follows. □

This controller (5.1)–(5.3) partially fulfils the requirements given in § 4.6. Notice in particular the following.

(1) the first term of the control law (5.2) is exactly the ZOH approximation of the continuous control law (2.11) and it also follows from the proof of the Theorem 3 that this term realizes the 1-linearization of (3.3).

(2) When $\delta \to 0$, the control law (5.1)–(5.3) tends to the continuous control law (2.11)

$$\lim_{\delta \to 0} u(x(k), \delta) = u_{\text{cont}}(t)$$

To derive the next theorem, we will need the following lemma.

Lemma 3: *For the system*

$$Dx(k) = Ax(k) + v(k)$$

if

(i) *$(I + \delta A)$ is a Schur-stable matrix*
$\Leftrightarrow \exists K(\delta)$ and $\lambda(\delta) > 0$ such that $\|(I + \delta A)^k\| \leq K(\delta)\,e^{-\lambda(\delta)k\delta}\ \forall k$ with
$\lim_{\delta \to 0} K(\delta) = K > 0$
$\lim_{\delta \to 0} \lambda(\delta) = \lambda > 0$

(ii) *$v(k)$ is uniformly bounded and hence $\exists C > 0$ such that $\|(k)\| \leq C\ \forall k$*

(iii) $\limsup_{k\to\infty} \|v(k)\| = c$

Then

(*a*) *$\|x(k)\|$ is uniformly bounded as follows*

$$\|x(k)\| \leq K(\delta)\|x(0)\| + \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}} C$$

(*b*) *the asymptotic behaviour of $x(k)$ fulfils*

$$\limsup_{k\to\infty} \|x(k)\| \leq \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}} c$$

(*c*)
$$\lim_{\delta\to 0} \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}} = \frac{K}{\lambda}$$

Proof: The equivalence in (i) and the proofs of (*a*), (*b*) and (*c*) follow from the solution of the linear system and from the unified theory of Middleton and Goodwin (1990) (see pp. 204–207). □

In the next theorem, we derive some asymptotic convergence properties of the discrete control law (5.1)–(5.3) and we describe the domain of admissible initial errors $x(0)$ ensuring that $x(k)$ remains always in B_x and therefore make the hypothesis of Theorem 3 valid.

Let us first consider that $x(l)$ lies in B_x for every $l: 0 \leq l \leq k$. $Dx(k)$ is then given by (5.5). We will now ensure that $x(k+1)$ still belongs to B_x.

Application of Lemma 3 to the closed loop (5.5) with $A = \Lambda + (\delta/2)\Lambda^2$, $v(k) = R_1(\delta^2)$ gives the following bounds

$$\|x(k+1)\| \leq \alpha_1(\delta)\|x(0)\| + \alpha_2(\delta)\frac{\delta^2}{3!} \leq \alpha_1\|x(0)\| + \alpha_2\frac{\delta^2}{3!} \tag{5.10}$$

with $K(\delta)$ and $\lambda(\delta)$ from Lemma 3

$\alpha_1(\delta) \triangleq K(\delta)$

$\alpha_1 \triangleq \sup\{\alpha_1(\delta): 0 < \delta < \delta_M\}$ which exists and > 0 from Lemma 3

$\alpha_2(\delta) \triangleq \dfrac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}} M_1(\delta)$

$\alpha_2 \triangleq \sup\{\alpha_2(\delta): 0 < \delta < \delta_M\}$ which exists and > 0 from Lemma 3.

We define

$$\delta_1 = \min\left\{\delta_M, \sqrt{\frac{k_1 3!}{\alpha_2}}\right\} \tag{5.11}$$

Then, for every $\delta < \delta_1$, $\eta_1(\delta) \triangleq k_1 - \alpha_2(\delta)(\delta^2/3!) > 0$ and the following set

$$B_0 = \left\{x: \alpha_1(\delta)\|x\| + \alpha_2(\delta)\frac{\delta^2}{3!} \leq k_1\right\} = B\left(0, \frac{\eta_1(\delta)}{\alpha_1(\delta)}\right) \tag{5.12}$$

is the closed sphere of radius $(\eta_1(\delta))/\alpha_1(\delta)$ centred on $x = 0$.

This sphere describes the set of admissible initial errors $x(0)$ as stated in Theorem 4.

Theorem 4: *Under Assumptions A1 and A2, for every* $\delta < \delta_1$ *given in* (5.11), *the control action* (5.1)–(5.3) *applied to the system* (3.3) *ensures that if* $x(0) \in B_0$ *given in* (5.12), *then*

(i) $$\forall k \geqslant 0,\ x(k) \in B_x \text{ and Theorem 3 holds.} \tag{5.13}$$

(ii) $$\limsup_{k\to\infty} \|x(k)\| \leqslant \alpha_2(\delta)\frac{\delta^2}{3!} \leqslant \alpha_2 \frac{\delta^2}{3!} \tag{5.14}$$

Proof:

From (5.10) and (5.12) we obtain that $\|x(k+1)\| \leqslant k_1$ and (i) then follows by induction.

(ii) follows directly from (*b*) of Lemma 3. □

All the requirement of § 4.6 are now fulfilled.

Remarks:

(1) For $r = 1$, *mutatis mutandis*, we can define a constant α_2' such that the 1-linearizing control which is given by (5.2) ensures similarly that

$$\limsup_{k\to\infty} \|x(k)\| \leqslant \alpha_2' \frac{\delta}{2!}$$

(2) Generalization of Theorems 3 and 4 for $r > 2$ is feasible only if $G(x)$ is square full rank. This generalization indeed leads to a controller of the form $u(k) = \sum_{i=0}^{r-1} \delta^i/(i+1)!\,u_i(k)$ that realizes the rth-degree linearization iff the u_i satisfy equations of the form

$$Gu_i = \text{fct}_i(x, u_0, u_1, \ldots, u_{i-1}) \quad \text{for } i = 0, \ldots, r-1$$

where the functions fct_i are obtained by identification of the closed loop with the desired closed loop (4.10). For $i = 0$ and 1, these equations are given by (5.7) and (5.8) and are solvable. For $i > 2$ the solvability is not ensured unless $G(x)$ is square full rank. □

6. Discrete parameter estimation

The discrete parameter estimation developed in this section is based on the regressor form (4.13) of the discrete model (3.3) and aims at estimating the first q parameter vectors $\beta_1, \ldots \beta_q$.

To be consistent with the continuous time approach, the expression (4.13) is first filtered, applying on both sides the operator $(D + \omega)^{-1}$ where ω is any positive constant such that $\omega\delta < 1$ (that is, the discrete counterpart of the filter $(s + \omega)^{-1}$). The discrete model is then easily shown to be equivalent to

$$x(k) = \sum_{i=1}^{\infty} \frac{\delta^{i-1}}{i!}\left[\varphi_{i0f}(x(k), u(k)) + \sum_{j=1}^{i} \varphi_{ijf}^{\mathrm{T}}(x(k), u(k))\beta j\right] \tag{6.1}$$

where $\varphi_{ijf}(x(k), u(k)) \triangleq \varphi_{ijf}(k)$ are obtained by filtering $x(k)$ and $\varphi_{ij}(x(k), u(k))$ as follows

$$\varphi_{10f}(k) = \frac{1}{D + \omega}[\omega x(k) + \varphi_{10}(x(k), u(k))] \tag{6.2}$$

$$\varphi_{ijf}(k) = \frac{1}{D+\omega}\varphi_{ij}(x(k), u(k)) \quad i = 1, \ldots, \infty;\ j = 0, \ldots, i \ (\text{and } ij \neq 10)$$

Using the linearity of the model (3.3) in the parameters $\beta_1, \ldots \beta_q$ it can be rewritten as

$$x(k) = \Psi_0(k) + \Psi^{\mathrm{T}}(k)\beta + R_2(\delta^q) \tag{6.3}$$

where

$$\Psi_0(k) = \sum_{i=1}^{q} \frac{\delta^{i-1}}{i!}\varphi_{i0f}(k) \tag{6.4}$$

$$\Psi^{\mathrm{T}}(k) = (\psi_1^{\mathrm{T}}(k), \psi_2^{\mathrm{T}}(k), \ldots \psi_q^{\mathrm{T}}(k)) \tag{6.5}$$

and

$$\psi_j^{\mathrm{T}}(k) = \sum_{i=j}^{q} \frac{\delta^{i-1}}{i!}\varphi_{ijf}(k) \tag{6.6}$$

$$\beta^{\mathrm{T}} = (\beta_1^{\mathrm{T}}, \beta_2^{\mathrm{T}}, \ldots \beta_q^{\mathrm{T}}) \tag{6.7}$$

$$R_2(\delta^q) = \frac{\delta^q}{(q+1)!}\sum_{i=q+1}^{\infty} r_i(x, u, \delta, \beta_1, \ldots \beta_i) \tag{6.8}$$

(6.8) highlights the fact that $R_2(\delta^q)$ gathers together all the remaining terms of (6.3) which are of order at least q in δ. r_i are nonlinear functions of $x(k)$, linear functions of β_i and polynomial functions of δ and of the components of $u(k)$.

The estimation algorithm will need bounds on $\Psi(k)$ and $R_2(\delta^q)$ which are given in the following lemma.

Lemma 4: *For a given* q *and for all* $\delta < \delta^*$, *there exist positive uniform bounds* $\Psi_{\max}(\delta)$ *and* $M_2(\delta)$ *such that for every* k *if* $\forall l: 0 \leqslant l \leqslant k$, $x(l) \in B_x$ *and* $u(x(l))$ *is uniformly bounded as in* (4.5), *then*

$$\|\Psi(k+1)\| \leqslant \Psi_{\max}(\delta) \tag{6.9}$$

and

$$\|R_2(\delta^q)\| \leqslant \frac{\delta^q}{(q+1)!}M_2(\delta) < \infty \tag{6.10}$$

with $M_2(\delta)$ *polynomial in* δ

Proof:

(i) The elements of $\|\Psi(k+1)\|$ are functions of $x(l)$ and $u(x(l))$ for $l < k+1$. As continuous functions on a compact set, they are uniformly bounded and (6.9) follows.

(ii) Equation (6.10) follows from Theorem 1 and filtering. □

We will now use expression (6.3) to derive the estimation algorithm. This estimation is intended to estimate the true value of the parameter β, which will be denoted β^*.

Without the term $R_2(\delta^q)$, a classic recursive algorithm could easily be implemented, the system being in linear regression form. To take this term into account, the algorithm will be modified, introducing a deadzone to render the estimation robust against the 'unmodelled' term $R_2(\delta^q)$.

The algorithm is presented here without *a priori* choice of the degree of estimation q.

With $\hat{\beta}$ denoting the estimate of β^*, the following recursive normalized least squares algorithm is designed.

prediction

$$\hat{x}(k) = \Psi_0(k) + \Psi^T(k)\hat{\beta}(k) \tag{6.11}$$

prediction error

$$e(k) = x(k) - \hat{x}(k) \tag{6.12}$$

parameter adaptation

$$D\hat{\beta}(k) = \begin{cases} \dfrac{\alpha P(k)\Psi(k)e(k)}{c + \|\Psi(k)\Psi^T(k)\|} & \text{if } \|e(k)\| > d(\delta)\dfrac{\delta^q}{(q+1)!} \\ 0 & \text{otherwise} \end{cases} \tag{6.13}$$

gain adaptation

$$DP(k) = \begin{cases} -\dfrac{\alpha P(k)\Psi(k)\Psi^T(k)P(k)}{c + \|\Psi(k)\Psi^T(k)\|} & \text{if } \|e(k)\| > d(\delta)\dfrac{\delta^q}{(q+1)!} \\ 0 & \text{otherwise} \end{cases} \tag{6.14}$$

$$\text{with } P(0) = \gamma I \ (\gamma > 0) \tag{6.15}$$

'q' will be called the estimation degree; c denotes any positive constant. The deadzone size $d(\delta)$ is defined as follows

$$d(\delta) = \frac{M_2(\delta)}{(1 - \alpha\delta\gamma)^{1/2}} \tag{6.16}$$

where the stepsize α must satisfy

$$0 < \alpha\delta\gamma < 1 \tag{6.17}$$

The following theorem establishes the properties of the estimation algorithm.

Theorem 5: *Under Assumptions A1–A3, for all q, for all $\delta < \delta^*$, if $\forall l: 0 \leq l \leq k$, $x(l) \in B_x$ and $u(x(l))$ is uniformly bounded as in (4.5), then, the estimation algorithm (6.11)–(6.15) applied to the system (3.3) has the following properties.*

(i) *The parameter estimation error $\tilde{\beta}(k) = \beta^* - \hat{\beta}(k)$ is bounded as follows*

$$\|\tilde{\beta}(k)\| \leq \|\tilde{\beta}(0)\| \tag{6.18}$$

(ii) *The prediction error is bounded as follows*

$$\|e(k)\| \leq \Psi_{\max}\|\tilde{\beta}(0)\| + M_2(\delta)\frac{\delta^q}{(q+1)!} \tag{6.19}$$

(iii)
$$\limsup_{k\to\infty} \|e(k)\| \leq d(\delta)\frac{\delta^q}{(q+1)!} \tag{6.20}$$

Proof:

(*a*) We first prove by induction that $P(k) > 0$, $\forall k$. Let $P(k)$ be positive definite. $P(k+1)$ is positive definite if

$$\frac{\sigma\delta P(k)\Psi(k)\Psi^T(k)P(k)}{c + \|\Psi(k)\Psi^T(k)\|}$$

is positive definite, i.e. (from Horn and Johnson 1985 p. 471) if the spectral radius of

$$\frac{\alpha\delta P(k)\Psi(k)\Psi^T(k)}{c + \|\Psi(k)\Psi^T(k)\|}$$

is < 1. This is verified since α satisfies (6.17).

Then, since $\gamma > 0$, obviously $0 < P(k+1) \leq P(k) \leq P(0)$ and therefore, since $P(k)$ is invertible

$$\frac{1}{\gamma}I = P(0)^{-1} \leq P(k)^{-1} \leq P(k+1)^{-1}, \quad \forall k \tag{6.21}$$

(*b*) Let $V(k) = \tilde{\beta}(k)^T P(k)^{-1}\tilde{\beta}(k)$, then after some calculations we obtain that

$$V(k+1) - V(k) = -\frac{\alpha\delta[e^T(k)e(k) - R_2^T(\delta^q)H(k)R_2(\delta^q)]}{c + \|\Psi(k)\Psi^T(k)\|}$$

were

$$H(k) \triangleq \left[I - \frac{\alpha\delta\Psi^T(k)P(k)\Psi(k)}{c + \|\Psi(k)\Psi^T(k)\|}\right]^{-1}$$

and hence

$$\|H(k)\| \leq [1 - \alpha\delta\gamma]^{-1}$$

Therefore, the deadzone (6.16) ensures that $\exists \varepsilon > 0$ such that

$$V(k+1) - V(k) < -\varepsilon < 0 \quad \text{if } \|e(k)\| > d(\delta)\frac{\delta^q}{(q+1)!} \tag{6.22}$$
$$= 0 \text{ otherwise}$$

(i) and (ii) follow then directly from (6.21), (6.22) and Lemma 4.

(*c*) It follows from (6.22) that after a finite number of adaptation steps ($\leq V(0)/\varepsilon$), the prediction error $e(k)$ definitely lies within the deadzone and that ensures (iii). □

Concluding this section, notice that we have obtained a discrete parameter estimation algorithm that fulfils almost all requirements of § 4.6. Notice also the following.

(1) The size of the deadzone decreases with decreasing δ and increasing q and so does the asymptotic value of the prediction error $e(k)$.

(2) For $\delta \to 0$, the deadzone tends, as desired, to zero, but there remains a slight lack of symmetry between the discrete and the continuous algorithms since we needed in discrete-time, a normalized least squares algorithm while Bastin and Campion (1989) used an unnormalized one.

Notice that, to derive the estimation algorithm, we work with the operator $Dx(k)$ defined in (3.4). This is the δ-operator introduced by Middleton and Goodwin (1990) precisely in order to get unified results between continuous and discrete linear systems. There, also, normalized least squares were required to derive unified results.

(3) The blind discretization of the estimation algorithm of Bastin and Campion (1989) should lead to estimation of $\beta_1 = \theta$ only, i.e. to $q = 1$, but without introducing a deadzone in the algorithm. It appears however from the analysis of this section that the existence of this deadzone is crucial to obtain acceptable performance results. This highlights the danger of sampling the continuous time algorithms without a careful analysis.

7. Adaptive 2-linearizing control

In this section, we exploit the results of the preceeding sections to build up an adaptive controller for the system (3.3) with unknown parameter vector θ.

An immediate design should consist of simply applying the so-called certainty equivalence principle, replacing in the control law (5.1) the true value of the parameter θ^* by its estimate $\hat{\beta}_1(k)$ given by the discrete estimation algorithm (6.11)–(6.15); that is the control law $u_0(x(k), \hat{\beta}_1(k)) + (\delta/2)u_1(x(k), \hat{\beta}_1(k))$.

Such a design is, however, not satisfactory. Indeed, it is easily shown that this controller should lead to a closed-loop behaviour with the following structure

$$Dx(k) = \left(\Lambda + \frac{\delta}{2}\Lambda^2\right)x(k) + \varphi_{11}^{\mathrm{T}}\tilde{\beta}_1(k) + \frac{\delta}{2}[\varphi_{21}^{\mathrm{T}}\tilde{\beta}_1(k) + \varphi_{22}^{\mathrm{T}}\tilde{\beta}_2(k)] + R(\delta^2)$$
$$+ \frac{\delta}{2}\varphi_{22}^{\mathrm{T}}[\hat{\beta}_2(k) - \hat{\beta}_1(k) \star \hat{\beta}_1(k)] \quad (7.1)$$

where $R(\delta^2)$ gathers together all terms of degree at least 2 in δ, φ_{ij} are defined in (4.13) and $\tilde{\beta}_i(k) = \beta_i^* - \hat{\beta}_i(k)$.

No acceptable performance results can be derived from this expression of the closed loop behaviour because of the third line of the RHS of (7.1). The two first lines could be related to a desired linear behaviour and (after inverse filtering) to the prediction error, and an appropriate analysis could be performed. No successful analysis could, however, be applied to the last line term. The existence of this term is related to the overparameterization of the model (3.3) which was necessary to recover linearity with respect to the parameters. More precisely, this term is due to the fact that $\beta_2^* = \beta_1^* \star \beta_1^*$ but that no constraint has been imposed between $\hat{\beta}_1(k)$ and $\hat{\beta}_2(k)$ and therefore there is no guarantee concerning their behaviour.

Nevertheless, as shown in the next lemma, Assumption A4 allows us to tackle this problem by modifying, in an appropriate manner, the control law.

Lemma 5: *Under Assumption A4, $\varphi_{22}^{\mathrm{T}}(k)[\theta \star \theta] \in \operatorname{span} G(x, \theta)$ for all $\theta \in B_\theta$.*

Proof: From Assumptions A2 and A4, $\forall \theta \in B_\theta$, $f_k(x)$ and $g_{ik}(x) \in \operatorname{span} G(x, \theta)$ and $\operatorname{span} G(x, \theta) = \operatorname{span} B$.

Hence, $\forall i = 1, \ldots m$, $\forall k = 1, \ldots p$, $\exists \lambda_k(x)$ and $\mu_{ik}(x) \in \mathbb{R}^{m\times 1}$ such that $f_k(x) = B\lambda_k(x)$ and $g_{ik}(x) = B\mu_{ik}(x)$.

Then, the term $\varphi_{22}^{\mathrm{T}}(k)[\theta \star \theta]$ can be rewritten as follows

$$\varphi_{22}^{\mathrm{T}}(k)[\theta \star \theta] = B\frac{\partial}{\partial x}\left[\sum_{k=1}^{p}\theta_k\left(\lambda_k(x) + \sum_{i=1}^{m}\mu_{ik}(x)u_i\right)\right]\left[\sum_{k=1}^{p}\theta_k\left(f_k(x) + \sum_{i=1}^{m}g_{ik}(x)u_i\right)\right] \quad (7.2)$$

and hence this term $\in \operatorname{span} B = \operatorname{span} G(x, \theta)$. □

Under Assumption A4, the last term of (7.1) can now be eliminated if the control law is modified and depends on both estimates $\hat{\beta}_1(k)$ and $\hat{\beta}_2(k)$.

We define the adaptive 2-linearizing control law by the following equations where the estimates $\hat{\beta}_1(k)$ and $\hat{\beta}_2(k)$ are given by the relations (6.11) to (6.15) of the discrete parameter estimation algorithm

$$u_{\mathrm{A}}(x, \delta, \hat{\beta}_1, \hat{\beta}_2) \triangleq u_{\mathrm{A}0}(x, \hat{\beta}_1) + \frac{\delta}{2}[u_{\mathrm{A}1}(x, \hat{\beta}_1) + u'_{\mathrm{A}1}(x, \hat{\beta}_1, \hat{\beta}_2)] \quad (7.3)$$

with

$$u_{\mathrm{A}0}(x, \hat{\beta}_1) = [G^{\mathrm{T}}G]^{-1}G^{\mathrm{T}}[\Lambda x - f(x, \hat{\beta}_1)] \quad (7.4)$$

$$u_{\mathrm{A}1}(x, \hat{\beta}_1) = \frac{\partial u_{\mathrm{A}0}(x, \hat{\beta}_1)}{\partial x}\Lambda x \quad (7.5)$$

$$u'_{\mathrm{A}1}(x, \hat{\beta}_1, \hat{\beta}_2) = [G^{\mathrm{T}}G]^{-1}G^{\mathrm{T}}\varphi_{22}^{\mathrm{T}}(x, u_0)(\hat{\beta}_1 \star \hat{\beta}_1 - \hat{\beta}_2) \quad (7.6)$$

where the notations of the (k) dependence have been dropped out and G stands for $G(x, \hat{\beta}_1)$ to improve readability.

Lemma 6: *Under Assumptions A1 to A4, for every $x(k) \in B_x$ and for every $\hat{\beta}_1(k)$ and $\hat{\beta}_2(k)$ such that*

$$\|\beta_1^* - \hat{\beta}_1(k)\| \leq k_2 \text{ and } \|\beta_2^* - \hat{\beta}_2(k)\| \leq k_2$$

the control law (7.3)–(7.6) is well-defined and there exists a uniform bounding function $\bar{U}_{\mathrm{A}}(\delta) = \bar{U}_{0\mathrm{A}} + (\delta/2)\bar{U}_{1\mathrm{A}}$ such that

$$\|u_{\mathrm{A}}(x(k), \delta, \hat{\beta}_1(k), \hat{\beta}_2(k))\| \leq \bar{U}_{\mathrm{A}}(\delta) = \bar{U}_{0\mathrm{A}} + \frac{\delta}{2}\bar{U}_{1\mathrm{A}} \quad (7.7)$$

Proof: The proof is analoguous to the proof of Lemma 2. □

The bounding function also satisfies here, Assumption (4.5) of Theorem 1. This allows us again to define δ_M in expression (4.11).

Applying the control law (7.3)–(7.6) to the system described by (3.3), we easily show that the closed-loop dynamics are written as follows

$$Dx(k) = \left(\Lambda + \frac{\delta}{2}\Lambda^2\right)x(k) + \delta^2 F_1(x) + \delta^3 F_2(x) + \varphi_{11}^{\mathrm{T}}\tilde{\beta}_1(k)$$
$$+ \frac{\delta}{2}[\varphi_{21}^{\mathrm{T}}\tilde{\beta}_1(k) + \varphi_{22}^{\mathrm{T}}\tilde{\beta}_2(k)] + \sum_{i=3}^{\infty}\frac{\delta^{i-1}}{i!}\left[\varphi_{i0} + \sum_{j=1}^{i}\varphi_{ij}\beta_j\right] \quad (7.8)$$

with

$$F_1(x) = \frac{1}{4}\frac{\partial f}{\partial x}g(u_{\mathrm{A}1} + u'_{\mathrm{A}1}) + \sum_{j=1}^{m}\left[\frac{\partial g_j}{\partial x}(u_{\mathrm{A}1j} + u'_{\mathrm{A}1j})\right](f + gu_0)$$

$$F_2(x) = \frac{1}{8}\sum_{j=1}^{m}\left[\frac{\partial g_j}{\partial x}(u_{A1j} + u'_{A1j})\right]g(u_{A1} + u'_{A1}) \tag{7.9}$$

where all the arguments have been dropped for the purpose of readability.
Theorem 6 then follows.

Theorem 6: *Under Assumptions A1 to A4, for every $\delta < \delta_M$, for every q, if $\forall l: 0 \leq l < k$, $x(l) \in B_x$ and $\|\tilde\beta(l)\| \leq k_2$, the control law (7.3)–(7.6) applied to the system (3.3) provides the closed-loop dynamics*

$$Dx(k) = \left(\Lambda + \frac{\delta}{2}\Lambda^2\right)x(k) + \varphi_{11}^{\mathrm{T}}\tilde\beta_1(k) + \frac{\delta}{2}[\varphi_{21}^{\mathrm{T}}\tilde\beta_1(k) + \varphi_{22}^{\mathrm{T}}\tilde\beta_2(k)] + R_3(\delta^2) \tag{7.10}$$

$$\text{where } \|R_3(\delta^2)\| \leq \frac{\delta^2}{3!}M_3(\delta) \tag{7.11}$$

with $M_3(\delta)$ polynomial in δ

Proof:

(i) Equation (7.10) follows immediately from (7.8) with the adequate definition of $R_3(\delta^2)$.

(ii) Equation (7.12) follows easily from (7.8), (7.9) and Theorem 1, similarly to the proof of Theorem 3. □

The results of this theorem justify the '2-linearizing' characteristic of the controller (7.3)–(7.6).

In the next theorem, we will study both the transient and the asymptotic behaviour of the adaptive controller using the parameter estimation algorithm (6.11)–(6.15).

To derive this theorem, we will now characterize the admissible initial state error $x(0)$ and parameter error $\tilde\beta(0)$, which will ensure that the state keeps in B_x using the same procedure as in § 5.

Let us first consider that $x(l)$ lies in B_x for every $l: 0 \leq l \leq k$. $Dx(k)$ is then given by (7.8) or (7.10). We will now ensure that $x(k+1)$ still belongs to B_x.

Let us apply Lemma 3 to the closed loop with $A = \Lambda + (\delta/2)\Lambda^2$ and $v(k)$ gathering together all nonlinear terms.

Since φ_{ij} are continuous functions of $x(k)$ and $u(k)$, $\exists \varphi_{\max}(\delta) > 0$ such that: $\|\varphi_{11}^{\mathrm{T}}\| + \delta/2(\|\varphi_{21}^{\mathrm{T}}\| + \|\varphi_{22}^{\mathrm{T}}\|) \leq \varphi_{\max}(\delta)$. We also have that $\|\tilde\beta_i(k)\| \leq \|\tilde\beta(k)\|$ $\forall l: 0 \leq l \leq k$, $v(l)$ can then be bounded as follows

$$\|v(l)\| \leq \varphi_{\max}(\delta)\|\tilde\beta(0)\| + M_3(\delta)\frac{\delta^2}{3!}$$

The following bounds are then available for $x(k+1)$

$$\|x(k+1)\| \leq \gamma_1(\delta)\|x(0)\| + \gamma_2(\delta)\|\tilde\beta(0)\| + \gamma_3(\delta)\frac{\delta^2}{3!}$$

$$\leq \gamma_1\|x(0)\| + \gamma_2\|\tilde\beta(0)\| + \gamma_3\frac{\delta^2}{3!} \tag{7.12}$$

with $K(\delta)$ and $\lambda(\delta)$ from Lemma 3

$$\gamma_1(\delta) \triangleq K(\delta)$$
$$\gamma_1 \triangleq \sup\{\gamma_1(\delta): 0 < \delta < \delta_M\}$$
$$\gamma_2(\delta) \triangleq \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}}\varphi_{\max}(\delta)$$
$$\gamma_2 \triangleq \sup\{\gamma_2(\delta): 0 < \delta < \delta_M\}$$
$$\gamma_3(\delta) \triangleq \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}}M_3(\delta)$$
$$\gamma_3 \triangleq \sup\{\gamma_3(\delta): 0 < \delta < \delta_M\}$$

We define

$$\delta_2 = \min\left\{\delta_M, \sqrt{\frac{k_1 3!}{\gamma_3}}\right\} \tag{7.13}$$

Then, for every $\delta < \delta_2$, $\eta_2(\delta) \triangleq k_1 - \gamma_3(\delta)(\delta^2/3!) > 0$ and $(I + \delta\Lambda + (\delta^2/2)\Lambda^2)$ is Schur-stable. The following set

$$B_{00} = \{(x, \tilde\beta): \gamma_1(\delta)\|x\| + \gamma_2(\delta)\|\tilde\beta\| \leq \eta_2(\delta) \text{ and } \|\tilde\beta\| \leq k_2\} \tag{7.14}$$

is hence a non-empty and non-infinitesimal set that will describe the admissible state and parameter errors as stated in Theorem 7.

Theorem 7: *Under Assumptions A1 to A4, for every $\delta < \delta_2$ given in (7.13), the control action (7.3)–(7.6) applied to the system (3.3) ensures that if $(x(0), \tilde\beta(0)) \in B_{00}$ given in (7.14)*

(i) *$\forall k$, $x(k) \in B_x$ and Theorem 6 holds.* (7.15)

(ii) $$\limsup_{k\to\infty}\|x(k)\| \leq \gamma_4(\delta)\frac{\delta^2}{3!} + \gamma_5(\delta)\frac{\delta^{q-1}}{(q+1)!} \leq \gamma_4\frac{\delta^2}{3!} + \gamma_5\frac{\delta^{q-1}}{(q+1)!} \tag{7.16}$$

where γ_i and $\gamma_i(\delta)$ are defined in the proof.

Proof:

(*a*) From (7.12) and (7.14), we obtain that $\|x(k+1)\| \leq k_1$ and (i) follows by induction.

(*b*) To establish the asymptotic result (ii), we will exploit the asymptotic results of the estimation algorithm given in Theorem 5. To this end, it is necessary that we rewrite the closed-loop (7.8) putting forward the dependence of the RHS in the prediction error. After some manipulation, we obtain

$$Dx(k) = \left(\Lambda + \frac{\delta}{2}\Lambda^2\right)x(k) - (D + \omega)e(k) + \Psi^{\mathrm{T}}(k+1)D\tilde\beta(k) + R_4(\delta^2) \tag{7.17}$$

$$\text{with } R_4(\delta^2) = \delta^2F_1(x) + \delta^3F_2(x) + \sum_{i=3}^{q}\frac{\delta^{i-1}}{i!}\left[\varphi_{i0} + \sum_{j=1}^{i}\varphi_{ij}\tilde\beta_j(k)\right] \tag{7.18}$$

and the asymptotic bounds

$$\limsup_{k\to\infty}\Psi^{\mathrm{T}}(k+1)D\tilde\beta(k) = 0$$

$$\limsup_{k\to\infty} R_4(\delta^2) \leq M_4(\delta)\frac{\delta^2}{3!}$$

$$\limsup_{k\to\infty} (D + \omega)e(k) \leq \left(\frac{2}{\delta} + \omega\right)d(\delta)\frac{\delta^q}{(q+1)!}$$

since $\tilde{\beta}(k)$ is constant for k large enough and $\Psi^T(k+1)$ is bounded from Lemma 4. $M_4(\delta)$ is a bound of a finite sum of terms on compact sets and is a polynomial in δ of degree $= \max\{1, q-3\}$.

Applying Lemma 3 leads then to (ii) with

$$\gamma_4(\delta) = \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}} M_4(\delta)$$

$$\gamma_5(\delta) = \frac{\delta K(\delta)}{1 - e^{-\lambda(\delta)\delta}}(2 + \omega\delta)d(\delta)$$

$$\gamma_i = \sup\{\gamma_i(\delta): 0 < \delta < \delta_M\}$$ □

Theorems 6 and 7 show that the requirements of § 4.6 are fulfilled.

Theorem 7 also shows how the performances of the estimation algorithm of degree q and of the 2-linearizing control combine.

To ensure the consistency of the choices of degrees in the control and estimation algorithm, i.e. to neglect terms of the same order of magntitude, it appears that the appropriate choice of degree of estimation is to choose

$$q = 3$$

so that all remaining bounds are of the order of δ^2. Notice that this choice is recommended although the estimation of $\hat{\beta}_3(k)$ will not directly affect the value of the 2-linearizing control $u(x(k), \delta, \hat{\beta}_1(k), \hat{\beta}_2(k))$ but will affect the precision of the prediction error as stated in Theorem 5 and therefore the values of $\hat{\beta}_1(k)$ and $\hat{\beta}_2(k)$, and so have an indirect effect on the control law.

8. Conclusions

Sampled-data control laws have been derived in both non-adaptive and adaptive cases for a class of nonlinear continuous systems. They perform 2-linearization of the sampled-data model of the continuous system.

Convergence properties have been studied thoroughly and detailed conditions have been derived regarding the allowable sampling period and domain of initial errors such that the control laws can be applied to the original continuous system.

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