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Control of Nonholonomic Wheeled Mobile Robots by State Feedback Linearization

Abstract

We are concerned in this article with the control of wheeled mobile robots, *which constitute a class of nonholonomic mechanical systems. More precisely, we are interested in solving the problem of* tracking *with stability of a reference trajectory, by means of linearizing* "static" *and* "dynamic" *state feedback laws. We give conditions to avoid possible singularities of the feedback laws.*

1. Introduction

In recent years, there has been a growing interest in the design of feedback control laws for mobile robots with nonholonomic constraints that arise from constraining the wheels of the robots to roll without slipping. Perhaps the most challenging issue from a theoretical viewpoint has been to find feedback control laws that can stabilize the robots about an equilibrium point. The reason is that a nonholonomic system cannot be stabilized by a smooth state feedback (Brockett 1983; Bloch et al. 1990), and it is therefore necessary to find more clever solutions involving nonstationary (time-varying) and/or singular feedback controls, as pointed out for example, in Samson (1990) and Samson and Ait-Abderrahim (1990). Various solutions have been published for particular types of mobile robots (Coron 1992; Coron and d'Andréa-Novel 1992; Pomet 1992; Pomet et al. 1992; Samson 1990; Samson and Ait-Abderrahim 1990; Samson 1992; Sordalen and Canudas de Wit 1992). Recent results concerning exponential convergence via continuous time-periodic feedback have also been obtained for nonholonomic systems in M'Closey and Murray (1993), and Pomet and Samson (1993).

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However, the problem is not yet completely solved in its entire complexity for the most general dynamical models of mobile robots.

In the present article, we will be concerned not with the stabilization problem about an equilibrium (which is a regulation problem) but with the tracking with stability of a reference motion (this is called also *stabilization about a trajectory* in Walsh et al. [1992]). Somewhat paradoxically, the tracking problem is easier to solve than the regulation problem for wheeled mobile robots. A solution based on a linear approximation of the system around sufficiently exciting reference trajectories is described and analyzed in Walsh et al. (1992).

Our purpose in the present article is to investigate the solvability of tracking problems for mobile robots by means of static and dynamic feedback linearization. Of course, feedback linearization is not the only technique to handle tracking problems, but it is of theoretical and practical significance, precisely when it leads to full linearization. When this is the case, the closed-loop behavior can be made linear and decoupled, and the trajectory error exponentially converges to zero. Unfortunately, dynamic linearizing control laws are singular at rest points, which is not surprising, since, as we have previously recalled, equilibrium points cannot be stabilized by means of smooth state feedback laws. Alternative methods, such as Lyapunov techniques, together with a parameterization relative to the followed path have also been developed (Samson 1990; Samson and Ait-Abderrahim 1990; Micaelli and Samson 1993), which can lead to larger stability domains but guarantee exponential convergence of the trajectory error, provided the longitudinal velocity is bounded away from zero.

We will focus here on the applicability of feedback linearization techniques to solve tracking control problems for mobile robots. In particular, we will elucidate the connection between the intrinsic structural mobility of the robots and their feedback linearizability. Indeed, all mobile robots do not have the same mobility, which strongly depends on the way they are designed and constructed. There exists, for instance, a category of wheeled mobile robots that are constructed to have maximal mobility. Such robots are often called *omnidirectional robots* and have been described in Campion and Bastin (1989). In contrast, mobile robots that are built on the model of a conventional car (often called *car-like robots* in the literature) have a rather poor mobility.

In Campion et al. (1993), we have shown that there exist only five generic types of wheeled mobile robots. This classification will be briefly recalled and motivated in Section 2. For each of these five types of robots, we will address two basic tracking problems: point tracking and posture tracking. These are described in Section 3. In Section 4, by means of static feedback linearization, we will solve the point and posture tracking problems for omnidirectional robots and the *point* tracking problem only for robots having a restricted mobility. In Section 5, we show how the posture tracking problem can be solved for all types of robots by dynamic feedback linearization, albeit with minor singularities (partial results in that sense have been obtained on specific examples in d'Andréa-Novel et al. [1992]). In Section 6, design specifications that guarantee the avoidance of the singularities are also given. The posture tracking problem is then illustrated on a type (2,0) robot. Finally, we discuss the applicability of dynamic feedback linearization together with time-varying feedback laws to globally solve the practical problem of tracking a reference trajectory ended by a rest configuration.

2. Kinematic and Dynamical Models

2.1. Robot Posture

The wheeled mobile robots that are considered in this article are assumed to be made up of a rigid frame equipped with nondeformable wheels. Without loss of generality, we suppose, furthermore, that they are moving on a horizontal plane. The position of the robot is completely described by the following vector ξ of posture coordinates:

$$\xi = (x\, y\, \theta)^T \tag{1}$$

where x, y are the coordinates of a reference point P on the frame in a fixed orthonormal inertial basis $\{0, \vec{I_1}, \vec{I_2}\}$, and θ is the orientation of an arbitrary basis $\{\vec{x_1}, \vec{x_2}\}$ attached to the frame with respect to the inertial basis $\{\vec{I_1}, \vec{I_2}\}$ (Fig. 1).

We will denote in the sequel $\mathbf{R}(\theta)$ the rotation matrix:

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

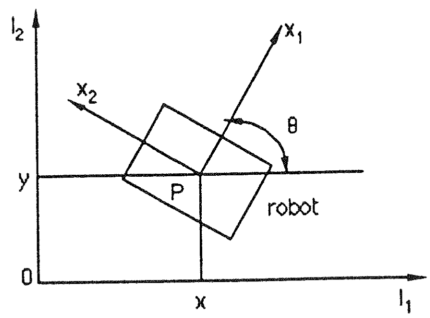

Fig. 1. Position of the robot in the plane.

2.2. Mobility

We have shown in a previous article (Campion et al. 1993) that the mobility of any mobile robot (whatever its constructive peculiarities) can be fully characterized by two characteristic integers δ_m and δ_s that are defined as follows:

DEFINITION 1. δ_m is the *degree of mobility*: it is the dimension of the distribution spanned by the velocity vector $\dot{\xi}(t)$. δ_s is the *degree of steeribility*: it is the number of steering wheels of the robot that can be oriented independently in such a way that the instantaneous center of rotation exists.

Remark 1. The degree of mobility δ_m satisfies the following inequalities:

$$1 \leq \delta_m \leq 3. \tag{3}$$

The upper bound is obvious. The lower bound means that we consider only the case where a motion is possible (i.e. $\delta_m \neq 0$).

The degree of steeribility δ_s satisfies the following inequalities:

$$0 \leq \delta_s \leq 2. \tag{4}$$

The lower bound is obvious. The upper bound means that, if a mobile robot is equipped with more than two steering wheels, the motion of these wheels must be coordinated to guarantee the existence of the instantaneous center of rotation at each time instant. A way of taking possible deadlocks into account is presented in Micaelli et al. (1992) and Thuilot et al. (1994).

Remark 2. In addition to the steering wheels, a mobile robot can be equipped with other types of wheels. Wheels that cannot be oriented will be called *fixed wheels* and wheels that can be self-oriented will be called *self-aligning* or *free wheels* (see Campion et al. [1993] for details concerning the description of the wheels).

2.3. *Kinematic State Space Models*

PROPOSITION 1. There exist only five generic types of wheeled mobile robots (Campion et al. 1993) corresponding to the five pairs of values of δ_m and δ_s that satisfy inequalities (3) and (4), together with the additional inequalities $2 \leq \delta_m + \delta_s \leq 3$:

δ_m	3	2	2	1	1
δ_s	0	0	1	1	2

We will often use the expression "robot of type (δ_m, δ_s)" in the sequel. The five possible types of robots are thus denoted: type (3,0), type (2,0), type (2,1), type (1,1), type (1,2).

We have also shown in Campion et al. (1993) that for each type of mobile robot, however they are constructed, it is always possible to select the reference point P on the frame (see Section 2.1) in such a way that the motion of the robot is described by a kinematic state space model, given in Table 1.

Remark 3. In these state space models, the dimension of the state vector is equal to $3 + \delta_s$. The state variables are:

- The posture coordinates $(x, y, \theta)^T$.
- δ_s (=1 or 2) additional steering coordinates (denoted β for types (1,1) and (2,1), and (β_1, β_2) for type (1,2)).

Remark 4. The number of control inputs (η_i and ζ_i) is equal to $\delta_m + \delta_s$, δ_m being the number of inputs η_i and δ_s the number of inputs ζ_i. These control inputs enter the model linearly and are homogeneous to velocities.

Remark 5. For robots of type (1,1) and type (1,2), the models are parameterized by a constant L, which is not arbitrary and can receive a geometric interpretation. Examples will be given in Section 4.2.

Remark 6. The posture kinematic models in Table 1 can be written in a more compact form:

$$\dot{\mathbf{X}} = \mathbf{B}(\mathbf{X})\mathbf{u} \tag{5}$$

with

$$\text{if } \delta_s = 0: \ \mathbf{X} = \xi, \quad \mathbf{B}(\mathbf{X}) = \mathbf{R}^T(\theta)\Sigma(\beta), \quad \mathbf{u} = \eta \tag{6}$$

or

$$\text{if } \delta_s \neq 0: \ \mathbf{X} = \begin{pmatrix} \xi \\ \beta \end{pmatrix}, \quad \mathbf{B}(\mathbf{X}) = \begin{pmatrix} \mathbf{R}^T(\theta)\Sigma(\beta) & 0 \\ 0 & \mathbf{I} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \tag{7}$$

with the definition of the matrix Σ or $\Sigma(\beta)$ for each type of robot given in Table 2. It has been shown in Campion et al. (1993) that these models are irreducible and that the matrix $\mathbf{B}(\mathbf{X})$ has rank $\delta_m + \delta_s$.

Remark 7. Robots of type (3,0) are called *omnidirectional* because they have a perfect mobility ($\delta_m = 3$) on the plane (see, for example, Muir and Newman 1987; Campion and Bastin 1989). Robots of the other types (with $\delta_m = 1$ or 2) are called *restricted mobility robots.*

2.4. *Dynamical Models*

In fact, the true physical control inputs of a mobile robot are the torques provided by the embarked motors for either the rotation of (some of) the wheels or the orientation of the steering wheels. The kinematic posture model (6), (7) is only a subsystem of the general dynamical model.

In Campion et al. (1993) it is shown that the general dynamical model of wheeled mobile robots can be reduced to the following simple structure, which is sufficient for control design purposes:

$$\begin{cases} \dot{\mathbf{X}} = B(X)u \\ \dot{\mathbf{u}} = v \end{cases} \tag{8}$$

where $\mathbf{v}$ denotes a new $(\delta_m + \delta_s)$-dimensional input vector whose components are homogeneous to torques.

Remark 8. We observe that this dynamical model (8) is simply constructed by augmenting the kinematic state space model (5) with a set of $(\delta_m + \delta_s)$ integrators. Such structures are often referred to as "cascaded systems" (Praly et al. 1991).

Remark 9. The simple dynamical model (8) is obtained from a general dynamical description of wheeled mobile robots by means of nonsingular static state feedback of the wheels' rotation and orientation angles and their derivatives. This feedback plays a role similar to that of the well-known "computed torque" compensation for robotic manipulators (see Campion et al. [1993] for a detailed justification).

3. The Feedback Control Problems

Let us now formulate the two main feedback control problems that will be considered in the sequel.

Table 1.

Type (δ_m, δ_s)	Kinematic State Space Model
(3,0)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$
(2,0)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$
(2,1)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\sin(\theta+\beta) & 0 \\ \cos(\theta+\beta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ $\dot{\beta} = \zeta_1$
(1,1)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -L\sin\theta\sin\beta \\ L\cos\theta\sin\beta \\ \cos\beta \end{pmatrix} \eta_1$ $\dot{\beta} = \zeta_1$
(1,2)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -2L\cos\theta\sin\beta_1\sin\beta_2 - L\sin\theta\sin(\beta_1+\beta_2) \\ -2L\sin\theta\sin\beta_1\sin\beta_2 + L\cos\theta\sin(\beta_1+\beta_2) \\ \sin(\beta_2-\beta_1) \end{pmatrix} \eta_1$ $\dot{\beta}_1 = \zeta_1$ $\dot{\beta}_2 = \zeta_2$

Table 2.

	Type (δ_m, δ_s)				
	(3,0)	(2,0)	(2,1)	(1,1)	(1,2)
Matrix Σ or $\Sigma(\beta)$	Id	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\sin\beta & 0 \\ \cos\beta & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ L\sin\beta \\ \cos\beta \end{pmatrix}$	$\begin{pmatrix} -2L\sin\beta_1\sin\beta_2 \\ L\sin(\beta_1+\beta_2) \\ \sin(\beta_2-\beta_1) \end{pmatrix}$

3.1. The Posture Tracking Problem

The problem is to find a state feedback controller that can achieve the tracking, with stability, of a given moving reference posture $\boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t})$:

$$\boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t}) : \mathbb{R}^+ \longrightarrow \mathbb{R}^3,$$

which will be assumed to be twice differentiable. More precisely, the objective is to find a state feedback control law $\mathbf{v}$ such that:

1. The tracking error $\tilde{\boldsymbol{\xi}}(\mathbf{t}) = \xi(t) - \xi_{\text{ref}}(t)$ and the control $\mathbf{v}$ is bounded for all t.
2. The tracking error asymptotically converges to zero: $\lim_{t\to\infty} \| \boldsymbol{\xi}(\mathbf{t}) - \boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t}) \| = 0$.
3. If $\boldsymbol{\xi}(\mathbf{0}) = \boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{0})$, then $\boldsymbol{\xi}(\mathbf{t}) = \boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t})$ for all t.

In more concrete terms, this can be seen as the problem of tracking the posture $\boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t})$ of a virtual reference robot of the same type.

For omnidirectional robots, this problem can be solved by a smooth static linearizing state feedback (see Section 4.1). For restricted mobility robots, we will show in Section 4.2 that this problem can be solved by a smooth dynamic linearizing state feedback, as long as the robot is moving (i.e. $\dot{\boldsymbol{\xi}}(\mathbf{t}) \neq 0 \ \forall t$).

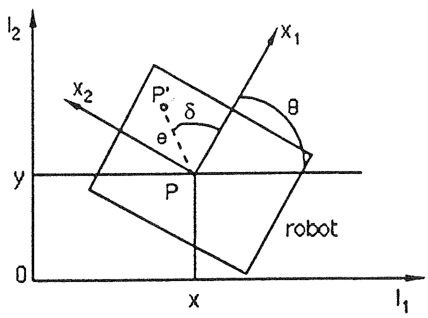

Fig. 2. The point tracking problem.

3.2. The Point Tracking Problem

In a number of instances, full control of the posture of the robot is not required; it is sufficient (or even desirable) to control only the position of a fixed point P' on the frame of the robot (Fig. 2). The polar coordinates of this point, in the basis $\{\vec{x_1}, \vec{x_2}\}$, are denoted (e, δ). The Cartesian coordinates of P' in the inertial basis $\{O, \vec{I_1}, \vec{I_2}\}$ are then expressed as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + e\cos(\theta + \delta) \\ y + e\sin(\theta + \delta) \end{pmatrix}. \qquad (9)$$

The point tracking problem is to find a state feedback controller that can achieve the tracking, with stability, of a given moving reference position $x'_{ref}(t), y'_{ref}(t)$:

$$\begin{pmatrix} x'_{ref}(t) \\ y'_{ref}(t) \end{pmatrix} : \mathbb{R}^+ \longrightarrow \mathbb{R}^2,$$

which is assumed to be twice differentiable.

More precisely, the objective is to find a state feedback control law v such that:

1. The tracking errors $\tilde{\mathbf{x}}'(\mathbf{t}) = x'(t) - x'_{ref}(t)$, $\tilde{\mathbf{y}}'(\mathbf{t}) = y'(t) - y'_{ref}(t)$, and the control $v(t)$ is bounded for all t.
2. $\lim_{t\to\infty} \| \mathbf{x}'(\mathbf{t}) - \mathbf{x}'_{\mathbf{ref}}(\mathbf{t}) \| = 0$; $\lim_{t\to\infty} \| \mathbf{y}'(\mathbf{t}) - \mathbf{y}'_{\mathbf{ref}}(\mathbf{t}) \| = 0$.
3. If $\mathbf{x}'(\mathbf{0}) = \mathbf{x}'_{\mathbf{ref}}(\mathbf{0})$ and $\mathbf{y}'(\mathbf{0}) = \mathbf{y}'_{\mathbf{ref}}(\mathbf{0})$, then $\mathbf{x}'(\mathbf{t}) = \mathbf{x}'_{\mathbf{ref}}(\mathbf{t})$ and $\mathbf{y}'(\mathbf{t}) = \mathbf{y}'_{\mathbf{ref}}(\mathbf{t})$ for all t.

Although restricted mobility robots are not full state feedback linearizable via static-state feedback, as we will see in Section 4, a partial feedback linearization can nevertheless help to solve the point tracking problem.

3.3. Velocity and Torque Control

The control problems, as they have been formulated above, are to find a feedback control law to achieve, for the dynamical model (8),

$$\begin{aligned} \dot{\mathbf{X}} &= B(X)u, \\ \dot{\mathbf{u}} &= \quad v, \end{aligned}$$

the tracking of a reference trajectory. This is called *feedback torque control*, because the control action $\mathbf{v}$ is homogeneous to a torque.

The same posture and point tracking problems can also be considered for the kinematic model (5) only:

$$\dot{\mathbf{X}} = B(X)u.$$

This is called *feedback velocity control*, because the control action u is homogeneous to a velocity.

4. Solving Feedback Control Problems by Means of Static State Feedback

We will use results from static feedback linearization (Isidori 1989; Nijmeijer and Vander Schaft 1990), to solve our feedback control problems. We first show the following property about the maximal subsystem we can linearize after static state feedback:

PROPOSITION 2. The largest linearizable subsystem of the kinematic model (5) has dimension $(\delta_m + \delta_s)$. The largest linearizable subsystem of the dynamical model (8) has dimension $2(\delta_m + \delta_s)$.

Proof. Using the results of Marino (1986), we can easily check that the largest linearizable subsystem of the kinematic state space model (5) has dimension $(\delta_m + \delta_s)$ and is obtained by choosing as "output functions" a vector $\mathbf{z}_1$ of $(\delta_m + \delta_s)$ functions, depending on $\mathbf{X}$. A vector $\mathbf{z}_3$ of $(3 - \delta_m)$ coordinates always remains nonlinearized. In a similar way, we can check that by choosing the same vector $\mathbf{z}_1$ of $(\delta_m + \delta_s)$ output functions, the largest linearizable subsystem of the dynamical state space model (8) has dimension $2(\delta_m + \delta_s)$. The same vector $\mathbf{z}_3$ of $(3 - \delta_m)$ coordinates remains nonlinearized. □

Remark 10. For omnidirectional robots $\delta_m = 3$, and consequently, Proposition 2 shows that the kinematic (resp. dynamical) model is fully linearizable by static state feedback. In contrast, restricted mobility robots are only partially feedback linearizable (a choice of appropriate output functions is presented in Table 3). Therefore, we will show in this section that by means of static feedback, we are able to solve the posture tracking problem for omnidirectional robots (Section 4.1) and only the point tracking problem for restricted mobility robots (Section 4.2).

Table 3.

Type (δ_m, δ_s)	Linearizing Outputs $z_1 = h(\xi, \beta)$	Nonlinear Coordinates $z_3 = k(\xi, \beta)$	Regularity Conditions $\det K(\xi, \beta) \neq 0$
(2,0)	$\begin{pmatrix} x + e\cos(\theta+\delta) \\ y + e\sin(\theta+\delta) \end{pmatrix}$	θ	$e \neq 0,\ \delta \neq k\pi$
(2,1)	$\begin{pmatrix} x + e\cos(\theta+\delta) \\ y + e\sin(\theta+\delta) \\ \beta \end{pmatrix}$	θ	$e \neq 0,\ \delta \neq \beta \bmod \pi$
(1,1)	$\begin{pmatrix} x + L\sin\theta + e\cos(\theta+\beta) \\ y - L\cos\theta + e\sin(\theta+\beta) \end{pmatrix}$	$\begin{pmatrix} \theta \\ \beta \end{pmatrix}$	$e \neq 0$
(1,2)	$\begin{pmatrix} x + L\cos\theta - e\sin(\theta+\beta_1) \\ y + L\sin\theta + e\cos(\theta+\beta_1) \\ \beta_2 \end{pmatrix}$	$\begin{pmatrix} \theta \\ \beta_1 \end{pmatrix}$	$e\sin\beta_2 \neq 0$

4.1. The Case of Omnidirectional Robots

From Table 1 and (8) we can see that the dynamical model of omnidirectional robots has the following form:

$$\begin{cases} \dot{\xi} = R^T(\theta)\eta, \\ \dot{\eta} = v. \end{cases} \tag{10}$$

Differentiating the first equation in $\dot{\xi}$ we obtain:

$$\ddot{\xi} = R^T(\theta)\dot{\eta} + \dot{R}^T(\theta)\eta. \tag{11}$$

Since the matrix $\mathbf{R}^T(\theta)$ is invertible for all θ, it is clear that we can use the control $\mathbf{v} = \dot{\eta}$ to assign any arbitrary dynamics to the posture $\xi(\mathbf{t})$. More precisely we have:

PROPOSITION 3. The posture tracking problem is solved for omnidirectional robots by choosing the following state feedback torque control:

$$\mathbf{v} = R^{-T}(\theta)\,[-\dot{R}^T(\theta)\eta + \ddot{\xi}_{ref} - (\Lambda_1 + \Lambda_2)\dot{\tilde{\xi}} - \Lambda_1\Lambda_2\tilde{\xi}] \tag{12}$$

where $\tilde{\xi} = \xi - \xi_{\mathbf{ref}}$ and the (3×3) matrices Λ_1 and Λ_2 are positive and diagonal.

Proof. Replacing v by (12) in (11) we see that the dynamics of the tracking error $\tilde{\xi}(\mathbf{t}) = \xi(t) - \xi_{ref}(t)$ are governed by the following linear stable differential equation:

$$\ddot{\tilde{\xi}}(t) + (\Lambda_1 + \Lambda_2)\dot{\tilde{\xi}}(t) + \Lambda_1\Lambda_2\tilde{\xi}(t) = 0 \tag{13}$$

with the (3×3) matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ positive and diagonal.

This equation (13) can be interpreted as a reference model for the tracking error—that is, a model of how we want the tracking error to decrease.

We observe that this control law is quite similar to the "computed torque" control that has been classically derived for rigid link manipulators. □

It is also of interest to write the reference model (13) in state space form:

$$\begin{cases} \dot{\tilde{\xi}} = -\Lambda_1\tilde{\xi} + \tilde{\sigma}, \\ \dot{\tilde{\sigma}} = -\Lambda_2\tilde{\sigma}, \end{cases}$$

with the new state $\tilde{\sigma} = \Lambda_1\tilde{\xi} - \dot{\xi}_{\mathbf{ref}} + R^T(\theta)\eta$.

It is easily checked that this new state $\tilde{\sigma}$ can also be written as follows:

$$\tilde{\sigma} = R^T(\theta)(\eta - \eta_{ref}), \tag{14}$$

with $\eta_{\mathbf{ref}} = R(\theta)[\dot{\xi}_{ref} - \Lambda_1\tilde{\xi}]$ being the linearizing velocity control of the kinematic state space submodel.

We can therefore interpret (12) as a torque control law that performs the tracking of the reference linearizing velocity control signal $\eta_{\mathbf{ref}}(\mathbf{t})$ together with the reference posture $\xi_{\mathbf{ref}}(\mathbf{t})$.

4.2. The Case of Restricted Mobility Robots

The dynamical model (8) can be written in the following form for restricted mobility robots:

$$\dot{\xi} = R^T(\theta)\Sigma(\beta)\eta \tag{15}$$

$$\dot{\beta} = \zeta \tag{16}$$

$$\dot{\eta} = v_1 \tag{17}$$

$$\dot{\zeta} = v_2 \tag{18}$$

We know from Proposition 2 that this model (15)–(18) is not full state linearizable by a smooth static time-invariant state feedback and that the largest linearizable subsystem has dimension $2(\delta_m + \delta_s)$. The purpose of this section is to examine how this property can be used to solve the point tracking problem by static state feedback linearization. Before that, we discuss the partial feedback linearization of the dynamical model (15)–(18) in a more detailed way in Propositions 4 and 5.

PROPOSITION 4. The general dynamical model (15)–(18) can be generically transformed by state feedback and diffeomorphism into a controllable linear subsystem of dimension $2(\delta_m + \delta_s)$ and a nonlinear subsystem of dimension $3 - \delta_m$ of the following form:

$$\begin{cases} \dot{\mathbf{z}}_1 = z_2 \\ \dot{\mathbf{z}}_2 = w \\ \dot{\mathbf{z}}_3 = \bar{Q}(z_1, z_3) z_2 \end{cases} \tag{19}$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ have dimension $\delta_m + \delta_s$, $\mathbf{z}_3$ has dimension $3 - \delta_m$, and $\mathbf{w}$ is an auxiliary torque control input.

Proof. As a consequence of Proposition 2, there exists a linearizing output vector function

$$\mathbf{z}_1 = h(\xi, \beta) = h(X) \tag{20}$$

of dimension $(\delta_m + \delta_s)$ that depends on the posture coordinates $\boldsymbol{\xi}$ and the angular coordinates β only, but *not* on the states η and $\boldsymbol{\zeta}$, such that the largest linearizable subsystem is obtained by twice differentiating $\mathbf{z}_1$ as follows:

$$\begin{aligned} \dot{\mathbf{z}}_1 &= \frac{\partial h}{\partial X} B(X) u = K(X) u, \\ \ddot{\mathbf{z}}_1 &= K(X) v + g(X, u), \end{aligned} \tag{21}$$

with the decoupling matrix $\mathbf{K(X)}$:

$$\mathbf{K(X)} = \begin{pmatrix} \frac{\partial h}{\partial \xi} R^T(\theta) \Sigma(\beta) & \frac{\partial h}{\partial \beta} \end{pmatrix} \tag{22}$$

and

$$\mathbf{g(X, u)} = \tag{23}$$

$$\frac{\partial}{\partial \xi} \left[K(\xi, \beta) \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \right] R^T(\theta) \Sigma(\beta) \eta + \frac{\partial}{\partial \beta} \left[K(\xi, \beta) \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \right]$$

The change of coordinates can then be defined as follows:

$$\begin{cases} \mathbf{z}_1 = h(X) \\ \mathbf{z}_2 = K(X) u \\ \mathbf{z}_3 = k(X) \end{cases}$$

where $\mathbf{k(X)}$ is selected such that the transformation:

$$\begin{pmatrix} \xi \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} h(X) \\ k(X) \end{pmatrix} \tag{24}$$

is a diffeomorphism on $\mathbb{R}^{\delta_s + 3}$.

In the new coordinates, the system dynamics are written:

$$\begin{cases} \dot{\mathbf{z}}_1 = z_2 \\ \dot{\mathbf{z}}_2 = \bar{g}(z_1, z_2, z_3) + \bar{K}(z_1, z_3) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \\ \dot{\mathbf{z}}_3 = \bar{Q}(z_1, z_3) z_2 \end{cases} \tag{25}$$

where $\bar{\mathbf{g}}$, $\bar{\mathbf{K}}$, and $\bar{\mathbf{Q}}$ represent $\mathbf{g}$, $\mathbf{K}$, and $\mathbf{Q}$ expressed in the new coordinates, with:

$$\mathbf{Q(X)} = \left[\frac{\partial k}{\partial \xi} R^T(\theta) \Sigma(\beta) \quad \frac{\partial k}{\partial \beta} \right] K^{-1}(X). \tag{26}$$

Since the decoupling matrix $\mathbf{K(X)}$ is generically invertible, by applying the control:

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \bar{K}^{-1}(z_1, z_3) [w - \bar{g}(z_1, z_2, z_3)] \tag{27}$$

to (25), we obtain the expected partially linear system (19), and this concludes the proof. □

Remark 11. It is easy to check that the kinematic model (7) is also partially static feedback linearizable with the same output functions $\mathbf{z}_1 = h(\xi, \beta)$, which trivially lead to the same decoupling matrix $\mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\beta})$.

Once we have obtained the partially linearized structure (25), the auxiliary input $\mathbf{w}$ can then be used to freely assign the dynamics of $\mathbf{z}_1$ as the dynamics of a second-order stable linear system. However, to perform the tracking with stability of some suitable reference trajectory, we have to ensure at least the boundedness of the nonlinear part $\mathbf{z}_3$. More precisely we have:

PROPOSITION 5. Let $\mathbf{z}_{1ref}$, $\dot{\mathbf{z}}_{1ref}$, $\ddot{\mathbf{z}}_{1ref}$ be a smooth reference trajectory such that $\| \mathbf{z}_{1ref}(t) \|$, $\| \dot{\mathbf{z}}_{1ref}(t) \|$, $\| \ddot{\mathbf{z}}_{1ref}(t) \|$ are bounded for every t and such that $\dot{\mathbf{z}}_{1ref}(t)$ is L_1. If the matrix $\bar{\mathbf{Q}}(\mathbf{z}_1, \mathbf{z}_3)$ defined in (25) is bounded for every $\mathbf{z}_1$, $\mathbf{z}_3$ then the following auxiliary control law:

$$\mathbf{w} = \ddot{z}_{1ref}(t) - (\Lambda_1 + \Lambda_2) \dot{\tilde{z}}_1 - \Lambda_1 \Lambda_2 \tilde{z}_1, \tag{28}$$

where $\tilde{\mathbf{z}}_1 = z_1 - z_{1ref}$, Λ_1 and Λ_2 are arbitrary positive diagonal $(\delta_m + \delta_s) \times (\delta_m + \delta_s)$ matrices, generically ensures that:

- $\tilde{\mathbf{z}}_1$, $\dot{\tilde{\mathbf{z}}}_1$ exponentially converge to zero.
- $\mathbf{z}_3(\mathbf{t})$ is bounded for every t.

Proof. Replacing $\mathbf{w}$ par its expression (28) in (19) leads to the following linear second-order stable decoupled equation in $\tilde{\mathbf{z}}_1$:

$$\ddot{\tilde{z}}_1 + (\Lambda_1 + \Lambda_2) \dot{\tilde{z}}_1 + \Lambda_1 \Lambda_2 \tilde{z}_1 = 0.$$

Therefore, there exist positive constants C and α such that:

$$\| \tilde{\mathbf{z}}_2(\mathbf{t}) \| \leq C e^{-\alpha t} \| \tilde{z}_2(0) \| . \tag{29}$$

Moreover, since, by assumption, $\bar{\mathbf{Q}}(\mathbf{z}_1, \mathbf{z}_3)$ defined in (25) is bounded for every $\mathbf{z}_1$, $\mathbf{z}_3$—say, by a positive constant K_1—we have:

$$\| \mathbf{z}_3(\mathbf{t}) \| \leq \| z_3(0) \| + K_1 \left[\int_0^t \| z_{2ref}(\tau) \| \, d\tau + \frac{C}{\alpha} (1 - e^{-\alpha t}) \| \tilde{z}_2(0) \| \right]$$

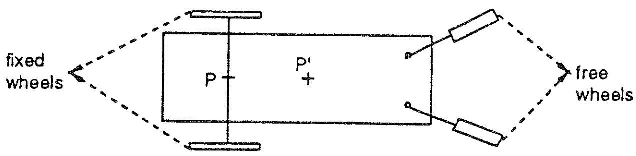

Fig. 3. A type (2,0) robot.

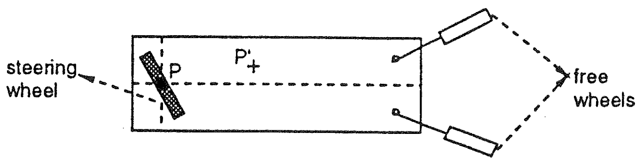

Fig. 4. A type (2,1) robot.

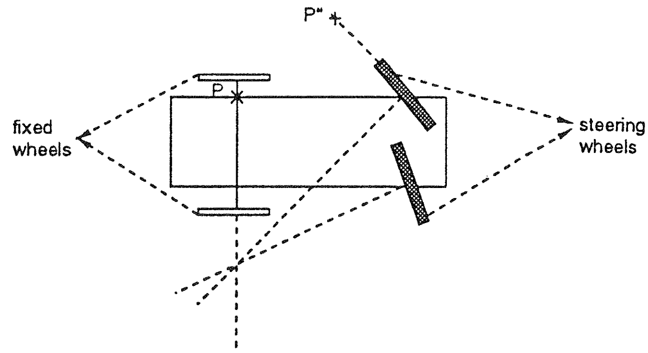

Fig. 5. A type (1,1) robot.

Since the reference trajectory is supposed to be such that $\dot{\mathbf{z}}_{1ref}(\mathbf{t}) = \mathbf{z}_{2ref}(\mathbf{t})$ is L_1, we can conclude that $\mathbf{z}_3$ remains bounded for all t. □

Remark 12. A function $f(t)$ is said to be L^1 if $\int_0^{+\infty} \mid f(t) \mid dt < +\infty$.

We now examine the possibility of using the above linearizing control approach to solve the point tracking problem for restricted mobility robots. For each type of robot, we apply Propositions 4 and 5 and give admissible choices of output functions $\mathbf{z}_1$ containing the Cartesian coordinates of a point P' on the frame of the robot (see Section 3.2), and of the nonlinearized coordinates z_3 in Table 3.

The last column of Table 3 also gives in each case the conditions for the decoupling matrix $\mathbf{K}(\xi, \beta)$ given by (22) to be not singular.

From Table 3 we see that the point tracking problem is solvable by static feedback linearization for type (2,0) and type (2,1) robots, because the Cartesian coordinates x', y' of a fixed point P' on the frame that we want to track are components of the vector $\mathbf{z}_1 = h(X)$ of linearizing output functions, in both cases. However, the regularity conditions may imply that some particular choices of the controlled point P' must be excluded. Let us examine this question on two concrete examples, shown in Figures 3 and 4.

Example of a Type (2,0) Robot The robot of Figure 3 has two fixed wheels and two free wheels. The kinematic state space model of this robot is given by Table 1, provided the point P is located on the axle of the fixed wheels. The regularity condition means that P' must not be located on the same axle.

In fact, any type (2,0) robot has no steering wheel, but it has either one or several fixed wheels, with a single common axle (Campion et al. 1993). Therefore, the regularity condition will always mean that P' must not be located on this axle.

Example of a Type (2,1) Robot The robot of Figure 4 has one steering wheel and two free wheels. The kinematic state space model of this robot is given by Table 1, provided the point P is the center of the steering wheel. In fact, any type (2,1) robot has only one steering wheel (Campion et al. 1993). The regularity conditions always mean that the point P' should not be the center of the steering wheel (e must be not zero) and that $\sin(\delta - \beta)$ must be not zero. Due to the block-triangular form of the corresponding decoupling matrix $\mathbf{K}(\theta, \beta)$, this singular configuration characterized by $\delta = \beta \, mod \, \pi$ can be avoided at each instant by an appropriate choice of the control input $\mathbf{v}_2$, whereas $\mathbf{v}_1$ ensures the linearization of the dynamics of the tracking error vector $\tilde{\mathbf{z}}_1$.

In contrast, for type (1,1) and type (1,2) robots, the point tracking problem for a fixed point P' of the frame defined by (9) cannot be solved by feedback linearization, the corresponding decoupling matrix being generically singular in this case. It is interesting, however, to notice that, for these robots, there are linearizing outputs that can be interpreted from a "physical" point of view.

Example of a Type (1,1) Robot The robot of Figure 5, classically referred to as the *front wheel drive car*, has two steering wheels with coordinated orientations, and two fixed wheels. The axles of the two steering wheels converge to the instantaneous center of rotation belonging to the axle of the fixed wheels. One of the steering wheels is selected (arbitrarily) as the "master" wheel, with an orientation characterized by the angle β, while the orientation of the other one is constrained in order to satisfy the above condition. The reference point P is located on the axle of the fixed wheels, at the intersection with the perpendicular passing through the center of the master wheel. Then, the constant L used in Table 2 (see Remark 5) is the distance between P and the center of the master wheel.

The linearizing outputs (see Table 3) correspond to the Cartesian coordinates of a material point P'' of the robot attached to the plane of one of the steering wheels,

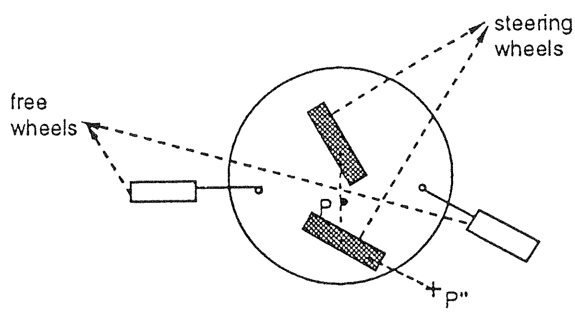

Fig. 6. A type (1,2) robot.

except the center of the wheel (belonging to the frame), for which $e = 0$. In fact, type (1,1) robots have one or several fixed wheels with a single common axle. They have also one or several steering wheels but with one independent steering orientation ($\delta_s = 1$), and the center of one of these steering wheels is not located on the axle of the fixed wheels (Campion et al. 1993). Consequently, for any type (1,1) robot, the linearizing outputs can be chosen as the Cartesian coordinates of a material point P'' of the robot attached to the plane of one of the steering wheels, except the center of the wheel.

Example of a Type (1,2) Robot The robot of Figure 6 has two independent steering wheels and two free wheels. The reference point P is the center of the axle joining the centers of the two steering wheels. Moreover, the constant L used in Table 2 (see Remark 5) is the half-length of this axle. There is once more a "physical" choice of linearizing outputs (see Table 3) that allows us to solve the point tracking problem for a material point P'' attached to the plane of one of the two independent steering wheels, except the center of the corresponding wheel. The regularity conditions also mean that $\beta_2 \neq 0 \bmod \pi$. As for the type (2,1) case, due to the block-triangular form of the corresponding decoupling matrix $\mathbf{K}(\theta, \beta)$, this singular configuration can be avoided at each instant by an appropriate choice of the control input $\mathbf{v}_2$, whereas $\mathbf{v}_1$ ensures the linearization of the dynamics of the tracking error vector $\tilde{\mathbf{z}}_1$. Type (1,2) robots have no fixed wheel and two or more steering wheels, two of them being oriented independently. Therefore, for any type (1,2) robot, a possible choice of linearizing outputs is the Cartesian coordinates of a point attached to the plane of one of the steering wheels, except the center of the corresponding wheel.

For each type of restricted mobility robot, it is easy to check from (22) that the decoupling matrix $\mathbf{K}(\xi, \beta)$ depends only on sine and cosine functions of the angles θ and β, and from Table 3 we see that $\mathbf{z}_3 = k(\xi, \beta)$ only depends linearly on θ and β. Consequently, we easily deduce that the matrix $\mathbf{Q}$ defined by (26) in (19) is bounded for every θ and β such that $\det\mathbf{K}(\theta, \beta) > \epsilon (> 0)$. Therefore, by applying Proposition 5, if the reference trajectory is such that $\mathbf{z}_{1\text{ref}}(\mathbf{t})$, $\dot{\mathbf{z}}_{1\text{ref}}(\mathbf{t})$, and $\ddot{\mathbf{z}}_{1\text{ref}}(\mathbf{t})$ are bounded for every t and $\dot{\mathbf{z}}_{1\text{ref}}(\mathbf{t})$ is L_1, we can conclude that the point tracking problem is solvable for any restricted mobility robot, with a fixed point P' on the frame when $\delta_m = 2$ and a fixed point P'' attached to one steering wheel when $\delta_m = 1$.

5. Solving Feedback Control Problems by Means of Dynamic State Feedback

5.1. Principle of Dynamic Feedback Linearization

We consider a dynamical system given in general state space form:

$$\dot{\mathbf{X}} = f(X) + \sum_{i=1}^{m} g_i(X) u_i, \tag{30}$$

where the state $\mathbf{X}$ is in $\mathbb{R}^n$, the input $\mathbf{u}$ in $\mathbb{R}^m$. The vector fields $\mathbf{f}$ and $\mathbf{g}_i$ are smooth.

When the system is not completely linearizable by diffeomorphism and static state feedback (as for restricted mobility robots, see the previous section), full linearization nevertheless can possibly be achieved by considering more general dynamic feedback laws of the form:

$$\begin{cases} \mathbf{u} = \alpha(X, \chi, w), \\ \dot{\chi} = a(X, \chi, w), \end{cases} \tag{31}$$

$\mathbf{w}$ being an auxiliary input.

Such a dynamic feedback is obtained through the choice of m suitable "output functions":

$$Y_i = h_i(X)\ ,\ i = 1, \ldots, m, \tag{32}$$

leading to a singular decoupling matrix. We apply on system (30)–(32) the so-called *dynamic extension algorithm* (Descusse and Moog 1985). The idea of this algorithm is to delay some "combinations of inputs" simultaneously affecting several outputs, via the addition of integrators, in order to enable other inputs to act in the meantime and therefore hopefully to obtain an extended decoupled system of the form:

$$Y_k^{(r_k)} = w_k\ ,\ k = 1, \ldots, m, \tag{33}$$

where $Y_k^{(i)}$ denotes the ith derivative of Y_k w.r.t. time, r_k is called the relative degree of Y_k, and w_k denotes the new auxiliary inputs. Moreover, in order to get full linearization, we must have for the n_e-dimensional extended system:

$$\sum_{i=1}^{m} r_i = n_e, \tag{34}$$

where n_e is the dimension of the extended state vector $\mathbf{X}_e = (X, \chi)^T$, and if (34) is satisfied, then:

$$\zeta = \Psi(X_e) = (Y_1 \cdots Y_1^{(r_1-1)} \cdots Y_m \cdots Y_m^{(r_m-1)})^T \tag{35}$$

is a local diffeomorphism.

The problem of finding sufficient conditions for dynamic feedback linearization is quite open. Nevertheless, a simple necessary condition for a system to be dynamically feedback linearizable in the neighborhood of an equilibrium point X_0 is given in Charlet (1989) (see also Charlet et al. [1989]).

PROPOSITION 6. If a system is dynamically feedback linearizable in the neighborhood of an equilibrium point X_0, then its tangent linearization at X_0 is controllable, in the linear sense.

Example of Type (2,0) Robots To be able to apply the dynamic extension algorithm, we have to choose linearizing output functions that correspond to a singular decoupling matrix when considering static state feedback laws. From Table 3, a possible choice consists in taking as output functions the coordinates of a point P' located on the axle of the fixed wheels (i.e., $\delta = k\pi$); for example:

$$\begin{cases} h_1 &= x + e\cos\theta \\ h_2 &= y + e\sin\theta \end{cases} \tag{36}$$

We first consider the kinematic state space model of type (2,0) robots given in Table 1. We easily check that the only combination of inputs appearing in $\dot{h}_1$ and $\dot{h}_2$ is $\chi_1 = \eta_1 + e\eta_2$, with χ_1 being the longitudinal velocity of P'. So, applying the dynamic extension algorithm, we delay χ_1—i.e., we introduce a new input U_1 such that

$$\dot{\chi}_1 = U_1. \tag{37}$$

Then, computing $\ddot{h}_1$ and $\ddot{h}_2$, the extended system with extended state vector

$$\mathbf{X_e} = (x, y, \theta, \chi_1)^T$$

is linearizable by static state feedback with new inputs U_1 and η_2 and diffeomorphism:

$$\Psi = (h_1, h_2, \dot{h}_1, \dot{h}_2)^T. \tag{38}$$

Moreover, since

$$\begin{cases} \ddot{h}_1 &= -\chi_1 \cos\theta\, \eta_2 - \sin\theta\, U_1, \\ \ddot{h}_2 &= -\chi_1 \sin\theta\, \eta_2 + \cos\theta\, U_1, \end{cases} \tag{39}$$

we notice that the new decoupling matrix is singular only when the longitudinal velocity χ_1 of P' is zero.

Let us now consider the dynamical model (8), which specializes here as follows:

$$\begin{cases} \dot{x} &= -\eta_1 \sin\theta \\ \dot{y} &= \eta_1 \cos\theta \\ \dot{\theta} &= \eta_2 \\ \dot{\eta}_1 &= v_1 \\ \dot{\eta}_2 &= v_2 \end{cases} \tag{40}$$

We easily see that the only combination of inputs appearing in $\ddot{h}_1$ and $\ddot{h}_2$ is now $\chi_2 = v_1 + ev_2 = \dot{\chi}_1$. So, applying the dynamic extension algorithm, we delay χ_2—i.e., we introduce a new input U_2 such that:

$$\dot{\chi}_2 = U_2. \tag{41}$$

Then, computing $h_1^{(3)}$ and $h_2^{(3)}$, the extended system with extended state vector

$$\mathbf{X_e} = (x, y, \theta, \eta_1, \eta_2, \chi_2)^T$$

is linearizable by static state feedback with new inputs U_2 and v_2 and diffeomorphism:

$$\Psi = (h_1, h_2, \dot{h}_1, \dot{h}_2, \ddot{h}_1, \ddot{h}_2)^T, \tag{42}$$

since:

$$\begin{cases} h_1^{(3)} = -U_2 \sin\theta - v_2 \cos\theta\, \chi_1 - 2\eta_2 \cos\theta\, \chi_2 + \eta_2^2 \sin\theta\, \chi_1, \\ h_2^{(3)} = U_2 \cos\theta - v_2 \sin\theta\, \chi_1 - 2\eta_2 \sin\theta\, \chi_2 - \eta_2^2 \cos\theta\, \chi_1. \end{cases} \tag{43}$$

We notice that the new decoupling matrix is also singular when the longitudinal velocity χ_1 of P' is zero. We summarize this discussion in the following proposition.

PROPOSITION 7. By considering as output functions the Cartesian coordinates given by (9) of a point P' on the axle of the fixed wheels (corresponding to $\delta = k\pi$), type (2,0) robots are generically fully dynamic feedback linearizable, and the posture tracking problem is solvable as long as the longitudinal velocity of P' is not zero.

This example has also been considered in Slotine and Li (1991) (with $e = 0$).

We have seen that the kinematic and dynamic state space models of type (2,0) robots are both fully dynamic feedback linearizable by considering the same linearizing output functions. In fact, this result can be generalized as follows:

LEMMA 1. If a nonlinear system (d'Andréa-Novel et al. 1993)

$$\dot{\mathbf{X}} = F(X, u) \tag{44}$$

(where $\mathbf{X}$ is the n-dimensional state vector and $\mathbf{u}$ is the m-dimensional input) is fully linearizable by means of endogenous dynamic feedback, then the cascaded system

$$\begin{cases} \dot{\mathbf{X}} = F(X, u) \\ \dot{\mathbf{u}} = v \end{cases} \tag{45}$$

(where the state vector is $(\mathbf{X}, \mathbf{u})^T$ and the input is v) is also fully linearizable by means of endogenous dynamic feedback, with the same linearizing outputs as the ones linearizing (44) and the same decoupling matrix.

The proof detailed in d'Andréa-Novel et al. (1993) uses results on "differentially flat" nonlinear systems (see Fliess et al. [1992] and Martin [1992]).

Remark 13. From Lemma 1, owing to the "cascaded systems" property (see Remark 8), it will be sufficient for restricted mobility robots to study the full dynamic feedback linearization property for the kinematic state space model (5).

Remark 14. From the linearization point of view, the control law becomes singular when the determinant of the square decoupling matrix $\mathbf{K}(\mathbf{X_e})$, is zero.

Generically, this corresponds in the extended state space to an $(n_e - 1)$–dimensional submanifold. From Charlet (1989), we know that this submanifold contains the domain where the diffeomorphism $\Psi(X_e)$ is singular. Consequently, at any point where the decoupling matrix is not singular, the diffeomorphism $\Psi(X_e)$ exists.

Since full linearization is obtained (at least generically), we will use this method to achieve the tracking with stability of a reference trajectory. More precisely, let $\mathbf{X_{ref}(t)}$ be a smooth reference trajectory; then, according to (32), we have

$$(Y_{ref})_i = h_i(X_{ref}), \qquad i = 1, \ldots, m \tag{46}$$

and according to (35), $\zeta_{\mathbf{ref}}(\mathbf{t})$ is defined as follows:

$$\zeta_{\mathbf{ref}} = ((Y_{ref})_1 \cdots (Y_{ref})_1^{(r_1-1)} \cdots (Y_{ref})_m \cdots (Y_{ref})_m^{(r_m-1)})^T. \tag{47}$$

This implies that

$$(\mathbf{X_e})_{\mathbf{ref}} = \Psi^{-1}(\zeta_{ref}). \tag{48}$$

The following proposition shows how to select the auxiliary control $\mathbf{w}$ (see (33)) in order to achieve the tracking with stability of $\zeta_{\mathbf{ref}}(\mathbf{t})$ and consequently of $\mathbf{X_{ref}(t)}$.

Proposition 8. The following auxiliary control laws w_i, $i = 1, \ldots m$:

$$w_i = (Y_{ref})_i^{(r_i)} + \sum_{j=0}^{r_i-1} \alpha_i^j \,(Y_i^{(j)} - (Y_{ref})_i^{(j)}), \tag{49}$$

generically ensure that the trajectory error vector $\tilde{\zeta} = \zeta - \zeta_{\mathbf{ref}}$ exponentially converges to zero, provided that the arbitrary constants α_i^j are suitably chosen. Moreover, $\tilde{\mathbf{X}} = \mathbf{X} - \mathbf{X_{ref}}$ also asymptotically converges to zero.

Proof. The exponential convergence of $\tilde{\zeta}$ to zero is easy to prove, provided a suitable choice of the constants α_i^j. Moreover, Ψ being a diffeomorphism, this implies that $\mathbf{X_e}$ tends to $(\mathbf{X_e})_{ref}$ and therefore that $\mathbf{X}$ tends to $\mathbf{X_{ref}}$. □

We will now apply the principles of dynamic feedback linearization to solve the posture tracking problem for all restricted mobility robots. We know from Lemma 1 (see also Remark 13), that it is sufficient to consider the kinematic models.

5.2. *Solving the Posture Tracking Problem for Restricted Mobility Robots*

By applying the dynamic extension algorithm to each kinematic model of restricted mobility robot given in Table 1, we have obtained Table 4, where we give in each case the linearizing output functions $h_i(X)$, the extended state vector $\mathbf{X_e}$, the diffeomorphism $\Psi(X_e)$, and the regularity conditions ensuring that the decoupling matrix of the extended system is not singular.

Where, for type (1,1) robots,

$$\chi_2 = U_1(L \sin\beta + e\cos\beta) + \eta_1\zeta_1(L\cos\beta - e\sin\beta)$$

with $\dot{\eta}_1 = U_1$.

We have already seen that for type (2,0) robots, the linearizing output functions are the coordinates of a point P' located on the axle of the fixed wheels, and the posture tracking problem is solvable as long as the longitudinal velocity χ_1 of P' is not zero.

For type (2,1) robots, the linearizing output functions are the coordinates of the center P' of the steering wheel completed by the orientation θ, and the posture tracking problem is solvable as long as the longitudinal velocity η_1 of P' is not zero.

For type (1,1) robots, the linearizing output functions are the Cartesian coordinates of a point P' on the axle of the fixed wheels, and the posture tracking problem is solvable as long as the longitudinal velocity η_1 of P' is not zero.

For type (1,2) robots, the linearizing output functions are the Cartesian coordinates given by (9) of a point P' of the frame completed by the orientation θ, and the posture tracking problem is solvable as long as η_1 and $\sin\beta_1 \sin\beta_2$ are not zero.

Therefore, using the notations of Section 5.1 and from Table 4, we can see that $\boldsymbol{\xi} = (x, y, \theta)^T$ is a part of $\mathbf{X_e}$. It is then a straightforward application of Proposition 8 to show the following proposition.

Table 4.

Type (δ_m, δ_s)	Linearizing Outputs $Y_i = h_i(X)$, $i = 1, \ldots, m,$	Extended Vector X_e	Diffeomorphism $\zeta = \Psi(X_e)$	Regularity Conditions $\det K(X_e) \neq 0$
(2,0)	$\begin{pmatrix} h_1 = x + e\cos\theta \\ h_2 = y + e\sin\theta \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ \theta \\ \chi_1 = \eta_1 + e\eta_2 \end{pmatrix}$	$\begin{pmatrix} h_1 \\ h_2 \\ \dot{h}_1 \\ \dot{h}_2 \end{pmatrix}$	$\chi_1 \neq 0$
(2,1)	$\begin{pmatrix} h_1 = x \\ h_2 = y \\ h_3 = \theta \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ \theta \\ \beta \\ \eta_1 \\ \eta_2 \end{pmatrix}$	$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{pmatrix}$	$\eta_1 \neq 0$
(1,1)	$\begin{pmatrix} h_1 = x + e\cos\theta \\ h_2 = y + e\sin\theta \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ \theta \\ \beta \\ \eta_1 \\ x_2 \end{pmatrix}$	$\begin{pmatrix} h_1 \\ h_2 \\ \dot{h}_1 \\ \dot{h}_2 \\ \ddot{h}_1 \\ \ddot{h}_2 \end{pmatrix}$	$\eta_1 \neq 0$
(1,2)	$\begin{pmatrix} h_1 = x + e\cos(\theta + \delta) \\ h_2 = y + e\sin(\theta + \delta) \\ h_3 = \theta \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ \theta \\ \beta_1 \\ \beta_2 \\ \eta_1 \end{pmatrix}$	$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{pmatrix}$	$L\eta_1 \sin\beta_1 \sin\beta_2 \neq 0$

PROPOSITION 9. Let $\boldsymbol{\xi}_{\mathbf{ref}}(\mathbf{t})$ be a smooth posture reference trajectory for a restricted mobility robot. There exists (at least generically) a suitable auxiliary control law w ensuring that the trajectory error vector $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - \boldsymbol{\xi}_{\mathbf{ref}}$ asymptotically converges to zero.

Remark 15. Not only $\tilde{\boldsymbol{\xi}}$, but all the components of $\mathbf{X_e} - (\mathbf{X_e})_{\mathbf{ref}}$ asymptotically converge to 0.

We have seen that the posture tracking problem can be solved for restricted mobility robots using dynamic state feedback linearization, but only "generically," since the rest point configurations are singular, as displayed in Table 4.

In the next section we discuss the problem of implementation of the dynamic feedback linearizing control.

6. Applicability of Dynamic Feedback Linearization and Simulation Results

The important drawback of the dynamic feedback linearization approach is the fact that it is not applicable around the singularities (i.e., around the rest positions). Our purpose in this section is to show how it can nevertheless be implemented in practical situations. More precisely, we give partial answers to the following two questions:

1. Given a reference trajectory and initial conditions, can we ensure a priori that the closed-loop trajectories avoid the singularities?
2. What do we do around a singularity, i.e., when the velocities become very low ? This discussion will be illustrated by the example of a type (2,0) robot.

The following lemma provides sufficient conditions on the reference trajectory and on the initial conditions that guarantee that the closed-loop trajectories are bounded away from the singularites.

LEMMA 2. Let system (30) be generically fully linearizable by dynamic state feedback and diffeomorphism:

$$\zeta = \Psi(X_e) \tag{50}$$

IF:

1. There exists an open simply connected domain D included in

$$S = \{\zeta \in \mathbb{R}^{n_e} \text{ such that } \mathbf{K}(\Psi^{-1}(\zeta)) \text{ is not singular}\}.$$

2. The reference trajectory $\zeta_{\mathbf{ref}}(\mathbf{t})$ belongs to D for all t, and moreover, there exists a positive continuous function $M_1(t)$ such that:

$$\| \tilde{\zeta}(\mathbf{t}) \| < M_1(t) \Rightarrow \zeta(t) \in D, \qquad (51)$$

where $\tilde{\zeta}(\mathbf{t}) = \zeta(\mathbf{t}) - \zeta_{\mathbf{ref}}(\mathbf{t})$.

3. The initial conditions satisfy:

$$\| \tilde{\zeta}(\mathbf{0}) \| \leq \min_t F(t) \text{ with } F(t) = \left(\frac{M_1(t) - \epsilon}{K_2(t)} \right) \qquad (52)$$

for some positive constant ϵ and where $K_2(t) = K_1 e^{-r_1 t} > 0$ characterizes the closed-loop exponentially stable dynamics (with $K_1 > 0$, $r_1 > 0$):

$$\| \tilde{\zeta}(\mathbf{t}) \| \leq K_2(t) \ \| \tilde{\zeta}(0) \|, \qquad \forall t. \qquad (53)$$

THEN:

$$\zeta(\mathbf{t}) \in D, \qquad \forall t \qquad (54)$$

Proof. To show (54) it is sufficient to show that:

$$\| \tilde{\zeta}(\mathbf{t}) \| < M_1(t), \qquad \forall t. \qquad (55)$$

By contradiction, let us suppose that there exists a positive time T such that:

$$\begin{aligned} \| \tilde{\zeta}(\mathbf{T}) \| &= M_1(T), \\ \| \tilde{\zeta}(\mathbf{t}) \| &< M_1(t), \qquad \forall t \in [0, T[. \end{aligned} \qquad (56)$$

On $[0, T[$ we have, from (53) and (52),

$$\| \tilde{\zeta}(\mathbf{t}) \| \leq M_1(t) - \epsilon \qquad (57)$$

by continuity of $\tilde{\zeta}(\mathbf{t})$ and $M_1(t)$, equation (57) is in contradiction with (56), and this concludes the proof. □

Remark 16. We can easily show that the following choice of $M_1(t)$ satisfies (51)

$$M_1(t) = \min_\zeta \| \tilde{\zeta}(\mathbf{t}) \| \qquad \text{when } det(K(\Psi^{-1}(\zeta)) = 0 \qquad (58)$$

The following lemma will now characterize the singularities and the function $M_1(t)$ of Lemma 2 for type (2,0) robots. As suggested by Proposition 7 (see also Table 4), we select as linearizing output functions the Cartesian coordinates of a point P' on the axle of the fixed wheels (given by equation (36)). Let us also recall from Table 4, that $\mathbf{X_e} = (\boldsymbol{\xi}, \chi_1)^T$ for these robots. Lemma 2 then specializes as follows:

Lemma 3. For type (2,0) robots we have:

$$M_1(t) = | (\chi_1)_{ref}(t) |, \qquad (59)$$

$(\chi_1)_{ref}(t)$ being the reference trajectory of the longitudinal velocity of P' given by (36). Consequently, if condition (52) is satisfied with (59), then the robot will follow the reference trajectory without singularity.

Proof. From (58) and (38) we easily obtain:

$$M_1(t) = \min | \tilde{h}_1^2 + \tilde{h}_2^2 + \dot{\tilde{h}}_1^2 + \dot{\tilde{h}}_2^2 |^{\frac{1}{2}} \quad \text{when } \chi_1 = 0;$$

i.e.,

$$M_1(t) = | (\chi_1)_{ref}(t) | .$$

Moreover, the diffeomorphism Ψ given by (50) is singular when $\chi_1 = 0$. Let us introduce:

$$S^+ = \left\{ \mathbf{X_e} \in \mathbb{R} \times \mathbb{R} \times \frac{\mathbb{R}}{2\pi\mathbb{Z}} \times \mathbb{R}^{*+} \right\},$$

and similarly,

$$S^- = \left\{ \mathbf{X_e} \in \mathbb{R} \times \mathbb{R} \times \frac{\mathbb{R}}{2\pi\mathbb{Z}} \times \mathbb{R}^{*-} \right\}.$$

Ψ can be inverted on $\Psi(S^+)$ and $\Psi(S^-)$, respectively, as follows:

$$\mathbf{X_e} = \Psi^{-1}(\zeta) = \begin{pmatrix} h_1 - e\cos(-\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + f(\dot{h}_2)\pi) \\ h_2 - e\sin(-\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + f(\dot{h}_2)\pi) \\ -\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + f(\dot{h}_2)\pi \\ \sqrt{\dot{h}_1^2 + \dot{h}_2^2} \end{pmatrix}$$

on $\Psi(S^+)$, and

$$\mathbf{Xe} = \Psi^{-1}(\zeta) = \begin{pmatrix} h_1 - e\cos(-\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + (1 - f(\dot{h}_2))\pi) \\ h_2 - e\sin(-\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + (1 - f(\dot{h}_2))\pi) \\ -\arctan(\frac{\dot{h}_1}{\dot{h}_2}) + (1 - f(\dot{h}_2))\pi \\ -\sqrt{\dot{h}_1^2 + \dot{h}_2^2} \end{pmatrix}$$

on $\Psi(S^-)$, where:

$$f(a) = \begin{cases} 0 & \text{if } a \geq 0 \\ 1 & \text{if } a < 0 \end{cases} \qquad (60)$$

and

$$\arctan\left(\frac{a}{b}\right) = sign(a)\frac{\pi}{2} \qquad \text{when } b = 0. \qquad (61)$$

Therefore, Ψ^{-1} is well defined as soon as the sign of $\chi_1(0)$ is defined. The conclusion then holds by applying Lemma 2. □

Remark 17. For a reference trajectory to be not singular, $(\chi_1)_{ref}(t)$ must always have the same sign, and the same holds for the actual closed-loop trajectory. Consequently, $\chi_1(0)$ and $(\chi_1)_{ref}(0)$ must have the same sign. Provided condition (52) is satisfied, if $(\chi_1)_{ref}(0) > 0$ (resp. $(\chi_1)_{ref}(0) < 0$), the robot will follow the reference trajectory with the free wheel backward (resp. forward).

We want to apply the results about the posture tracking problem to a type (2,0) robot with two fixed wheels and only one free wheel (see Fig. 3). We select as point P' the center of the axle of the fixed wheels. Therefore, the linearizing output functions are simply given by $h_1 = x$ and $h_2 = y$, corresponding to $e = 0$ in (36), which implies $\chi_1 = \eta_1$.

We give simulation results for this type (2,0) robot corresponding to the case of a nonsingular trajectory (Section 6.1) and to the case where the reference trajectory stops at the origin (Section 6.2). In this case, the dynamic feedback law becomes singular, of course, but we show that it is possible to switch to a stabilizing time-varying feedback law near the origin.

6.1. Posture Tracking of a Nonsingular Reference Trajectory for a Type (2,0) Robot

We consider the posture tracking problem of the following moving reference trajectory (see Fig. 8), which is not singular (i.e., such that $(\chi_1)_{ref}(t) = (\eta_1)_{ref}(t) \neq 0$ for all $t \geq 0$):

$$x_{ref}(t) = y_{ref}(t) = \frac{\sqrt{2}.10^{-2}}{3}(10-2t)^3 - \frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{6} \quad (62)$$

such that the point P' will follow a straight line, with initial conditions:

$$\begin{cases} x(0) &= x_{ref}(0) - 0.54 = 5.35\,\text{m} \\ y(0) &= y_{ref}(0) + 0.54 = 6.43\,\text{m} \\ \dot{x}(0) &= \dot{y}(0) = \dot{x}_{ref}(0) = \dot{y}_{ref}(0) = -3.53\,\text{m/s} \end{cases} \quad (63)$$

On the other hand, the second-order linear dynamics of the closed-loop errors in $(x(t) - x_{ref}(t))$ and $(y(t) - y_{ref}(t))$ have been chosen with two real eigenvalues equal to -1. Therefore, by Lyapunov arguments it is easy to show that a possible $K_2(t)$ function in Lemma 2 is the following:

$$K_2(t) = 2\sqrt{2}\left(1 + \frac{1}{\sqrt{2}}\right) exp\left(-\left(1 - \frac{1}{\sqrt{2}}\right)t\right). \quad (64)$$

Moreover, from (59) we obtain:

$$M_1(t) = 0.04(10-2t)^2 + 1.0. \quad (65)$$

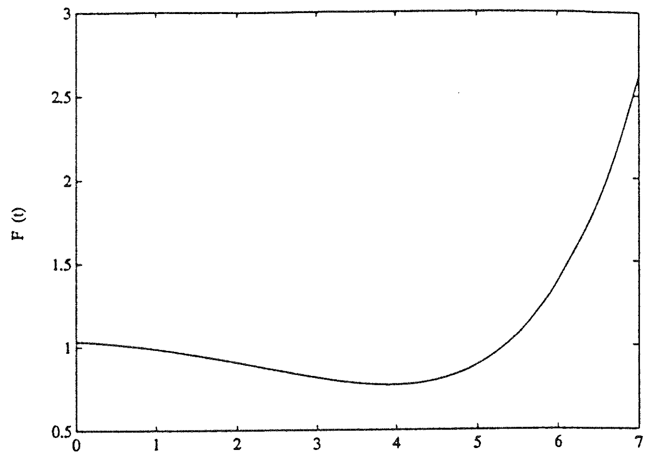

Fig. 7. The function $F(t)$.

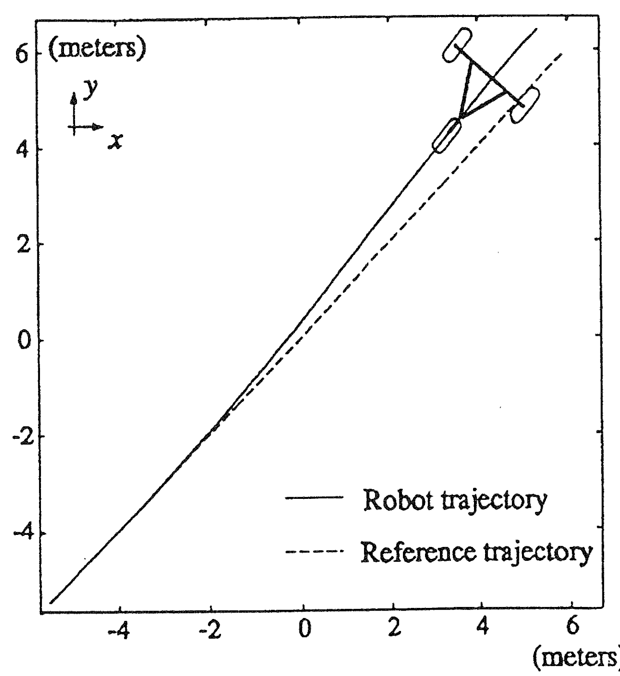

Fig. 8. The reference trajectory and the robot trajectory.

Consequently, with $\epsilon = 10^{-2}$, it is easy to obtain the function $F(t)$ introduced in Lemma 2 (see Fig. 7) and to check that with our choice (63), condition (52) is satisfied, and no singularity will be encountered. The simulation results are displayed in Figures 8 and 9.

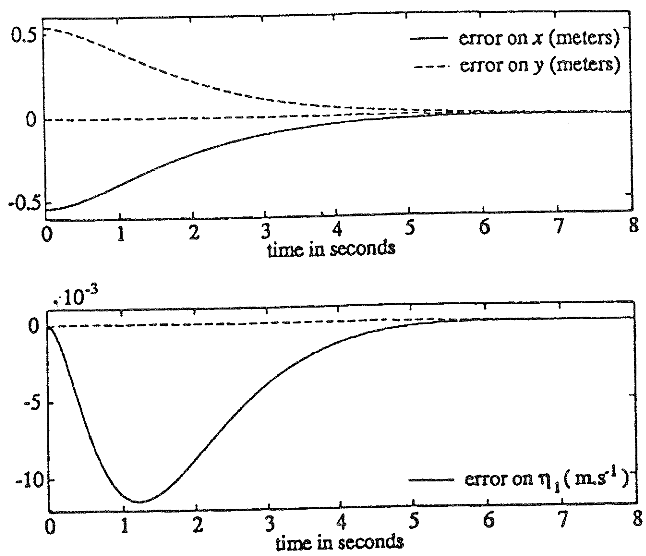

Fig. 9. *The errors in x, y, and η_1.*

6.2. Posture Tracking of a Singular Reference Trajectory for a Type (2,0) Robot

As already mentioned in the introduction, by application of the necessary condition of Brockett (1983), we know that restricted mobility robots are not stabilizable at equilibrium points by at least continuous state feedback laws. To deal with this stabilization problem at rest points, many authors have developed time-periodic laws, proposed, for example, in Pomet (1992), Pomet and Samson (1993), Pomet et al. (1992), M'Closey and Murray (1993), Coron (1992), Coron and d'Andréa-Novel (1992), Samson and Ait-Abderrahim (1990), Samson (1992), and Thuilot et al. (1994).

We now consider a singular reference trajectory that stops at the origin. As expected, the dynamic feedback law tends to infinity when we approach the origin. Nevertheless, to achieve this final step and to avoid singularities, we can use hybrid strategy and switch to a stabilizing time-varying feedback law when the trajectory enters a sufficiently small disk centered at the origin. This kind of hybrid strategy has already been used, for example, in Micaelli et al. (1992), Pomet et al. (1992), and Thuilot et al. (1994).

The moving reference trajectory that stops at the origin and is now considered is given by:

$$x_{ref}(t) = y_{ref}(t) = 10^{-2}\,\frac{\sqrt{2}}{6}(10-t)^3, \qquad (66)$$

with the same eigenvalues as before for the closed-loop behavior of $(x(t) - x_{ref}(t))$ and $(y(t) - y_{ref}(t))$, so that $K_2(t)$ is unchanged and the following initial conditions

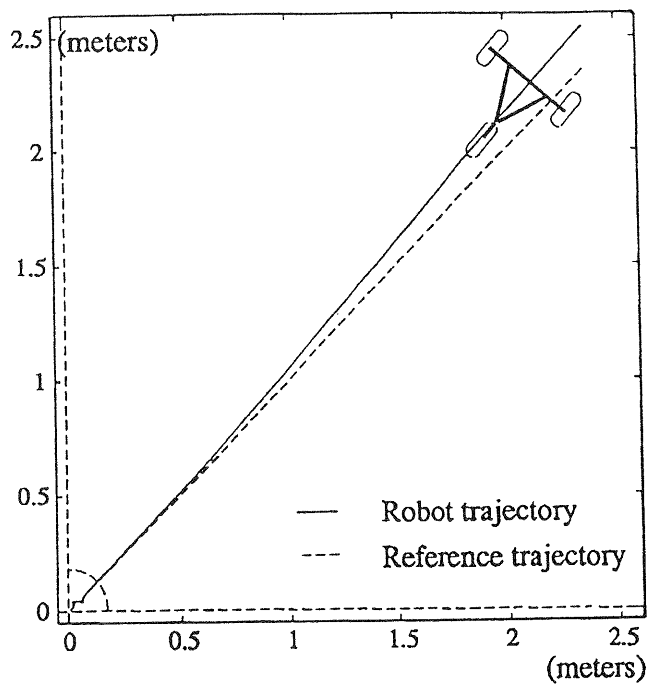

Fig. 10. *Reference and robot Cartesian trajectory.*

hold:

$$\begin{cases} x(0) = 10\,\dfrac{\sqrt{2}}{6}\,\text{m}, \\ y(0) = 2.54\,\text{m}, \\ \dot{x}(0) = \dot{y}(0) = -0.5\,\text{m/s}. \end{cases}$$

The simulation experiment is displayed in Figures 10 and 11. When

$$x^2 + y^2 = \frac{\sqrt{2}}{10},$$

η_1 is sufficiently small, and we have to change our control strategy to obtain in a final step the stabilization of the robot at the origin. We can see on Figure 11 that η_1 is not zero before the commutation time. To stabilize the origin in the final step, we have used the periodic-time control law CL3 proposed in Pomet et al. (1992) (with $\lambda = 40$ and $\psi(\theta) = \theta$).

Finally, we have elaborated global control strategy by using hybrid feedback laws combining dynamic linearizing ones to track arbitrary nonsingular reference trajectories and time-varying ones to achieve final stabilization. This allows us to solve the posture tracking problem for any restricted mobility wheeled mobile robot, in the practical case of a nonsingular reference trajectory ended by a rest position. Moreover, since the closed-loop system is linearized during the first step, the trajectory error exponentially converges to zero. We could have considered such a continuous periodic-time control law proposed in M'Closkey and Murray (1993) and Pomet

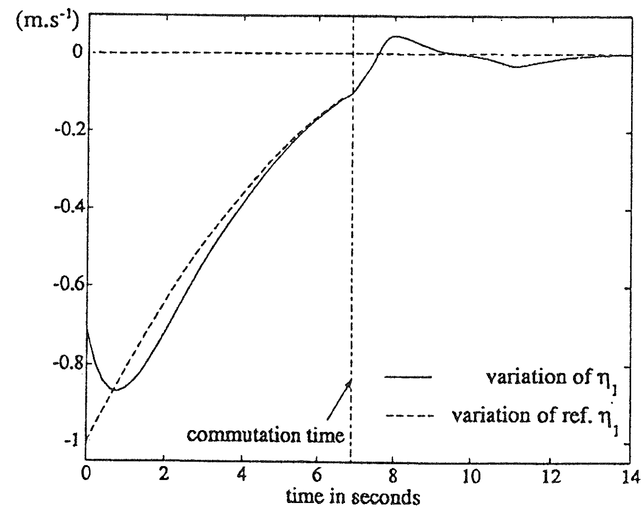

Fig. 11. The error in η_1.

and Samson (1993) to also ensure exponential convergence during the final step.

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