Stabilizability and Dead-Beat Controllers for Two Classes of Wiener–Hammerstein Models

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***Abstract*— Two classes of block oriented models of the Wiener–Hammerstein type are considered. We prove that a generic condition is sufficient for a null controllable discrete-time system of this form to have a stabilizing minimum-time dead-beat controller. When the condition is violated, we show how to design a nonminimum time stabilizing (dynamic) dead-beat controller. The result is used to obtain stabilizability conditions for these systems.**

***Index Terms*—Dead-beat, discrete-time, stabilizing, nonlinear, Wiener–Hammerstein systems.**

I. Introduction

Because of its simplicity, the linear time-optimal controller is an easy-to-design option for the control designer [12]. It is well known that a minimum-time dead-beat controller that stabilizes a completely controllable linear discrete-time system can always be designed when the system is null controllable—just place all the poles of the closed-loop system inside the unit disc, that is, at the origin. However, in the nonlinear context, all minimum-time dead-beat controllers may render the origin of the closed loop system attractive but not stable, violating in this way the most basic requirement for their implementation.

Stability analysis of minimum-time dead-beat controllers for general discrete-time nonlinear systems leads to computationally intractable problems even for "mild" nonlinearities and low order systems. Therefore, it appears to be necessary to consider simpler classes of discrete-time nonlinear systems in order to carry out stability analysis successfully. Such an important class of models, which are very often used in black-box identification of nonlinear systems, are of the Wiener–Hammerstein type [3]. Some applications of these models can be found in [1] and [4]. The basic building blocks for these systems are parallel and series connections of linear dynamical blocks and static nonlinearities, which are often of the form $N(\cdot) = (\cdot)^q,\ q \in \mathbb{N}$.

The result on controllability of linear systems with positive controls in [2] was recently used to prove null controllability conditions for several classes of Wiener–Hammerstein systems in [5]–[8]. Some related results on output controllability of a class of polynomial systems can be found in [10]. The design of minimum-time controllers for general polynomial systems was presented in [9] and [11]. In this correspondence, we address for the first time the important issue of the existence of stabilizing minimum-time dead-beat controllers for two classes of Wiener–Hammerstein systems. For the considered systems, we show that if the system is null controllable (this is always assumed) and a generic condition is satisfied, then there exists a minimum-time dead-beat controller which is globally stabilizing. If the condition is violated, we show how it is possible to design a stabilizing dynamic nonminimum-time dead-beat controller. The results are, to the best of our knowledge, the first of this kind for a class of nonlinear systems. Their importance is reflected in the fact that we use it in the second part of the correspondence to prove necessary and sufficient conditions for stabilizability of these systems. The stabilizability conditions are the same as in the linear case: all uncontrollable modes should be stable.

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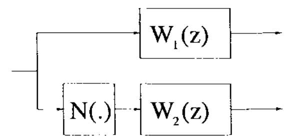

Fig. 1. Generalized Hammerstein model.

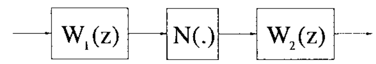

Fig. 2. Simple Wiener–Hammerstein model.

II. Preliminaries

Sets of real, natural, and complex numbers are, respectively, denoted as $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$. We consider SISO generalized Hammerstein discrete-time systems of the form (Fig. 1):

$$\begin{aligned} \Sigma_1:\quad & x_1(k+1) = Ax_1(k) + bu(k) \\ \Sigma_2:\quad & x_2(k+1) = Fx_2(k) + g(u(k))^q \\ & y(k) = cx_1(k) + hx_2(k) \end{aligned} \tag{1}$$

or SISO simple Wiener–Hammerstein discrete-time systems (Fig. 2):

$$\begin{aligned} \Sigma_1:\quad & x_1(k+1) = Ax_1(k) + bu(k) \\ \Sigma_2:\quad & x_2(k+1) = Fx_2(k) + g(cx_1(k))^q \\ & y(k) = hx_2(k) \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$, $n_1 + n_2 = n$, $u, y \in \mathbb{R}$, $q \in \mathbb{N}$, $q > 1$, and matrices A, F, b, g, c, h are of appropriate dimensions.

The systems (1) and (2) consist, respectively, of a parallel and series connection of two linear dynamical blocks

$$\begin{aligned} W_1(z) &= \frac{y_1(z)}{u_1(z)} = c(zI - A)^{-1}b = \frac{b_1(z)}{a_1(z)} \\ W_2(z) &= \frac{y_2(z)}{u_2(z)} = h(zI - F)^{-1}g = \frac{b_2(z)}{a_2(z)} \end{aligned} \tag{3}$$

interconnected via the static monomial nonlinearity $(\cdot)^q$. To simplify the exposition of the results, we assume without loss of generality that $W_i(z)$, $i = 1, 2$ are strictly proper rational transfer functions.

We denote a sequence of controls $\{u(0), u(1), \cdots\}$ as U, where $u(i) \in \mathbb{R}$ and its truncation of length N, that is, $\{u(0), \cdots, u(N-1)\}$, as U_N. The state of the system (1) or (2) at time step N, which is obtained when a sequence U_N is applied to the system and which emanates from the initial state $x(0)$, is denoted as $x(N, x(0), U_N)$. We give below the definitions that are used in the sequel.

Definition 1: The system is null (completely) controllable if, for any initial state $x(0) \in \mathbb{R}^n$ (any states $x^*, x(0) \in \mathbb{R}^n$), there exists an integer $N \in \mathbb{N}$ and a control sequence U_N such that $x(N, x(0), U_N) = 0$ (such that $x(N, x(0), U_N) = x^*$).

The characteristic polynomial of a matrix F is denoted as $P_F(\lambda) = \det(\lambda I - F)$. Given a polynomial $P(\lambda) = \lambda^t + a_{t-1}\lambda^{t-1} + \cdots + a_1\lambda + a_0$, we introduce a new polynomial $P^{[q]}(\lambda)$ which is obtained from $P(\lambda)$ when all the coefficients a_i are taken with a power $q \in \mathbb{N}$, that is, we write $P^{[q]}(\lambda) = \lambda^t + a_{t-1}^q\lambda^{t-1} + \cdots + a_1^q\lambda + a_0^q$. Also, if we are given a polynomial $H = \lambda^s + h_{s-1}\lambda^{s-1} + \cdots + h_1\lambda + h_0$, we use the notation:

$$(P \cdot H)^{[q]}(\lambda) = \lambda^{t+s} + (h_{s-1} + a_{t-1})^q\lambda^{t+s-1} + \cdots + (h_1 a_0 + h_0 a_1)^q\lambda + (a_0 h_0)^q.$$

Hence, the polynomial $(P \cdot H)^{[q]}(\lambda)$ is obtained when we first multiply the polynomials P and H and then take qth powers of all the coefficients of the product polynomial. Notice that $(P \cdot H)^{[q]}(\lambda) = (H \cdot P)^{[q]}(\lambda)$.

Minimum-time dead-beat controllers for polynomial systems can be designed by using a procedure based on the QEPCAD symbolic computation package [9], [11]. For a system $x(k+1) = f(x(k), u(k))$, QEPCAD is used to compute the sets:

$$S_0 = \{x : \exists u \in \mathbb{R},\ f(x,u) = 0\}$$
$$S_k = \{x : \exists u \in \mathbb{R},\ f(x,u) \in S_{k-1}\}.$$

The set S_k is a set of states $x \in \mathbb{R}^n$ for which the minimum time necessary to transfer x to the origin is at most $k+1$. The following sets are also important:

$$\hat{S}_0 = S_0, \quad \hat{S}_k = S_k - S_{k-1}, \qquad \forall k = 1, 2, \cdots, N \tag{4}$$

since they represent the sets of states for which the minimum time to transfer them to the origin is equal to $k+1$. It was shown in [5], [6], and [8] that if a system (1) or (2) is null controllable, then there exists a uniform bound on the dead-beat time. In other words, there exists a number N such that the sets (4) satisfy $\cup_{i=0}^{N}\hat{S}_i = \mathbb{R}^n$. This fact is exploited in the sequel.

Once we have computed the sets S_k, the design of a minimum time dead-beat feedback controller follows easily. Indeed, we know that $\forall x \in \hat{S}_0$ there exists (in general, nonunique) $u_0(x)$ such that $f(x, u_0(x)) = 0$. Moreover, $\forall x \in \hat{S}_k,\ k \geq 1$, there exists $u_k(x)$ such that $f(x, u_k(x)) \in S_{k-1}$. This defines a static state feedback control law which is expressed as follows (see Example 1 and [9]–[11]):

$$u(x) = \begin{cases} u_0(x), & \text{if } x \in \hat{S}_0 \\ u_1(x), & \text{if } x \in \hat{S}_1 \\ \cdots & \cdots \\ u_N(x), & \text{if } x \in \hat{S}_N \end{cases}$$

which we call a "minimum-time dead-beat controller." Note that given any $x_0 \in \mathbb{R}^n$, the minimum-time controller transfers it to the origin in minimum-time. This controller is called stabilizing if the origin of the closed-loop system is asymptotically stable in the Lyapunov sense.

III. Stabilizing Properties of Dead-Beat Controllers

In this section we present and prove the main results of the correspondence. In Theorem 1 we give a sufficient condition for the existence of a stabilizing minimum-time dead-beat controller for systems (1) and (2). The condition is generic for null controllable systems[1] (1) and (2). Then, in Theorem 2, we show that if the condition is violated, but the system (1) or (2) is null controllable, we can still design a dynamic state feedback dead-beat controller which is stabilizing. With this result we prove necessary and sufficient conditions for stabilizability of systems (1) and (2) (Theorem 3).

[1] Null controllability tests for Wiener–Hammerstein systems that we consider can be found in [6] and [8].

Theorem 1: There exists a stabilizing minimum-time dead-beat controller for the generalized Hammerstein system (1) [the simple Wiener–Hammerstein system (2)] if the system is null controllable and polynomials $b_1^{[q]}(z)$ and $a_2(z)$ are coprime (polynomials $a_1^{[q]}(z)$ and $a_2(z)$ are coprime).

Before proving the main results, we need the following.

Proposition 1: Consider the equation:

$$u^q + t_1 u^{q-1} + \cdots + t_{q-1}u + t_q = 0 \tag{5}$$

where $t_i = t_i(x),\ i = 1, \cdots, q$, with $t_i(0) = 0$, are functions in $x \in \mathbb{R}^n$. Suppose that the following holds: $\forall E > 0,\ \exists \delta_u > 0$ such that if $\|x\| < \delta_u$ then $|t_i| < E,\ \forall i = 0, 1, \cdots, q-1$. Denote the set of roots u_i to (5) as Λ. Then it holds that $\forall \epsilon_u > 0,\ \exists \delta_u > 0$ such that if $\|x\| < \delta_u$ then $|u_i| < \epsilon_u,\ \forall u_i \in \Lambda$.

Proposition 1 can be interpreted in the following way: if we can make coefficients $t_i = t_i(x)$ in (5) arbitrarily small by choosing x small enough, then all the roots $u_i \in \Lambda$ to (5) can be made arbitrarily small. Due to space constraints, we omit the proof of Proposition 1. We introduce the "small control property" (see [13]).

Definition 2: A control law $u = u(x)$ is said to have the small control property (SCP) if $\forall \epsilon_u > 0,\ \exists \delta_u > 0$ such that if $\|x\| < \delta_u$ then $|u(x)| < \epsilon_u$.

The following proposition shows that if the conditions of Theorem 1 are satisfied, then there exists a minimum-time dead-beat controller, respectively, for system (1) or (2), $u = u(x)$, which has SCP. The statement and proof are given only for the case of simple Wiener–Hammerstein systems. The proof for generalized Hammerstein systems follows the same arguments.

Proposition 2: Consider a null controllable system (2). If $b_1^{[q]}(z)$ and $a_2(z)$ are coprime, then there exists a minimum-time dead-beat controller $u = u(x)$, which has SCP.

Proof of Proposition 2: We assume without loss of generality that the matrices $A,\ F$ are nonsingular (see, for instance, [5]) and (A, b, c) and (F, g, h) are in controllability canonical form. In order to simplify the considerations, we introduce the nonsingular feedback transformation

$$u(k) = \bar{K}x_1(k) + v(k) \tag{6}$$

where $\bar{K}x_1(k)$ is the (unique) minimum-time dead-beat controller for the linear subsystem Σ_1 in (2). The state equations for the system become

$$\begin{aligned} x_1(k+1) &= Jx_1(k) + bv(k) \\ x_2(k+1) &= Fx_2(k) + g(cx_1(k))^q \end{aligned} \tag{7}$$

where $J = A + b\bar{K}$ has elements equal to 1 on the first superdiagonal and 0 everywhere else.

From the feedback transformation (6) we see that a minimum-time dead-beat controller has SCP if and only if $v = v(x)$ has SCP. We investigate now which values of $v(x)$ should be applied on the sets $\hat{S}_k$ to have SCP. We denote as $v_k(x)$ or v_k control actions that need to be applied on the set $\hat{S}_k$ [$v_k(x)$ or, equivalently, v_k are not the same as $v(k)$ in (7)].

A minimum-time dead-beat control law v_0 on the set $\hat{S}_0$ must satisfy the equations

$$0 = Jx_1 + bv_0$$
$$0 = Fx_2 + g(cx_1)^q.$$

It is easily seen that necessarily we have $v_0 = 0$ in order to zero the last equation of the first subsystem. Hence, any minimum-time dead-beat controller has SCP on the set $\hat{S}_0$. Consider now the set $\hat{S}_1$. Since the controller should drive any state from the set $\hat{S}_1$ to the origin in two steps, we have

$$\begin{aligned} 0 &= J^2 x_1 + Jbv_1 + bv_0 \\ 0 &= F^2 x_2 + Fg(cJx_1 + cbv_1)^q + g(cx_1)^q. \end{aligned} \tag{8}$$

If $n_1 \geq 2$, then necessarily we have that $v_1 = 0$, since the second-to-last equation of the first subsystem should be equal to zero. In the same way, we obtain that $v_{i-1}(x) = 0, \forall x \in \hat{S}_{i-1}, i = 1, \cdots, n_1$.

Consider now the control law $v_{n_1}(x)$ on the set $\hat{S}_{n_1}$. We have the formula

$$\begin{aligned} 0 &= \underbrace{J^{n_1+1}}_{=0} x_1 + \sum_{i=0}^{n_1} J^i b v_i \\ 0 &= F^{n_1+1} x_2 + \sum_{i=0}^{n_1} F^i g(cx_1(i))^q \end{aligned}$$

where $cx_1(i) = cJ^i x_1 + \sum_{i=0}^{i-1} cJ^i bv_i$. The first set of equations is identically equal to zero because $v_i = 0, \forall i = 1, \ldots, n_1 - 1$ and $J^{n_1} = 0$. Notice that $cJ^i b$ are coefficients of the polynomial $b_1(z)$ and the second set of equations can be rewritten as follows:

$$0 = F^{n_1+1} x_2 + b_1^{[q]}(F) g v_{n_1}^q + \text{lower order terms of } v_{n_1}.$$

If $b_1^{[q]}(z)$ and $a_2(z)$, which is the characteristic polynomial of F, are coprime, we have that the matrix $b_1^{[q]}(F)$ is nonsingular and hence $b_1^{[q]}(F)g$ is full column rank. As a result, we have that v_{n_1} should satisfy (at least) one equation of the form

$$v_{n_1}^q + \sum_{i=0}^{q-1} t_i v_{n_1}^i = 0$$

and coefficients $t_i = t_i(x, v_0, \cdots, v_{n_1-1})$ are easily seen from the construction to be polynomials in $v_i, i = 0, 1, \cdots, n_1 - 1$ and x, with $t_i(0, 0, \cdots, 0) = 0$. SCP of v_{n_1} on $\hat{S}_{n_1}$ follows from Proposition 1.

The proof follows by induction. We have checked by direct computations that $v_k(x), k = 0, 1, \cdots, n_1$ have SCP. Suppose that $v_i(x), i = 0, 1, \cdots, k-1, k-1 \geq n_1$ have SCP on sets $\hat{S}_i$ and conditions of Theorem 1 hold. For control $v_k(x), k \geq n_1 + 1$ on the set $\hat{S}_k$ we obtain the equations

$$\begin{aligned} 0 = b_1^{[q]}(F)[F^{k-n_1-1}g \quad &: \quad F^{k-n_1-2}g \quad : \quad \cdots \quad : \quad g] \\ &\times \begin{bmatrix} v_k^q & v_{k-1}^q & \cdots & v_{n_1}^q \end{bmatrix}^T + \text{other terms} \end{aligned}$$

where "other terms" are polynomials in lower powers of $v_i, i = n_1, \cdots, k$ and some powers of entries of x. We have that SCP holds for all $v_j(x), j = 0, 1, \cdots, k-1$. Also, since $b_1^{[q]}(F)$ is nonsingular (since $b_1^{[q]}(z)$ and $a_2(z)$ are coprime), F is nonsingular and (F, g) is a controllable pair, the rank of matrix $b_1^{[q]}(F)[F^{k-n_1-1}g \; : \; F^{k-n_1-2}g \; : \; \cdots \; : \; g]$ is full and, moreover, $b_1^{[q]}(F)F^i g \neq 0$, $\forall i \geq 0$. Hence, there exists at least one equation of the form

$$v_k^q + \sum_{i=0}^{q-1} t_i v_k^i = 0$$

which v_k must satisfy. As before, $t_i = t_i(x, v_0, \cdots, v_{k-1})$, $t_i(0, 0, \cdots, 0) = 0, \forall i = 0, 1, \cdots, q$. From Proposition 1, it follows that there exists $v_k(x)$ which also has SCP. Q.E.D.

Proof of Theorem 1: Suppose that the conditions of Theorem 1 are satisfied. Also, without loss of generality we suppose that the polynomials $a_1(z)$ and $a_2(z)$ have no zero roots.[2] We apply a minimum-time dead-beat controller which has SCP, $u = u(x)$, and the closed-loop system becomes

$$\begin{aligned} x_1(k+1) &= Ax_1(k) + bu(x(k)) \\ x_2(k+1) &= Fx_2(k) + g(cx_1(k))^q. \end{aligned} \tag{9}$$

We denote in the sequel the state of the closed-loop system (9) at time step k emanating from the initial state $x(0) \in \mathbb{R}^n$ as $x(k, x(0))$.

Notice that the origin of the closed-loop system (9) is globally attractive in finite time. Hence, there exists $N \in \mathbb{N}$ such that

$$\forall x(0) \in \mathbb{R}^n, \; x(k, x(0)) = 0, \qquad \forall k \geq N. \tag{10}$$

This also implies that the origin is a unique equilibrium of the closed-loop system. Hence, we need to check only stability of the closed-loop system.

We consider a controller which has SCP. The SCP implies that $\forall \epsilon_k > 0, \exists \delta_k > 0, k = 0, 1, 2, \cdots, N$ such that $\|x(k)\| < \delta_k, x(k) \in \hat{S}_k$ implies that $\|x(k+1, x(k))\| < \epsilon_k, \forall k = 0, 1, 2, \cdots, N$. Take arbitrary $\epsilon_N > 0$ and let $\epsilon_{k-1} = \delta_k, k = 1, \cdots, N$. We obtain that $\forall \epsilon_N > 0, \exists \delta_0 > 0$ such that if $\|x(0)\| < \delta_0$, then $\|x(k, x(0))\| < \epsilon_N, \forall k = 0, \cdots, N$. Finally, using (10) and letting $\epsilon_N = \epsilon, \delta_0 = \delta$, we have that $\forall \epsilon > 0, \exists \delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(k, x(0))\| < \epsilon, \forall k = 0, 1, 2, \cdots$, which proves stability of (9) by definition. Q.E.D.

A similar proof can be carried out for generalized Hammerstein systems and is not given here. In Example 1 we show that violation of the coprimeness condition of Theorem 1 may result in all minimum-time dead-beat controllers being destabilizing. A natural question that arises is whether it is possible to recover stability if a nonminimum-time dead-beat controller is used. The following theorem says that indeed it is always possible to do so.

Theorem 2: There exists a (nonminimum-time) dynamic stabilizing dead-beat controller for a system (1) [respectively, system (2)] if and only if the the system (1) [system (2)] is null controllable.

In order to prove Theorem 2, we cite a technical lemma, first proved in [6].

Lemma 1: Consider polynomials $P_1(\lambda) = \sum_{i=0}^{n_1} b_i \lambda^i$, $P_2(\lambda) = \sum_{i=0}^{n_2} a_i \lambda^i$ where $a_i, b_i \in \mathbb{R}$, $a_1 = b_1 = 1$, $b_0 \neq 0$, $a_0 \neq 0$. Suppose that $P_1(\lambda)$ and $P_2^{[q]}(\lambda)$, $q \in \mathbb{N}$, $1 \leq q$ are not coprime. There exists a polynomial $H(\lambda)$ with real coefficients of the degree at most n_2 such that the polynomials $P_1(\lambda)$ and $(P_2 \cdot H)^{[q]}(\lambda)$ are coprime if and only if $q > 1$.

Proof of Theorem 2: Consider a null controllable simple Wiener–Hammerstein model (2). If $b_1^{[q]}(z)$ and $a_2(z)$ are not coprime, from Lemma 1 it follows that we can find a transfer block of the form $W^*(z) = H(z)/H_1(z)$ such that $(b_1 \cdot H)^{[q]}(z)$ and $a_2(z)$ are coprime and there are no pole-zero cancellations in the transfer function of the augmented first subsystem $W^*(z)W_1(z)$. Hence, we have that the new augmented system is null controllable and it satisfies the conditions of Theorem 1. Therefore, there exists a minimum-time dead-beat controller for the augmented system which is stabilizing. Q.E.D.

Theorem 2 shows that we can make a tradeoff between the performance (stability of the closed loop) and the dead-beat time in cases when minimum-time dead-beat controller is not stabilizing. In [6] it was shown that a polynomial of the form $H = \lambda^{n_2} + h$, $h \in \mathbb{R}$, where n_2 is the degree of $P_2(\lambda)$, can always be found to satisfy the conditions of Lemma 1. Hence, we need to augment a null controllable system (1) or (2) with $W^*(z)$ whose order does not have to be greater than the degree of $b_1(z)$ in order to obtain a stable closed loop with finite settling time.

We now show that the stabilizing dead-beat controllers can be used in a constructive proof of stabilizability for systems (1) and (2). Hence, the result on stabilizing properties of dead-beat controllers is

[2] If the matrixes A or F are singular, we can design a minimum-time dead-beat controller for the nonzero modes only, since the zero modes die out in finite time when applying zero control. It is not difficult to show that such a controller would be minimum-time dead-beat for the overall system.

used to close the gap between controllability and stabilizability for the Wiener–Hammerstein systems we consider.

Introduce the notation D for the open unit disc. We say that the systems (1) or (2) are *asymptotically controllable* if: when q is odd rank$[\lambda I - A: b] = n_1$, $\forall \lambda \in \mathbb{C} - D$, and rank$[\lambda I - F: g] = n_2$, $\forall, \lambda \in \mathbb{C} - D$; when q is even rank$[\lambda I - A: b] = n_1$, $\forall \lambda \in \mathbb{C} - D$, rank$[\lambda I - F: g] = n_2$, $\forall \lambda \in \mathbb{C} - D$, and F has no real positive eigenvalues $\lambda_i(F) \in \mathbb{R}$, $\lambda_i(F) \geq 1$. Note that the above conditions can be interpreted as "all the uncontrollable modes are stable" for systems (1) or (2) (for controllability conditions see [5]–[8]).

Theorem 3: The system (1) or (2) is stabilizable (by dynamic feedback) if and only if it is asymptotically controllable.

Proof of Theorem 3: We decompose the system into its unstable (controllable) nonzero modes, zero modes, and uncontrollable modes. We can find a coordinate transformation so that in the new coordinates the system becomes

$$\begin{aligned} \xi_1(k+1) &= A_1\xi_1(k) + b_1 u(k) \\ \xi_2(k+1) &= A_2\xi_2(k) + b_2 u(k) \\ \eta_1(k+1) &= F_1\eta_1(k) + g_1(\bar{c}\xi(k))^q \\ \eta_2(k+1) &= F_2\eta_2(k) + g_2(\bar{c}\xi(k))^q. \end{aligned}$$

A_1 and F_1 are nonsingular and contain all controllable modes. A_2 and F_2 are Shur matrices. By designing the dead-beat controller $u(x)$ (perhaps nonminimum-time) for the subsystem ξ_1, η_1, stability is proved in a straightforward manner.

Indeed, the dead-beat controller yields for the closed loop system $\xi_1(N) = 0$ and $\eta_1(N) = 0$ and also $u(x(k)) = 0$, $\forall k > N$. Also, we have that $\lim_{k\to\infty} \|\xi_2(k)\|^2 + \|\eta_2(k)\|^2 = 0$ because of stability of matrices A_2, F_2. Hence, the origin of the closed loop system is asymptotically attractive. Now we use SCP to show stability, but only for the time steps $k = 0, 1, \cdots, N-1$, and the proof follows. Q.E.D.

Example 1: Consider the generalized Hammerstein system that is null controllable $(b \leq 0)$:

$$\begin{aligned} x_1(k+1) &= -x_1(k) + u(k) \\ x_2(k+1) &= bx_2(k) + u^2(k). \end{aligned} \tag{11}$$

From Theorem 1, it follows that there exists a minimum time dead-beat controller which is stabilizing if $b \neq -1$. For instance, the family of minimum-time dead-beat controllers for $b = -2$ is

$$u(x) = \begin{cases} x_1, & x \in \hat{S}_0 \\ \dfrac{-2x_1 \pm \sqrt{8x_1^2 + 16x_2}}{-2}, & x \in \hat{S}_1 \\ \Delta(x), & x \in \hat{S}_2 \end{cases}$$

where $\hat{S}_0 = \{x\colon -2x_2 + x_1^2 = 0\}$; $\hat{S}_1 = \{x\colon x_1^2 + 2x_2 \geq 0\} - \hat{S}_0$; $\hat{S}_2 = \mathbb{R}^2 - \hat{S}_0 - \hat{S}_1$, and $\Delta(x)$ is (any) solution to the inequality $x_1^2 - 4x_2 - 2x_1\Delta + 3\Delta^2 \geq 0$, $x \in \hat{S}_2$. There is lots of freedom in choosing $\Delta(x)$. However, it is obvious that we can choose it so that it has SCP. Notice that $\hat{S}_2 \subset \{x : x_2 < 0\}$ and one choice for Δ which has SCP is $\Delta(x) = 0$. SCP holds also on sets $\hat{S}_0$ and $\hat{S}_1$, and we conclude that there exists a minimum-time dead-beat controller which renders the origin of the closed-loop system globally asymptotically stable (in the Lyapunov sense). For instance, the minimum-time dead-beat controller with $\Delta(x) \equiv 0$ is stabilizing.

Suppose now that $b = -1$. Then the family of minimum-time dead-beat controllers is

$$u(x) = \begin{cases} x_1, & x \in \hat{S}_0 \\ \dfrac{x_2 - x_1^2}{2x_1}, & x \in \hat{S}_1 \\ \Delta(x), & x \in \hat{S}_2 \end{cases}$$

where the sets are computed to be $\hat{S}_0 = \{x\colon -x_2 + x_1^2 = 0\}$; $\hat{S}_1 = \{x\colon x_1 \neq 0\} - \hat{S}_0$; $\hat{S}_2 = \mathbb{R}^2 - \hat{S}_0 - \hat{S}_1$. The function $\Delta(x)$ should satisfy $\Delta(x) \neq 0$, $x \neq 0$ and $\Delta(0) = 0$. Hence, we can choose the function so that it satisfies SCP. However, on the set $\hat{S}_1$ we have for any $x_2 \neq 0$ and $x_1 \to 0$ that $|u(x)| \to \infty$. We prove instability of the closed-loop system. Fix any $\epsilon^* > 0$. Consider any $\delta > 0$. By choosing $x_2(0) = \delta/2$ and letting $x_1(0) \to 0$, we have that $u(x) \approx \delta/4x_1(0)$ and hence for small enough $x_1(0)$ we have have found $x(0)$ such that $\|x(0)\| \leq \delta$ implies $\|x(1, x(0))\| > \epsilon^*$. Hence, the origin of the closed-loop system is unstable in the Lyapunov sense by definition.

IV. Conclusion

Two basic models arising in black-box identification of nonlinear systems were considered. We presented conditions for existence of minimum-time dead-beat controllers that are stabilizing. If the conditions are violated, we showed how it is possible to design a dynamic dead-beat controller, which is stabilizing but not time-optimal. The results are then used to state necessary and sufficient conditions for stabilizability of these models.

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