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ORIGINAL ARTICLE

Robust boundary control of systems of conservation laws

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Abstract The stability problem of a system of conservation laws perturbed by non-homogeneous terms is investigated. These non-homogeneous terms are assumed to have a small C^1-norm. By a Riemann coordinates approach a sufficient stability criterion is established in terms of the boundary conditions. This criterion can be interpreted as a robust stabilization condition by means of a boundary control, for systems of conservation laws subject to external disturbances. This stability result is then applied to the problem of the regulation of the water level and the flow rate in an open channel. The flow in the channel is described by the Saint-Venant equations perturbed by small non-homogeneous terms that account for the friction effects as well as external water supplies or withdrawals.

Keywords Conservation laws · Nonlinear PDE · Robust stability · Hydraulic applications

1 Introduction

Many distributed parameter physical systems are described by hyperbolic partial differential equations (PDE). The main property of this class of PDE is the existence of

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the so-called Riemann coordinates which are useful for the proof of classical solutions existence, as well as the analysis and the design of control methods among other properties, see e.g. [2,11]. In this paper, we investigate the problem of the stability of such hyperbolic equations in presence of small non-homogeneous terms.

The stability of homogeneous hyperbolic systems has been analyzed for a long time in the literature. A sufficient condition is that the Jacobian matrix of the boundary conditions has a spectral radius less than 1, see [8]. For non-homogeneous systems, an additional condition on the size of the non-homogeneous term is needed. In [11, Chap. 5, Theorem 1.3], it is assumed that the non-homogeneous terms are two times continuously differentiable and at least quadratic at the equilibrium. In this paper, a stronger result is proved since it is only assumed that the non-homogeneous terms can be linear, with a sufficiently small non-zero slope at the origin. This extension is definitely non-trivial. The proof is based on a repetitive use of the existence of the solutions over a finite time interval (see [2,11,12]) combined with an extension of the state evolution in the Riemann coordinates.

The main result of this paper is applied to systems of conservation laws for the design of a stabilizing boundary control of a channel, subject to small external perturbations. More precisely we address the problem of the regulation of the water level and the water flow rate in open channels by using the gate opening as control action.

The model used is not strictly hyperbolic since we consider also the case of the presence of friction and some external small supplies or removal of water along the canal. The model is written in terms of Saint-Venant equations introduced in [19] and commonly used in hydraulics to describe the flow of water in open-channels (see e.g. the textbooks [3] or [7]). Here the Saint-Venant equations are perturbed by small non-homogeneous terms that account for the friction effects as well as external water supplies or withdrawals.

This stability problem for the regulation of the flow in a channel has been considered for a long time in the literature as reported in the survey paper [18] which involves a comprehensive bibliography. For advanced control methods, see [6,17] where discrete linear approximations of the perturbed Saint-Venant equations are used. See also [14–16] where an H_∞ control design is developed. In [13] the perturbed Saint-Venant equations are linearized and an infinite dimensional controller is designed to suppress the oscillating modes over the canal.

This paper can be seen as a non-trivial generalization of [9], since the Saint-Venant equations that are considered in that reference do not involve neither friction nor water supply/removal along the canal. However assuming that there is neither friction nor external supply is a strong and unrealistic assumption. Here any system of conservation laws (of any given size) is considered, as well as the case of the presence of small disturbances. As it is explained in Sect. 3, the analysis developed here can be applied e.g. to fluid networks (taking into account a friction coefficient and fluid supply/removal along the canal), piper-line networks, or traffic road (and not only to the homogeneous Saint-Venant equations as in [9]). Observe that taking into account friction and supply/removal needs to redefine the equilibrium and to modify the boundary control actions. Finally note that the approach developed in this paper has been successfully tested by numerical simulations using the data of a real river, viz. the Sambre river in Belgium, and by physical experiments on a micro-channel in [5].

The paper is organized as follows. First in Sect. 2, the main result, namely a sufficient condition for the stability of conservation laws, is stated. It is written in terms of the boundary conditions and non-homogeneous terms with small C^1-norm are considered. In Sect. 3, the main result is applied to the boundary regulation of the water level and the water flow rate in open channels in presence of small friction and small external supply or removal of water distributed along its length. Section 4 contains a detailed proof of the main result. Section 5 contains some concluding remarks.

2 Stability of hyperbolic systems with non-homogeneous terms

We consider the class of hyperbolic PDE defined in Riemann coordinates as follows

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = h(\xi) \tag{1}$$

with $\xi : [0, L]\times[0, +\infty) \to \mathbb{R}^n : (x, t) \mapsto \xi(x, t)$, and $\Lambda(\xi) = \mathrm{diag}(\lambda_1(\xi), \dots, \lambda_n(\xi))$, under the following assumptions:

- The λ_i's and h are continuously differentiable functions on a neighborhood of the origin;
- There exists an integer m, with $0 < m < n$, such that

$$\lambda_i(0) < 0 < \lambda_j(0)\ ,\ \ \forall i,\ \ 1 \le i \le m,\ \ \forall j,\ \ m+1 \le j \le n; \tag{2}$$

- and

$$h(0) = 0. \tag{3}$$

Remark 1 The results of this paper will be presented in the case where $0 < m < n$. We should however stress that the results also hold in the particular cases where $m = 0$ or $m = n$ (i.e. all the eigenvalues λ_i have the same sign). We will come back on this in the conclusion.

As usual, for each component ξ_i of the PDE (1), one can define the characteristic curve solution of the differential equation

$$\dot{x}(t) = \lambda_i(\xi(x(t), t)).$$

Then, for each function $(x, t) \mapsto \xi_i(x, t)$, the left-hand side of (1) can be interpreted as the total time-derivative of ξ_i along the corresponding curve, which implies that the system (1) can be written in "differential" form as:

$$d_t \xi_i(x(t), t)) = h_i(\xi(x(t), t)), \tag{4}$$

This fact is illustrated in Fig. 1.

For all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, let $\xi_- = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ and $\xi_+ = (\xi_{m+1}, \dots, \xi_n) \in \mathbb{R}^{n-m}$. The functions $h_-(\xi)$, $h_+(\xi)$, $\Lambda_-(\xi)$, and $\Lambda_+(\xi)$ are defined in a similar way.

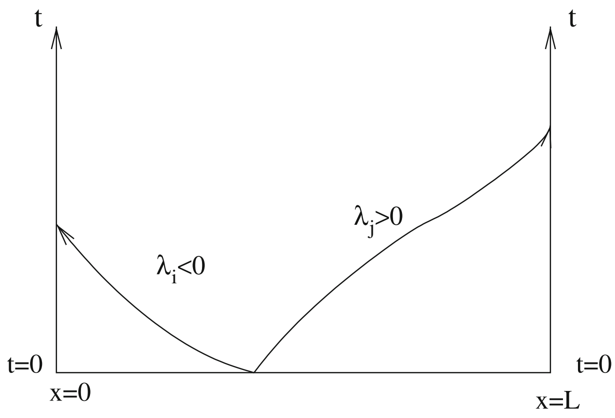

Fig. 1 Characteristic curves

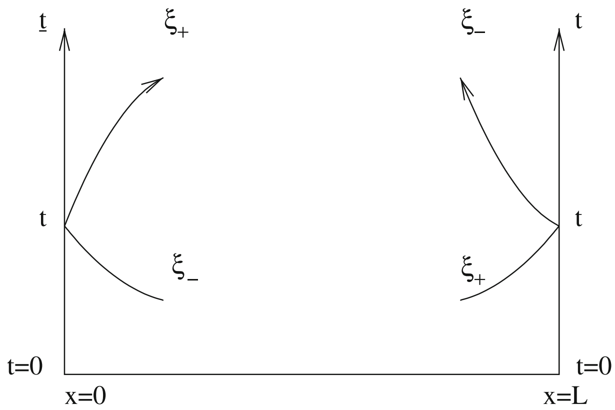

Fig. 2 Illustration of the boundary conditions

In order to complete the problem statement, boundary conditions (BC) are obviously needed. Here we consider the system (1) under the BC of the form

$$\begin{pmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{pmatrix} = g \begin{pmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{pmatrix}, \tag{5}$$

where g is a continuously differentiable function defined on a neighborhood of the origin, and satisfying $g(0) = 0$. Its Jacobian matrix at $\xi \in \mathbb{R}^n$ is denoted by $\nabla g(\xi)$.

This form of the BC is illustrated in Fig. 2 and can be interpreted as follows. The characteristic solution ξ_{+j} (or ξ_{-i}) that "leaves" the boundary at $x = 0$ (or at $x = L$) is a function of the characteristic solutions that "arrive" at the boundaries at the same instant. This form of the BC will be further motivated in the next section.

In order to state our main result, we need the following compatibility condition between the system (1) and the BC (5).

Definition 1 A function $\xi^\# \in C^1(0, L; \mathbb{R}^n)$ satisfies the compatibility condition $\mathcal{C}$ if

$$\begin{pmatrix} \xi_-^\#(L) \\ \xi_+^\#(0) \end{pmatrix} = g\begin{pmatrix} \xi_-^\#(0) \\ \xi_+^\#(L) \end{pmatrix},$$

and

$$\begin{pmatrix} \Lambda_-(\xi^\#(L))\partial_x\xi_-^\#(L) - h_-(\xi^\#(L)) \\ \Lambda_+(\xi^\#(0))\partial_x\xi_+^\#(0) - h_+(\xi^\#(0)) \end{pmatrix} = \nabla g\begin{pmatrix} \xi_-^\#(0) \\ \xi_+^\#(L) \end{pmatrix}\begin{pmatrix} \Lambda_-(\xi^\#(0))\partial_x\xi_-^\#(0) - h_-(\xi^\#(0)) \\ \Lambda_+(\xi^\#(L))\partial_x\xi_+^\#(L) - h_+(\xi^\#(L)) \end{pmatrix}.$$

Some additional notations and definitions are also needed:

– The norm $|\cdot|$ in $\mathbb{R}^n$ is defined, for all $\xi \in \mathbb{R}^n$, by

$$|\xi| = \max(|\xi_i|,\ i \in \{1, \ldots, n\}).$$

$B(\varepsilon)$ denotes the ball centered in $0 \in \mathbb{R}^n$ with radius $\varepsilon > 0$.

– Given Φ continuous on $[0, L]$ and Ψ continuously differentiable on $[0, L]$, we denote

$$|\Phi|_{C^0(0,L)} = \max_{x\in[0,L]}|\Phi(x)|,$$
$$|\Psi|_{C^1(0,L)} = |\Psi|_{C^0(0,L)} + |\Psi\,'|_{C^0(0,L)};$$

– $B_{\mathcal{C}}(\varepsilon)$ denotes the set of continuously differentiable functions $\xi^\#$: $[0, L] \to \mathbb{R}^n$ satisfying the compatibility assumption $\mathcal{C}$ and $|\xi^\#|_{C^1(0,L)} \le \varepsilon$;
– For a given matrix $A = (a_{ij})$, $\rho(A)$ denotes its spectral radius and $\text{abs}(A)$ is the matrix defined by $\text{abs}(A) = (|a_{ij}|)$.

The main result of this paper is the following

Theorem 1 *Let $\varepsilon_0 > 0$. If*

$$\rho(abs(\nabla g(0))) < 1, \tag{6}$$

then there exist $\varepsilon_1 \in (0, \varepsilon_0)$, $H_1 > 0$, $\mu > 0$ and $C_1 > 0$ such that, for all continuously differentiable functions $h : B(\varepsilon_1) \to \mathbb{R}^n$ such that (3) holds together with

$$|\nabla h(0)| \le H_1, \tag{7}$$

for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_1)$, there exists an unique function $\xi \in C^1([0, L] \times [0, +\infty)\ ; \mathbb{R}^n)$ satisfying the PDE (1), the boundary conditions (5) and the initial condition

$$\xi(x, 0) = \xi^\#(x)\ , \quad \forall x \in [0, L]. \tag{8}$$

Moreover, this function satisfies

$$|\xi(.,t)|_{C^1(0,L)} \le C_1 e^{-\mu\, t}|\xi^\#|_{C^1(0,L)}\ , \quad \forall t \ge 0. \tag{9}$$

This theorem generalizes previous results [11, Chap. 5, Theo. 1.3] and [9, Theo. 6] in various ways:

- On one hand [11, Chap. 5, Theo. 1.3] where
  - h is assumed to be at least quadratic at the origin ($h(\xi) = O(|\xi|^2)$) whereas it is assumed here that it can be linear, with a small enough non-zero slope at the origin;
  - the boundary conditions are less general and have the following form

$$\begin{pmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{pmatrix} = \begin{pmatrix} g_-(\xi_+(L,t)) \\ g_+(\xi_-(0,t)) \end{pmatrix};$$

- on the other hand [9, Theorem 6] uses the same boundary conditions but for the homogeneous form (i.e. $h(\xi) = 0$).

The proof of this result will be based on an estimation of the influence of the non-homogeneous terms on the evolution of the Riemann coordinates. In particular, we have to prove that the damping condition (6) is strong enough to manage the non-homogeneous terms, whose derivative is assumed to be small at the origin due to (7). This result will be proved in Sect. 4 for a particular structure of the boundary conditions (see (22)); next the result will be extended to the boundary conditions (5) in Sect. 4.5.

Observe that this result can be extended to the case of a non-homogeneous term h depending not only on ξ but also on x, i.e. the PDE (1) where the term $h(\xi)$ is replaced by $h(\xi, x)$. In this case, Theorem 1 still holds provided that conditions (3) and (7) be replaced respectively by the following two conditions on the non-homogeneous term h:

$$h(0, x) = 0, \quad \forall x \in [0, L]$$

and

$$|\nabla_\xi h(0, x)| \leq H_1, \quad \forall x \in [0, L]$$

for some constant $H_1 > 0$, where $\nabla_\xi h$ denotes the Jacobian matrix of h as a function of ξ. This generalization can also be derived by following the evolution of the Riemann coordinates as in Sect. 4. However, for simplicity, here we consider a non-homogeneous term as in the PDE (1).

3 Application to fluid networks

The class of hyperbolic PDE systems considered in this paper has a wide range of applications to physical and engineering systems described by conservation laws. In this section, the analysis will be applied to a general class of fluid networks including for instance networks of open-channels [3,7] (like waterways or irrigation canals), pipeline networks [1] and fluid models for road traffic networks [10].

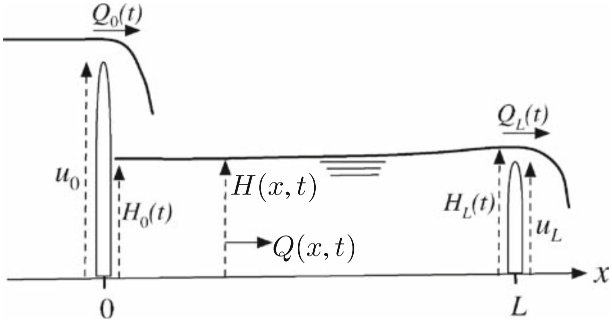

Fig. 3 A reach of an open channel delimited by two adjustable overflow spillways

The analysis will be illustrated with an hydraulic application: the regulation of the water level and the flow rate in networks of open channels. Our presentation will be organized in two stages: in a first step, we consider a single reach described by a system of two scalar conservation laws (i.e. $n = 2$). In a second step, we shall explain how the analysis can be extended to general fluid networks.

A system of two conservation laws

In the field of hydraulics, the flow in open-channels is generally described by the so-called Saint-Venant equations. We consider the case of a reach of an open channel delimited by two overflow spillways as depicted in Fig. 3.

It is assumed that :

1. The channel is horizontal,
2. the channel is prismatic with a constant rectangular section and a **unit** width,
3. the channel is subject to time-invariant spatially distributed water supplies or removals that do not modify the momentum conservation.

The flow dynamics are described by a non-homogeneous system of two laws of conservation (Saint-Venant or shallow water equations), namely a law of mass conservation:

$$\partial_t H(x,t) + \partial_x (Q(x,t)) = q, \tag{10}$$

and a law of momentum conservation:

$$\partial_t Q(x,t) + \partial_x \left(\frac{Q^2(x,t)}{H(x,t)} + \frac{gH^2(x,t)}{2} \right) = -C_f \frac{Q^2(x,t)}{H^2(x,t)}. \tag{11}$$

where $H(x,t)$ represents the water level and $Q(x,t)$ the water flow rate in the reach, g denotes the gravitation constant, q is a constant water supply/removal along the canal, C_f the friction coefficient. The system is written in matrix form as follows:

$$\partial_t \begin{pmatrix} H \\ Q \end{pmatrix} + A(H,Q)\partial_x \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} q \\ -C_f Q^2/H^2 \end{pmatrix}$$

with the matrix $A(H, Q)$ defined as:

$$A(H, Q) = \begin{pmatrix} 0 & 1 \\ gH - (Q^2/H^2) & 2Q/H \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix $A(H, Q)$:

$$c_1(H, Q) = (Q/H) - \sqrt{gH}$$
$$c_2(H, Q) = (Q/H) + \sqrt{gH}$$

are generally called *characteristic velocities*. The flow is said to be *fluvial* (or subcritical) when the characteristic velocities have opposite signs:

$$c_1(H, Q) < 0 < c_2(H, Q).$$

3.1 Steady-state solution

Under constant boundary conditions

$$H(L, t) = \bar{H}_L, \;\; Q(0, t) = \bar{Q}_0\,, \quad \forall t, \tag{12}$$

such that

$$c_1(\bar{H}_L, \bar{Q}_0) < 0 < c_2(\bar{H}_L, \bar{Q}_0), \tag{13}$$

if $|(C_f, q)|$ is sufficiently small, there exists a steady-state solution:

$$H(x, t) = \bar{H}(x) \quad \text{and} \quad Q(x, t) = \bar{Q}(x) \quad \forall x \in [0, L], \quad \forall t$$

satisfying

$$\begin{aligned} d_x \bar{Q}(x) &= q, \\ d_x \bar{H}(x) &= \frac{C_f \dfrac{\bar{Q}^2(x)}{\bar{H}^2(x)} + \dfrac{2\bar{Q}(x)q}{\bar{H}}}{-g\bar{H}(x) + \dfrac{\bar{Q}^2(x)}{\bar{H}^2(x)}}. \end{aligned} \tag{14}$$

Indeed, if $C_f = 0$, and $q = 0$, then, for all $(\bar{H}_L, \bar{Q}_0)$ satisfying (13), there exists a unique equilibrium $(\bar{H}, \bar{Q}) : [0, L] \to [0, +\infty) \times \mathbb{R}$ satisfying (14) and the boundary conditions (12). (It reads $(\bar{H}(x), \bar{Q}(x)) = (\bar{H}_L, \bar{Q}_0)$.) Thus, thanks to the continuity of the solution of the differential equation (14) with respect to parameters (C_f, q), for all $(\bar{H}_L, \bar{Q}_0) \in [0, +\infty) \times \mathbb{R}$ satisfying (13), for all (C_f, q) such that $|(C_f, q)|$ is sufficiently small, there exists a unique $(\bar{H}, \bar{Q}) : [0, L] \to [0, +\infty) \times \mathbb{R}$ satisfying (14) and the boundary conditions (12).

3.2 Control design

The control objective is to stabilize the level $H(x, t)$ and the flow rate $Q(x, t)$ at the steady state profiles $\bar{H}(x)$ and $\bar{Q}(x)$ corresponding to set points $\bar{H}_L$ and $\bar{Q}_0$. We assume that the boundary flow rates $Q(0, t)$ and $Q(L, t)$ are the control actions at the user's disposal because they can be assigned by the positions u_0 and u_L of the spillways. It is also assumed that the water levels at the boundaries $H_0(t) = H(0, t)$ and $H_L(t) = H(L, t)$ are the only available on-line measurements. In order to satisfy this control objective, the following control laws were introduced in [9]:

$$Q_0 = \frac{\bar{Q}_0}{\bar{H}_0} H_0 - \alpha_0 H_0 \left(2\sqrt{gH_0} - 2\sqrt{g\bar{H}_0}\right) \tag{15}$$

$$Q_L = \frac{\bar{Q}_L}{\bar{H}_L} H_L + \alpha_L H_L \left(2\sqrt{gH_L} - 2\sqrt{g\bar{H}_L}\right) \tag{16}$$

with:

$$0 < \alpha_0 < 1 \quad \text{and} \quad 0 < \alpha_L < 1.$$

The parameters α_0 and α_L are tuning parameters at the user's disposal. It can be seen that both controls have the form of a state feedback at the two boundaries. In addition, it can be emphasized that the implementation of the controls is particularly simple since only measurements of the levels $H_0(t)$ et $H_L(t)$ at the two spillways are required. This means that the feedback implementation does not require neither level measurements inside the pool nor any flow rate measurements.

3.3 Stability analysis

We shall now show that the stability of this control system can be analyzed with the theorem presented in Sect. 2. The analysis is made easier with the state vector

$$\mathbf{Y} = \begin{pmatrix} H \\ Q \end{pmatrix}$$

and the model (10) and (11) rewritten in compact form:

$$\partial_t \mathbf{Y} + A(\mathbf{Y})\partial_x \mathbf{Y} = \mathbf{f}(\mathbf{Y}) \tag{17}$$

with obvious definitions of $A(\mathbf{Y})$ and $\mathbf{f}(\mathbf{Y})$. This system can be diagonalized in the Riemann coordinates. This means that there exists a change of coordinates

$$\mathbf{Z}(\mathbf{Y}) = \begin{pmatrix} z_1(\mathbf{Y}) \\ z_2(\mathbf{Y}) \end{pmatrix} = \begin{pmatrix} (Q/H) - 2\sqrt{gH} \\ (Q/H) + 2\sqrt{gH} \end{pmatrix}$$

whose Jacobian matrix $\nabla\mathbf{Y} = \partial\mathbf{Z}/\partial\mathbf{Y}$ diagonalizes $A(\mathbf{Y})$ and therefore satisfies:

$$\nabla\mathbf{Y}\ A(\mathbf{Y}) = C(\mathbf{Y})\ \nabla\mathbf{Y}$$

with $C(\mathbf{Y}) = \mathrm{diag}\ (c_1(\mathbf{Y}), c_2(\mathbf{Y}))$. This change of coordinates can be inverted as:

$$\mathbf{Y}(\mathbf{Z}) = \begin{pmatrix} H(\mathbf{Z}) \\ Q(\mathbf{Z}) \end{pmatrix} = \begin{pmatrix} (z_2 - z_1)^2/16g \\ (z_1 + z_2)(z_2 - z_1)^2/32g \end{pmatrix}. \tag{18}$$

By left-multiplying equation (17) with the matrix $\nabla\mathbf{Y}$ and using the inverse coordinate change (18), we get the model in the Riemann coordinates as

$$\partial_t\mathbf{Z} + B(\mathbf{Z})\partial_x\mathbf{Z} = \mathbf{g}(\mathbf{Z})$$

where $B(\mathbf{Z}) = \mathrm{diag}\ (c_1(\mathbf{Y}(\mathbf{Z})), c_2(\mathbf{Y}(\mathbf{Z}))$ and $\mathbf{g}(\mathbf{Z}) = \nabla\mathbf{Y}(\mathbf{Z})\ \mathbf{f}(\mathbf{Y}(\mathbf{Z}))$.

Let us define the steady state solution in the Riemann coordinates as $\bar{\mathbf{Z}}(x) = \mathbf{Z}(\bar{\mathbf{Y}}(x))$ which obviously satisfies

$$B(\bar{\mathbf{Z}})\partial_x\bar{\mathbf{Z}} = \mathbf{g}(\bar{\mathbf{Z}}).$$

Then in order to transform the closed-loop system into the characteristic form (1), the ξ characteristic coordinates are defined as

$$\xi(\mathbf{Z}) = \mathbf{Z} - \bar{\mathbf{Z}} \ \text{ or componentwise } \ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}.$$

In these coordinates, the model (17) is finally written in the form:

$$\partial_t\xi + \Lambda(\xi)\partial_x\xi = h(\xi) \tag{19}$$

with $\Lambda(\xi) = B(\xi + \bar{\mathbf{Z}})$ and $h(\xi) = \mathbf{g}(\xi + \bar{\mathbf{Z}}) - B(\xi + \bar{\mathbf{Z}})\partial_x\bar{\mathbf{Z}}$. We observe that $h(0) = 0$ as requested in the assumptions of Theorem 1.

To check that, by prescribing q and C_f, $|\nabla h(0)|$ could be made as small as we want, let us first note that $\nabla h(\xi) = \nabla g(\xi + \bar{\mathbf{Z}}) - \nabla B(\xi + \bar{\mathbf{Z}})\partial_x\bar{\mathbf{Z}}$. Due to (14), $|\partial_x\bar{\mathbf{Z}}|$ vanishes when q and C_f vanish. Moreover note that $g(\xi + \bar{\mathbf{Z}}) = \nabla\mathbf{Y}(\xi + \bar{\mathbf{Z}})\ \mathbf{f}(\mathbf{Y}(\xi + \bar{\mathbf{Z}}))$, and that $\mathbf{f}(\bar{\mathbf{Z}})$ and $\nabla\mathbf{f}(\bar{\mathbf{Z}})$ vanish when q and C_f vanish.

Thus, by continuity of ∇h at the origin, for each $H > 0$, there exist $\bar{q}$ and $\bar{C}_f$ such that for all $|q| < \bar{q}$ and $|C_f| < \bar{C}_f$, we have $|\nabla h(0)| < H$ as requested in the assumptions of Theorem 1.

Moreover, in the characteristic coordinates, the control laws (15) and (16) can be shown to be equivalent to the following boundary conditions:

$$\xi_1(L,t) = -k_L\xi_2(L,t) \quad \text{with} \quad k_L = \frac{1-\alpha_L}{1+\alpha_L}$$

and

$$\xi_2(0,t) = -k_L\xi_1(0,t) \quad \text{with} \quad k_0 = \frac{1-\alpha_0}{1+\alpha_0}.$$

Observe that these boundary conditions are in the form (5) and that, in this special case:

$$\rho(\text{abs}(\nabla g(0))) = k_0 k_L.$$

Hence the closed-loop system is then exactly set in a form which allows to apply Theorem 1. This means that, provided the conditions of Theorem 1 are satisfied (in particular provided the tuning parameters α_0 and α_L are chosen such that $|k_0 k_L| < 1$, and the friction coefficient and the water supply/removal are sufficiently small), under the control laws (15) and (16), the level $H(x,t)$ and the flow rate $Q(x,t)$ are guaranteed to smoothly exponentially converge to the desired steady-state profile $\bar{H}(x)$ and $\bar{Q}(x)$.

General fluid networks

The analysis can be directly generalized to fluid networks that are made up of a set of interconnected fluid transportation devices in one space dimension (canals, pipes, etc ...). The dynamics of such networks are naturally represented by a set of subsystems of two conservations laws of the following general form:

$$\partial_t \begin{pmatrix} H_i \\ Q_i \end{pmatrix} + A_i(H_i, Q_i)\partial_x \begin{pmatrix} H_i \\ Q_i \end{pmatrix} = \mathbf{f}_i(H_i, Q_i) \quad i = 1, \ldots, m. \tag{20}$$

The matrices $A_i(H_i, Q_i)$ have two non-zero real distinct eigenvalues $c_i(H_i, Q_i)$ and $c_{m+i}(H_i, Q_i)$ with opposite signs: $c_i(H_i, Q_i) < 0 < c_{m+i}(H_i, Q_i)$. Each subsystem (20) can obviously be transformed into characteristic form exactly as we did above. This means that characteristic coordinates ξ_i and ξ_{m+i} can be defined such that the system (20) is equivalent to a system of the form:

$$\begin{aligned} &\partial_t \xi_i + \lambda_i(\xi_i, \xi_{m+i})\partial_x \xi_i = h_i(\xi_i, \xi_{m+i}) \\ &\partial_t \xi_{m+i} + \lambda_{m+i}(\xi_i, \xi_{m+i})\partial_x \xi_{m+i} = h_{m+i}(\xi_i, \xi_{m+i}) \end{aligned} \qquad i = 1, \ldots, m. \tag{21}$$

with $\lambda_i(\xi_i, \xi_{m+i}) < 0 < \lambda_{m+i}(\xi_i, \xi_{m+i})$ being the eigenvalues c_i and c_{m+i} expressed in the characteristic coordinates. Hence, this system network is exactly in the form required for the application of Theorem 1 (with $n = 2m$, i.e. with m positive and m negative eigenvalues λ_i).

4 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. First we assume that the boundary conditions (BC) have a particular form (see (22)). It allows us to set down a more natural machinery to prove intermediate technical lemmas. In Sect. 4.1, we state an existence result of a solution in finite time. In Sects. 4.2 and 4.3, estimates of $|\xi(\cdot,t)|_{C^0(0,L)}$ and $|\partial_x\xi(\cdot,t)|_{C^0(0,L)}$ are derived, and we conclude the proof of Theorem 1 in Sect. 4.5 (in particular the assumption on the form of the boundary conditions is removed).

In this section, except for Sect. 4.5, it is assumed that the boundary conditions (5) are of the particular form

$$\begin{pmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{pmatrix} = \begin{pmatrix} g_-(\xi_+(L,t)) \\ g_+(\xi_-(0,t)) \end{pmatrix}, \tag{22}$$

where the functions g_- and g_+ are continuously differentiable on a neighborhood of 0.

4.1 Existence result

The following existence result on a finite time interval is a basic tool for the proof of Theorem 1.

Lemma 1 *Let $T_2 > T_1 > 0$ and $T = T_2 - T_1$. Assume that the BC satisfy (6). Then there exist $\varepsilon(T) > 0$, $c(T) > 0$ and $H(T)$ such that, for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon(T))$ and for all continuously differentiable functions h: $B(\varepsilon(T)) \to \mathbb{R}^n$ such that (3) holds and*

$$|\nabla h(0)| \le H(T), \tag{23}$$

there exists a unique function $\xi \in C^1([0,L]\times[T_1,T_2],\mathbb{R}^n)$ satisfying the PDE (1) with boundary conditions (22) and initial condition (8). Moreover, this function ξ satisfies, $\forall t \in [T_1,T_2]$,

$$|\xi(.,t)|_{C^0(0,L)} \le c(T)|\xi^\#|_{C^0(0,L)}, \tag{24}$$

$$|\xi(.,t)|_{C^1(0,L)} \le c(T)|\xi^\#|_{C^1(0,L)}. \tag{25}$$

Proof By [12, Theorem 3.3, p 180], the existence of a unique solution ξ to the PDE (1) with B.C. (22) and initial condition (IC) (8) is guaranteed on a time interval $(0,\delta)$ for some δ sufficiently small.

Now the thesis of Lemma 1 holds for the particular case where $\nabla h(0) = 0$ (see [11, Chap. 5, Theorem 1.1]). Hence, by using the continuous dependence of the solution with respect to parameters, namely here $\nabla h(0)$, and with respect to the initial condition (i.e. by the proof of [2, Chap. 3, Theorem 3.5] extended to quasilinear hyperbolic systems), Lemma 1 holds on $(0,\delta)$. This leads to the *a priori* estimates (24) and (25) of the solution on that interval, and more generally on its existence domain as well.

Finally one can conclude that the lemma is established on (T_1,T_2), by a repeated application of the existence of solutions on intervals of the form $(0,n\delta)$, $n = 1,2,\dots$ up to N such that $N\delta > T$, by choosing ε sufficiently small, i.e. by reducing the C^1-norm of the initial condition $\xi^\sharp$ accordingly, and by using the *a priori* estimates (24) and (25). □

In the following, Lemma 1 is applied several times on intervals which will be defined with the help of two decreasing sequences of positive numbers $\varepsilon_2, \varepsilon_3, \dots$and $H_2, H_3, \dots$We consider initial conditions $\xi^\#$ successively in $B_{\mathcal{C}}(\varepsilon_2)$, $B_{\mathcal{C}}(\varepsilon_3)$, …

Let, for $i \in \{1, \dots, n\}$,

$$s_i = \frac{L}{|\lambda_i(0)|}, \tag{26}$$

$$\tau_1 > \max\{s_i,\ i \in \{1, \dots, n\}\}. \tag{27}$$

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $a > 1$ such that

$$|(\nabla g)_{ij}(0)| < a_{ij} < a, \quad \forall (i, j) \in \{1, \dots, n\}^2, \tag{28}$$

$$\rho(A) < 1. \tag{29}$$

From (29), there exists a sufficiently larger integer $K \geq 2$ such that $c(2\tau_1) \sum_{k \geq K} |A^k| < 1$, where $c(2\tau_1)$ is given by Lemma 1 applied on $[0, 2\tau_1]$. Let

$$\tau_2 := (K + 2)\tau_1. \tag{30}$$

Let ν and $\omega > 1$ be such that

$$\nu = c(2\tau_1) \sum_{k \geq K} |A^k| < 1, \tag{31}$$

$$\omega \geq |(\nabla \Lambda)_{ij}(0)|, \forall (i, j) \in \{1, \dots, n\}^2, \tag{32}$$

$$\omega \geq |(\bar{\Lambda})_{ij}|, \forall (i, j) \in \{1, \dots, n\}^2. \tag{33}$$

where $\bar{\Lambda} = \Lambda(0)$.

4.2 Estimation of $|\xi(., t)|_{C^0(0,L)}$

Let $\varepsilon_2 = \varepsilon(\tau_2)$ and $H_2 = H(\tau_2)$ given by Lemma 1 applied on $[0, \tau_2]$. For all $0 < H < H_2$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and

$$|\nabla h(0)| \leq H, \tag{34}$$

for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_2)$, the PDE (1), with the boundary condition (22) and the initial condition (8), admits a unique solution $\xi \in C^1([0, L] \times [0, \tau_2]; \mathbb{R}^n)$.

In view of (2), and by using a continuity argument, we may assume without loss of generality (i.e. with ε_2 sufficiently small) that

$$\lambda_i(\xi(x, t)) < 0 < \lambda_j(\xi(x, t)), \ \forall i \in \{1, \dots, m\}, \ \forall j \in \{m + 1, \dots, n\}. \tag{35}$$

The aim of this section is to establish the following

Lemma 2 *There exist $\bar{H} > 0$, $\bar{\varepsilon} \in (0, \varepsilon_2)$ and $\bar{\mu} > 0$ such that, for all $0 < H < \bar{H}$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$, the following inequality holds*:

$$|\xi(., \tau_2)|_{C^0(0,L)} \leq (\nu + \bar{\mu})(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{36}$$

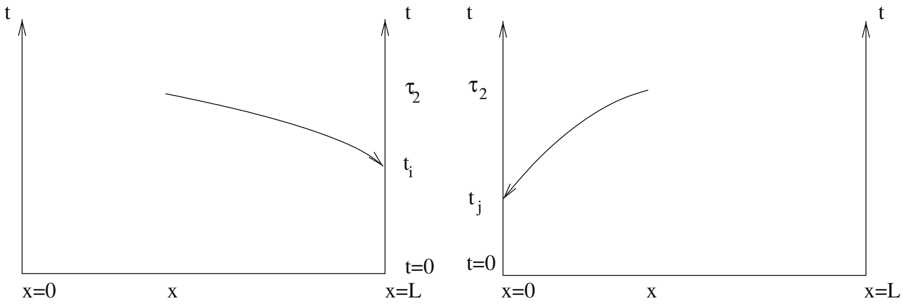

Fig. 4 Characteristic curves defining p_i and p_j. *Left*, for $i \in \{1, \dots, m\}$, *right*, for $j \in \{m+1, \dots, n\}$

Before proving this lemma, let us state a series of intermediate results from Claim 3 to Claim 7.

Let $x \in [0, L]$. Hereafter the characteristics are defined backwards in time from (x, τ_2).

In what follows, $\mathbb{N}$ denotes the set of nonnegative integers. For $k \in \mathbb{N}$ and for $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$, we define $t_{i_1 \dots i_k} \in [0, T]$ and $p_{i_1 \dots i_k} \in [0, L] \times \{t_{i_1 \dots i_k}\}$ by induction on k as follows (see Fig. 4).

Initial step $k = 1$

– For $i \in \{1, \dots, m\}$, let us consider the solution y_i of the Cauchy problem

$$d_t y_i(t) = \lambda_i(\xi(y_i(t), t)), \quad y_i(\tau_2) = x.$$

In view of (35), it is allowed to define the time instant $t_i \le \tau_2$ by

$$y_i(t_i) = L$$

and we set

$$p_i = (L, t_i).$$

– For $j \in \{m+1, \dots, n\}$, let us consider the solution y_j of the Cauchy problem

$$d_t y_j(t) = \lambda_j(\xi(y_j(t), t)), \quad y_j(\tau_2) = x.$$

Similarly, we define the time instant $t_j \le \tau_2$ by

$$y_j(t_j) = 0$$

and we set

$$p_j = (0, t_j).$$

The following claim states the initial estimation of the Riemann coordinates along the characteristics:

Claim 3 There exist $\varepsilon_3 \in (0, \varepsilon_2)$ and $H_3 \in (0, H_2)$ such that, for all $0 < H < H_3$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_3)$, and for all $i_1 \in \{1, \dots, n\}$,

$$|\xi_{i_1}(x, \tau_2)| \leq |\xi_{i_1}(p_{i_1})| + c(\tau_2)\tau_2(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{37}$$

Proof of Claim 3 Note that using (1), (24) of Lemma 1 and by construction of p_{i_1}, there exist $\varepsilon_3 \in (0, \varepsilon_2)$ and $H_3 \in (0, H_2)$ such that, for all $0 < H < H_3$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_3)$, and for all $i_1 \in \{1, \dots, n\}$,

$$\begin{aligned} |\xi_{i_1}(x, \tau_2)| &= |\xi_{i_1}(p_{i_1}) + \int_{t_{i_1}}^{\tau_2} h_{i_1}(\xi(y_{i_1}(s), s))ds|, \\ &\leq |\xi_{i_1}(p_{i_1})| + \int_{t_{i_1}}^{\tau_2} |h_{i_1}(\xi(y_{i_1}(s), s))ds|. \end{aligned} \tag{38}$$

Moreover, by the Mean-Value Inequality, also called Finite-Increment Theorem (see [21, Propoaition 2. p. 78]), and conditions (3) and (34), there exists an increasing function $w : (0, \varepsilon_2) \to (0, +\infty)$, satisfying $w(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that, for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_3)$, for all $s \in [t_{i_1}, \tau_2]$,

$$|h_{i_1}(\xi(y_{i_1}(s), s))| \leq (H + w(c(\tau_2)\varepsilon_3))|\xi(y_{i_1}(s), s)|.$$

It follows by using inequalities (24) and (38) that

$$|\xi_{i_1}(x, \tau_2)| \leq |\xi_{i_1}(p_{i_1})| + \tau_2(H + w(c(\tau_2)\varepsilon_3))c(\tau_2)|\xi^\#|_{C^0(0,L)}.$$

Therefore, up to reducing $\varepsilon_3 \in (0, \varepsilon_2)$, we may conclude that (37) holds. □

General induction step

Now let $k \in \mathbb{N}$ be arbitrarily fixed and assume that $t_{i_1 \dots i_k} \in [0, \tau_2]$ and $p_{i_1 \dots i_k} \in [0, L] \times \{t_{i_1 \dots i_k}\}$ are defined. Then, for $i_{k+1} \in \{1, \dots, n\}$, we define $t_{i_1 \dots i_{k+1}} \in [0, \tau_2]$ and $p_{i_1 \dots i_{k+1}} \in [0, L] \times \{t_{i_1 \dots i_{k+1}}\}$, by considering two cases (as done above for $k = 1$):

– For all $i \in \{1, \dots, m\}$, consider the Cauchy problem

$$d_t y_i(t) = \lambda_i(\xi(y_i(t), t)), \quad y_i(t_{i_1 \dots i_k}) = 0$$

and define $t_{i_1 \dots i_k i} \in [0, t_{i_1 \dots i_k})$ by

$$y_1(t_{i_1 \dots i_k i}) = L.$$

If such $t_{i_1\dots i_k i}$ exists, it is unique and we define $p_{i_1\dots i_k i}$ by

$$p_{i_1\dots i_k i} = (L, t_{i_1\dots i_k i}).$$

In contrast, if such $t_{i_1\dots i_k i}$ does not exist, we do not define $t_{i_1\dots i_k i}$, neither $p_{i_1\dots i_k i}$, nor $t_{i_1\dots i_k i\dots i_l}$ and $p_{i_1\dots i_k i\dots i_l}$ for $l > k+1$.

– For all $j \in \{m+1,\dots,n\}$, consider the Cauchy problem

$$d_t y_j(t) = \lambda_j(\xi(y_j(t),t)), \quad y_j(t_{i_1\dots i_k}) = L,$$

and define $t_{i_1\dots i_k j} \in [0, t_{i_1\dots i_k})$ by

$$y_j(t_{i_1\dots i_k j}) = 0.$$

Again, if such $t_{i_1\dots i_k j}$ exists, it is unique and then we define $p_{i_1\dots i_k j}$ by

$$p_{i_1\dots i_k j} = (0, t_{i_1\dots i_k j}).$$

However if such $t_{i_1\dots i_k j}$ does not exist, we do not define $t_{i_1\dots i_k j}$, neither $p_{i_1\dots i_k j}$, nor $t_{i_1\dots i_k j\dots i_l}$ and $p_{i_1\dots i_k j\dots i_l}$ for $l > k+1$.

Similarly to Claim 3, using (1), (3), (24) of Lemma 1 and by construction of $t_{i_1\dots i_{k+1}}$ and $p_{i_1\dots i_{k+1}}$, we get the following result.

Claim 4 There exist $\varepsilon_4 \in (0, \varepsilon_3)$ and $H_4 \in (0, H_3)$ such that, for all $0 < H < H_4$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), and for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_4)$,

$$|\xi_i(0, t_{i_1\dots i_k})| \le |\xi_i(p_{i_1\dots i_k i})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^\#|_{C^0(0,L)}, \quad \forall i \in \{1,\dots,m\} \tag{39}$$

and

$$|\xi_j(L, t_{i_1\dots i_k})| \le |\xi_j(p_{i_1\dots i_k j})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^\#|_{C^0(0,L)} \quad \forall j \in \{m+1,\dots,n\}. \tag{40}$$

Note that, in view of (26), (27) and (30) there exists a finite number of $k \ge 1$ such that

$$s_{i_1} + \dots + s_{i_k} \le \tau_2 - \tau_1\ , \quad \forall i_1,\dots,i_k \in \{1,\dots,n\}.$$

With the previous two Claims, the influence of the boundary conditions on the Riemann coordinates are estimated as follows:

Claim 5 There exist $\varepsilon_5 \in (0, \varepsilon_4)$ and $H_5 \in (0, H_4)$, such that, for all $0 < H < H_5$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_5)$, for all integer $k \ge 1$, and for all $(i_1,\dots,i_k,i_{k+1}) \in \{1,\dots,n\}^{k+1}$ such that $s_{i_1} + \dots + s_{i_k} \le \tau_2 - \tau_1$, the time instant $t_{i_1\dots i_k i_{k+1}}$ and the point $p_{i_1\dots i_k i_{k+1}}$ exist.

Moreover

$$|\xi_{i_k}(p_{i_1\ldots i_k})| \le \sum_{j\neq i_k} a_{i_k j}|\xi_j(p_{i_1\ldots i_k j})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{41}$$

Proof of Claim 5 Let $(i_1,\ldots,i_k,i_{k+1}) \in \{1,\ldots,n\}^{k+1}$ such that $s_{i_1}+\cdots+s_{i_k} \le \tau_2-\tau_1$. Due to (26), (27), the time instant $t_{i_1\ldots i_k i_{k+1}}$ and the point $p_{i_1\ldots i_k i_{k+1}}$ exist. Inequality (41) is a consequence of Claims 3 and 4 and the boundary conditions (22), and (28). □

The following result will also be useful.

Claim 6 There exist $\bar\varepsilon \in (0,\varepsilon_5)$ and $\bar H \in (0,H_5)$, such that, for all $0<H<\bar H$, for all continuously differentiable functions $h\colon B(\varepsilon_2)\to\mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\bar\varepsilon)$, for all integer $k\ge 1$, and for all $(i_1,\ldots,i_k,i_{k+1})\in\{1,\ldots,n\}^{k+1}$ such that

$$\tau_2-2\tau_1 \le s_{i_1}+\cdots+s_{i_k} \le \tau_2-\tau_1,$$

the existence of $t_{i_1\ldots i_k i_{k+1}}$ is guaranteed and the time instant $t_{i_1\ldots i_k}$ is in the interval $[0,2\tau_1]$. Moreover

$$|\xi(p_{i_1\ldots i_k})| \le c(2\tau_1)|\xi^\#|_{C^0(0,L)}. \tag{42}$$

Proof of Claim 6 The existence of $t_{i_1\ldots i_k i_{k+1}}$ follows from Claim 5. The estimation $t_{i_1\ldots i_k} \le 2\tau_1$ follows from $\tau_2-2\tau_1\le s_{i_1}+\cdots+s_{i_k}$ and the definition of the time instant $t_{i_1\ldots i_k}$.

Estimation (42) is a consequence of Lemma 1 applied on $[0,2\tau_1]$. □

Finally, let us state the following

Claim 7 For all l in $\mathbb{N}$, there exists $\delta_l>0$ such that the following assertion $(\mathcal{P}_l)$ holds:

For all $0<H<\bar H$, for all continuously differentiable functions $h\colon B(\varepsilon_2)\to\mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\#\in B_{\mathcal{C}}(\bar\varepsilon)$, $\forall(i_1,\ldots,i_l)\in\{1,\ldots,n\}^l$ such that $s_{i_1}+\cdots+s_{i_l}\le\tau_2-2\tau_1$, we have

$$|\xi_{i_l}(p_{i_1\ldots i_l})| \le \sum_{k\ge l}\sum_{I_k}\sum_{j=1}^{n} a_{i_l i_{l+1}}a_{i_{l+1}i_{l+2}}\cdots a_{i_k j}|\xi_j(p_{i_1\cdots i_k j})| + \delta_l(H+\varepsilon_2)|\xi^\#|_{C^0(0,L)}.$$

where I_k denotes the set of indices i_j, $j\in\{1,\ldots,k\}$ such that

$$\tau_2-2\tau_1\le s_{i_1}+\cdots+s_{i_l}+s_{i_{l+1}}+\cdots+s_{i_k}\le\tau_2-\tau_1.$$

The proof of Claim 7 is based on a decreasing induction on l. See Appendix.

We are now in a position to prove Lemma 2.

Proof of Lemma 2 Due to Claim 7, $(\mathcal{P}_1)$ is true and thus with Claim 3, for all $0 < H < \bar{H}$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $i_1 \in \{1, \dots, n\}$, for all $x \in [0, L]$ and $\xi^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$,

$$|\xi_{i_1}(x, \tau_2)| \leq \sum_{k \geq 1} \sum_{\tau_2 - 2\tau_1 \leq s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1} \sum_{j=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k j} |\xi_j(p_{i_1 \dots i_k j})|$$
$$+ (\delta_1 + c(\tau_2)\tau_2)(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)},$$

which, with (42), gives

$$|\xi_{i_1}(x, \tau_2)| \leq c(2\tau_1)|\xi^\#|_{C^0(0,L)} \sum_{k \geq 1} \sum_{\tau_2 - 2\tau_1 \leq s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1} \sum_{j=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k j}$$
$$+ (\delta_1 + c(\tau_2)\tau_2)(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{43}$$

Note that the sums in (43) are finite. Moreover, due to (27) and (30)

$$\left(s_{i_1} + \dots + s_{i_k} \geq \tau_2 - 2\tau_1 = K\tau_1\right) \Rightarrow k \geq K. \tag{44}$$

Observe also that, by the definition of matrix product, we have, for all $N \in \mathbb{N}$,

$$\sum_{(i_2, \dots, i_k, j) \in \{1, \dots, N\}^k} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k j} = \sum_{j=1}^{N} (A^k)_{i_1 j} \leq |A^N|. \tag{45}$$

From (43)–(45), we get, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$, we have

$$|\xi_{i_1}(x, \tau_2)| \leq c(2\tau_1)|\xi^\#|_{C^0(0,L)} \sum_{k \geq K} |A^k| + (\delta_1 + c(\tau_2)\tau_2)(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{46}$$

Let $\bar{\mu} = \delta_1 + c(\tau_2)\tau_2$. Recall that δ_l is recursively defined by (58) and (65) (see the Appendix) and thus it does not depend on H. We get with (31), (46) and Claim 7,

$$|\xi_{i_1}(x, \tau_2)| \leq (\nu + \bar{\mu})(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)},$$

for all i_1 in $\{1, \dots, n\}$ and for all x in $[0, L]$. This is (36). This concludes the proof of Lemma 2. □

4.3 Estimation of $|\partial_x \xi(., t)|_{C^0(0,L)}$

Let η: $[0, L] \times [0, \tau_2] \to \mathbb{R}^n$ be defined by $\eta = \bar{\Lambda} \partial_x \xi$ where $\xi \in C^1([0, L] \times [0, \tau_2]; \mathbb{R}^n)$ is defined by $\xi^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$, the PDE (1), the boundary condition (8), and the initial condition (22).

Similarly let us define η_1: $[0, L] \times [0, \tau_2] \to \mathbb{R}^n$ and η_2: $[0, L] \times [0, \tau_2] \to \mathbb{R}^n$ defined respectively by $\eta = (\eta_1,\ \eta_2)^T$.

Differentiating (1) with respect to x, it follows that

$$\partial_t \eta + \bar{\Lambda}\Lambda(\xi)\bar{\Lambda}^{-1}\partial_x \eta = -\bar{\Lambda}(\nabla\Lambda(\xi)\partial_x\xi)\partial_x\xi + \bar{\Lambda}\nabla h(\xi)\partial_x\xi, \tag{47}$$

along the characteristics.

Moreover, differentiating (22) and using (1), it gives

$$\begin{aligned}
&\begin{pmatrix} (-\Lambda(\xi)\bar{\Lambda}^{-1}\eta + h(\xi))_-(L,t) \\ (-\Lambda(\xi)\bar{\Lambda}^{-1}\eta + h(\xi))_+(0,t) \end{pmatrix} \\
&\quad = \nabla g \begin{pmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{pmatrix} \begin{pmatrix} (-\Lambda(\xi)\bar{\Lambda}^{-1}\eta + h(\xi))_-(0,t) \\ (-\Lambda(\xi)\bar{\Lambda}^{-1}\eta + h(\xi))_+(L,t) \end{pmatrix}.
\end{aligned}$$

A development similar to ξ can be used as for ξ_i along the trajectories of (47). It can be shown from Lemma 1, and (25), (32) and (33), that there exist $\varepsilon_6 \in (0, \bar{\varepsilon})$ and $H_6 \in (0, \bar{H})$ such that, for all $0 < H < H_6$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_6)$ then, we have, for all $i \in \{1, \ldots, n\}$ and for all $t_1 < t_2$ along a characteristic of (47).

$$\begin{aligned}
|\eta_i(y(t_2), t_2) - \eta_i(y(t_1), t_1)| &\le \omega^2\, c(2\tau_1)^2\, |\xi^\#|^2_{C^1(0,L)}\, |t_2 - t_1| \\
&\quad + \omega\, (H + \varepsilon_2)\, c(2\tau_1)\, |\xi^\#|_{C^1(0,L)}\, |t_2 - t_1|, \\
&\le \omega\, (H + 2\max(1, \omega c(2\tau_1))\varepsilon_2)\, c(2\tau_1) \\
&\quad \times |\xi^\#|_{C^1(0,L)} |t_2 - t_1|.
\end{aligned}$$

The last inequality allows us to prove the analogous of Claim 3 for the variable η. Using the computations of Sect. 4.2, we deduce the following

Lemma 8 *There exist $\tilde{\varepsilon}$, $\tilde{H}$, and $\tilde{\mu} > 0$ such that, for all $0 < H < \tilde{H}$, for all continuously differentiable functions h: $B(\varepsilon_2) \to \mathbb{R}^n$ satisfying* (3) *and* (34)*, for all $\xi^\# \in B_{\mathcal{C}}(\tilde{\varepsilon})$, we have*

$$|\eta(., \tau_2)|_{C^0(0,L)} \le (\nu + \tilde{\mu})\, (H + 2\varepsilon_2 \max(1, \omega c(2\tau_1)))\, |\xi^\#|_{C^1(0,L)}. \tag{48}$$

4.4 Proof of Theorem 1 for the particular boundary conditions (22)

In this section, we conclude the proof of Theorem 1 for the special boundary conditions (22) (instead of (5)). To deduce Theorem 1 for the boundary conditions given by (5), we need to double the size of the state as done in [9, Proof of Theorem 6]. This is done in Sect. 4.5.

Let $\nu' \in (0, 1)$. Up to reducing ε_2, there exists $H_7 \in (0, \tilde{H})$ such that

$$(\nu + \max(\mu, \tilde{\mu}))(H_7 + 2\varepsilon_2 \max(1, \omega c(2\tau_1))) < \nu'.$$

We combine (36) and (48) to get the existence of $\varepsilon_7 \in (0, \varepsilon_6)$, such that, for all $0 < H < H_7$, for all continuously differentiable functions $h: B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_7)$, we have

$$|\xi(., \tau_2)|_{C^1(0,L)} \leq \nu' |\xi^\#|_{C^1(0,L)}.$$

This estimate allows a repeated application of Lemma 1 on intervals of length τ_2 to give, for all $0 < H < H_7$, for all continuously differentiable functions $h: B(\varepsilon_2) \to \mathbb{R}^n$ satisfying (3) and (34), for all $\xi^\# \in B_{\mathcal{C}}(\varepsilon_7)$, the existence of a unique solution of (1), (8) and (22) over any interval $[0, N\tau_2]$ with $N \in \mathbb{N} \setminus \{0\}$ and

$$|\xi(., N\tau_2)|_{C^1(0,L)} \leq \nu'^N |\xi^\#|_{C^1(0,L)}.$$

Thus, by letting $C_1 = \max(c(\tau_2), 1)e^{-\ln \nu'}$ and $\mu = -\frac{\ln(\nu')}{\tau_2}$, we get (9).

4.5 Proof of Theorem 1 for the general boundary conditions (5)

In the previous section, we have proved Theorem 1 when the boundary conditions have the special form (22). It turns out that Theorem 1 remains valid, when the boundary conditions have the more general form (5).

To prove this, we adapt to our more general situation by doubling the size of the state as done in [9, Proof of Theorem 6]. More precisely, consider the hyperbolic system

$$\partial_t \widetilde{\xi} + \widetilde{\Lambda}(\widetilde{\xi}) \partial_x \widetilde{\xi} = 0, \tag{49}$$

with

$$\widetilde{\xi} = (\xi_{1-}^T, \ \xi_{2-}^T, \ \xi_{1+}^T, \ \xi_{2+}^T)^T,$$

where $\xi_{1-} \in \mathbb{R}^m$, $\xi_{2-} \in \mathbb{R}^{n-m}$, $\xi_{1+} \in \mathbb{R}^{n-m}$, $\xi_{2+} \in \mathbb{R}^m$ and $\widetilde{\Lambda}: \mathbb{R}^{2n} \to \mathbb{R}^{2n \times 2n}$ is defined by

$$\widetilde{\Lambda}(\widetilde{\xi}) = \operatorname{diag} \begin{pmatrix} \Lambda_-((\xi_{1-}^T, \xi_{1+}^T)^T) \\ -\Lambda_+((\xi_{2+}^T, \xi_{2-}^T)^T) \\ \Lambda_+((\xi_{1-}^T, \xi_{1+}^T)^T) \\ -\Lambda_-((\xi_{2+}^T, \xi_{2-}^T)^T) \end{pmatrix}.$$

The boundary conditions for (49) are defined by

$$\begin{pmatrix} \xi_{1-}(L,t) \\ \xi_{2-}(L,t) \end{pmatrix} = g\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_{1+}(L,t) \\ \xi_{2+}(L,t) \end{pmatrix} \right),$$
$$\begin{pmatrix} \xi_{1+}(0,t) \\ \xi_{2+}(0,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} \xi_{1-}(0,t) \\ \xi_{2-}(0,t) \end{pmatrix}.$$

This boundary conditions can be written in the following form

$$\begin{pmatrix} \widetilde{\xi}_-(L,t) \\ \widetilde{\xi}_+(0,t) \end{pmatrix} = \widetilde{g}\begin{pmatrix} \widetilde{\xi}_+(L,t) \\ \widetilde{\xi}_-(0,t) \end{pmatrix} = \begin{pmatrix} \widetilde{g}_1(\widetilde{\xi}_+(L,t)) \\ \widetilde{g}_2(\widetilde{\xi}_-(0,t)) \end{pmatrix}, \tag{50}$$

with

$$\widetilde{\xi}_- = (\xi_{1-}^T, \xi_{2-}^T)^T,\ \widetilde{\xi}_+ = (\xi_{1+}^T, \xi_{2+}^T)^T, \tag{51}$$

$$\widetilde{g}_1 = g \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \widetilde{g}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ g. \tag{52}$$

In particular the boundary conditions for $\widetilde{\xi}$ have the special form (22) and

$$\rho(\text{abs}(\nabla\widetilde{g}(0))) = \rho(\text{abs}(\nabla g(0)))^2. \tag{53}$$

Let $\xi^\# \in C^1([0,L];\mathbb{R}^n)$ satisfying the compatibility condition $\mathcal{C}$ and such that $|\xi^\#|_{C^1(0,L)}$ is small enough. We choose as initial condition for $\widetilde{\xi}$ at $t=0$,

$$\begin{aligned} \xi_{1-}^\#(x) &= \xi_-^\#(x), \quad \xi_{2-}^\#(x) = \xi_+^\#(L-x), \\ \xi_{1+}^\#(x) &= \xi_+^\#(x), \quad \xi_{2+}^\#(x) = \xi_-^\#(L-x). \end{aligned} \tag{54}$$

One easily sees that $\widetilde{\xi}^\# := (\xi_{1-}^{\#T}, \xi_{2-}^{\#T}, \xi_{1+}^{\#T}, \xi_{2+}^{\#T})^T$ satisfies the compatibility condition associated to (49) and (50). Hence there exists a unique C^1-solution $\widetilde{\xi}$ of (49) and (50) such that

$$\widetilde{\xi}(x,0) = \widetilde{\xi}^\#(x). \tag{55}$$

Let

$$\widetilde{\xi}^*(x,t) = \begin{pmatrix} \xi_{2+}(L-x,t)^T \\ \xi_{1+}(L-x,t)^T \\ \xi_{2-}(L-x,t)^T \\ \xi_{1-}(L-x,t)^T \end{pmatrix}.$$

Then, as one easily checks, $\widetilde{\xi}^*$ satisfies as $\widetilde{\xi}$ the hyperbolic system (49), the boundary conditions (50) and the initial condition (55). Hence by the uniqueness of the C^1-solution of the Cauchy problem associated to (49) and (50), one has $\widetilde{\xi}^* = \widetilde{\xi}$. In particular $\xi_{1-}(x,t) = \xi_{2+}(L-x,t)$, and $\xi_{1+}(x,t) = \xi_{2-}(L-x,t)$. Hence, if $\xi_-(x,t) := \xi_{1-}(x,t)$, and $\xi_+(x,t) := \xi_{1+}(x,t)$, then $\xi = (\xi_-^T, \xi_+^T)^T$ satisfies (1), (5) and (8).

Conversely, if $\xi = (\xi_-^T, \xi_+^T)^T$ satisfies (1), (5) and (8), then $\widetilde{\xi}$ defined by

$$\begin{aligned} \xi_{1-}(x,t) &:= \xi_-(x,t), & \xi_{1+}(x,t) &:= \xi_+(x,t), \\ \xi_{2+}(x,t) &:= \xi_-(L-x,t), & \xi_{2-}(x,t) &:= \xi_+(L-x,t), \end{aligned}$$

satisfies the hyperbolic system (49), the boundary conditions (50) and the initial condition (55). Hence, see also (53), Theorem 1 for the hyperbolic system (1) and the boundary conditions (5) are a consequence of Sect. 4.4 for the hyperbolic system (49) and the special boundary conditions (50).

This concludes the proof of Theorem 1.

5 Conclusion

The aim of this paper was to address the following two problems.

Firstly, a sufficient condition was stated for the stability of systems of conservation laws, perturbed by non-homogeneous terms which are assumed to be small in C^1-norm. This criterion is written in terms of the boundary conditions and it was proved thanks to an analysis of the Riemann coordinates.

This result has been presented in the case where all the eigenvalues of the function Λ defining the PDE (1) do not have the same sign. However this result remains true in the case where all the eigenvalues have the same sign. In this case, only one kind of characteristic curves should be considered and some parts of the proof in Sect. 4 can be removed or simplified (e.g. the boundary conditions (5) become simpler, or for the definition of the sequence p_i in Sect. 4.2, we do not have to make the distinction between $i \in \{1, \dots, m\}$ and $j \in \{m + 1, \dots, n\}$).

This main result can be seen as a robustness property of the results of [9] and [11, Chap. 5] for the stability of hyperbolic equations. To the best to our knowledge, this kind of stability results is not usual in the study of PDE.

Secondly, this general condition was applied to the case of the regulation of the water level and the water flow rate in open channels. The evolution of the flow is described by the Saint-Venant equations perturbed by small non-homogeneous terms that account for the friction effects as well as external supplies or withdrawals. The general sufficient condition established in Sect. 3 leads to design of stabilizing boundary controls of the canal.

An interesting approach for the proof of an analoguous result and its hydraulic application could be the use of a Lyapunov function for boundary control of hyperbolic systems as computed in [4]. Using this Lyapunov function for a hyperbolic system perturbed by sufficiently small terms may be possible. Note however that in [4], the stability result and the Lyapunov function have been computed with the H^2-norm instead of the C^1-norm as in this paper. Thus, the robustness issue would be different, if it is true. Another potentially interesting approach could be to linearize the PDE describing the model, together with the boundary conditions, around a given equilibrium profile. The well-posedness and the stability of the resulting linear boundary control system might then be analyzed by means of known techniques: see e.g. [20] and the references therein. However the next step would then be to analyze the nominal nonlinear model by interpreting it as a linear boundary control system perturbed by nonlinear terms: this problem is clearly nontrivial and is an interesting topic for further research.

The main result of this paper has been successfully tested in [5] on numerical simulations using the data of a real river (more precisely the Sambre river in Belgium), and experimented on an experimental micro-channel in Valence. Other applications in the control of fluid networks are in progress.

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Appendix: Proof of Claim 7

We prove $(\mathcal{P}_l)$ by a decreasing induction on l.

By defining $\tilde{l}$ as the smaller integer such that

$$\tilde{l} > \frac{\tau_2 - \tau_1}{\min\{s_i,\ i = 1, \dots, n\}}, \tag{56}$$

it follows that

$$\left\{(i_1, \dots, i_l) \in \{1, \dots, n\}^l;\ s_{i_1} + \dots + s_{i_l} \leq \tau_2 - \tau_1\right\} = \emptyset, \quad \forall l \geq \tilde{l}. \tag{57}$$

Since $\tilde{l}$ is minimally defined,

$$\tilde{l} \leq \frac{\tau_2 - \tau_1}{\min\{s_i,\ i = 1, \dots, n\}} + 1. \tag{58}$$

Due to (24) of Lemma 1, by letting

$$\delta_l = \frac{c(\tau_2)}{\varepsilon_2}, \quad \forall l \geq \tilde{l}, \tag{59}$$

we have

$$|\xi_{i_l}(p_{i_1 \dots i_l})| \leq \delta_l \varepsilon_2 |\xi^\#|_{C^0(0,L)}$$

and thus, due to (57), $(\mathcal{P}_l)$ holds, for $l \geq \tilde{l}$.

Let $l \geq 2$ be such that $(\mathcal{P}_l)$ is true. Let us prove that $(\mathcal{P}_{l-1})$ is true. Let $(i_1, \dots, i_{l-1}) \in \{1, \dots, n\}^{l-1}$ be such that

$$s_{i_1} + \dots + s_{i_{l-1}} \leq \tau_2 - 2\tau_1. \tag{60}$$

From (41) in Claim 5,

$$|\xi_{i_{l-1}}(p_{i_1 \dots i_{l-1}})| \leq \sum_{j \neq i_{l-1}} a_{i_{l-1}j} |\xi_j(p_{i_1 \dots i_{l-1}j})| + c(\tau_2)\tau_2(H + \varepsilon_2)|\xi^\#|_{C^0(0,L)}. \tag{61}$$

Due to (27) and (60), for all $j \in \{1, \dots, n\}$,

$$s_{i_1} + \dots + s_{i_{l-1}} + s_j \leq \tau_2 - \tau_1.$$

Therefore, it follows from (61) that

$$|\xi_{i_{l-1}}(p_{i_1\ldots i_{l-1}})| \le \sum_{j,\ s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-\tau_1} a_{i_{l-1}j}|\xi_j(p_{i_1\ldots i_{l-1}j})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}. \tag{62}$$

Now let us split this sum into two parts as follows:

$$|\xi_{i_{l-1}}(p_{i_1\ldots i_{l-1}})| \le \sum_{j,\ s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-2\tau_1} a_{i_{l-1}j}|\xi_j(p_{i_1\ldots i_{l-1}j})| \tag{63}$$

$$+ \sum_{j,\ \tau_2-2\tau_1\le s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-\tau_1} a_{i_{l-1}j}|\xi_j(p_{i_1\ldots i_{l-1}j})| \tag{64}$$

$$+c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}.$$

For all $j\in\{1,\ldots,n\}$ such that $s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-2\tau_1$, $(\mathcal{P}_l)$ applies and thus, for such a j,

$$|\xi_j(p_{i_1\ldots i_{l-1}j})| \le \sum_{k\ge l}\sum_{I_k}\sum_{j=1}^{n} a_{i_l i_{l+1}}a_{i_{l+1}i_{l+2}}\cdots a_{i_k j}|\xi_j(p_{i_1\cdots i_k j})| + \delta_l(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}.$$

Therefore, in view of (28) and (63),

$$\begin{aligned}|\xi_{i_{l-1}}(p_{i_1\ldots i_{l-1}})| \le{}& \sum_{j,\ s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-2\tau_1} a_{i_{l-1}j}\sum_{k\ge l}\sum_{I_k}\\ &\times\sum_{j=1}^{n} a_{i_l i_{l+1}}a_{i_{l+1}i_{l+2}}\cdots a_{i_k j}|\xi_j(p_{i_1\cdots i_k j})|\\ &+2an\delta_l(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}\\ &+\sum_{j,\ \tau_2-2\tau_1\le s_{i_1}+\cdots+s_{i_{l-1}}+s_j\le\tau_2-\tau_1} a_{i_{l-1}j}|\xi_j(p_{i_1\ldots i_{l-1}j})|\\ &+c(\tau_2)\tau_2(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}.\end{aligned}$$

By rewriting the first sum and by estimating the second sum, it follows that

$$\begin{aligned}|\xi_{i_{l-1}}(p_{i_1\ldots i_{l-1}})| \le{}& \sum_{k\ge l}\sum_{I_k}\sum_{j=1}^{n} a_{i_{l-1}j}a_{i_l i_{l+1}}a_{i_{l+1}i_{l+2}}\cdots a_{i_k j}|\xi_j(p_{i_1\cdots i_k j})|\\ &+\sum_{j=1}^{n} a_{i_{l-1}j}|\xi_j(p_{i_1\ldots i_{l-1}r})|+(c(\tau_2)\tau_2+2an\delta_l)(H+\varepsilon_2)|\xi^{\#}|_{C^0(0,L)}.\end{aligned}$$

Thus assertion $(\mathcal{P}_{l-1})$ holds with

$$\delta_{l-1} = c(\tau_2)\tau_2 + 2n\delta_l. \tag{65}$$

This concludes the proof of Claim 7.

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