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ON LYAPUNOV STABILITY OF LINEARISED SAINT-VENANT EQUATIONS FOR A SLOPING CHANNEL

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Abstract. We address the issue of the exponential stability (in L^2-norm) of the classical solutions of the linearised Saint-Venant equations for a sloping channel. We give an explicit sufficient dissipative condition which guarantees the exponential stability under subcritical flow conditions without additional assumptions on the size of the bottom and friction slopes. The stability analysis relies on the same strict Lyapunov function as in our previous paper [5]. The special case of a single pool is first treated. Then, the analysis is extended to the case of the boundary feedback control of a general channel with a cascade of n pools.

1. **Introduction.** Dynamics of open-water channels are usually described by Saint-Venant equations which are nonlinear PDEs representing mass and momentum balance along the channel. The Saint-Venant equations therefore constitute a so-called 2×2 system of one-dimensional balance laws. Our concern in this paper is to discuss the exponential stability (in L^2-norm) of the classical solutions of the linearised Saint-Venant equations. The stability of systems of one-dimensional conservation laws has been analyzed for a long time in the literature. The most recent results can be found in the reference [4] where it is shown that the stability is guaranteed if the Jacobian matrix of the boundary conditions satisfies an appropriate sufficient dissipativity condition. Under the same dissipativity condition, the stability is preserved for systems of balance laws that are small perturbations of conservations laws. More precisely, in the special case of Saint-Venant equations, under the assumption that the bottom and friction slopes are sufficiently small in C^1-norm, the stability may be proved using the method of characteristics as in [8]-[9] or using a

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Lyapunov approach as in [1]-[4]. Our contribution in this paper is to give a sufficient condition for exponential stability of the classical solutions for the linearised Saint-Venant equations in fluvial regime without any condition on the smallness of the bottom and friction slopes. The stability analysis relies on the same strict Lyapunov function as in our previous paper [5]. We shall first deal with the special case of a single pool. Then, the analysis will be extended to the case of the boundary feedback control of a general channel with a cascade of n pools.

2. **Saint-Venant equations.** We consider a pool of a prismatic open channel with a rectangular cross section and a constant non-zero slope. The dynamics of the system are described by the Saint-Venant equations

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} HV \\ \frac{1}{2}V^2 + gH \end{pmatrix} + \begin{pmatrix} 0 \\ g[S_f(H,V) - S_b] \end{pmatrix} = \mathbf{0}, t \in [0, +\infty), x \in [0, L],$$

with the state variables $H(t,x)$ = water depth and $V(t,x)$ = horizontal water velocity at the time instant t and the location x along the channel. L is the length of the pool, S_b the bottom slope and g the gravity constant. $S_f(H,V)$ is the so-called friction slope for which various empirical models are available in the engineering literature. In this article, we adopt the simple model

$$S_f(H,V) = C\frac{V^2}{H} \tag{1}$$

with C being a constant friction coefficient.

2.1. **Steady-state.** A steady-state (or equilibrium) is a constant state H^*, V^* that satisfies the relation

$$S_f(H^*, V^*) = S_b \qquad \text{or} \qquad S_b H^* = C(V^*)^2.$$

2.2. **Linearisation.** In order to linearise the model, we define the deviations of the states $H(t,x)$ and $V(t,x)$ with respect to the steady-states H^* and V^* :

$$h(t,x) \triangleq H(t,x) - H^*, \qquad v(t,x) \triangleq V(t,x) - V^*.$$

Then the linearised Saint-Venant equations around the steady-state are

$$\begin{aligned} &\partial_t h + V^* \partial_x h + H^* \partial_x v = 0, \\ &\partial_t v + g\partial_x h + V^* \partial_x v - \left[\frac{g}{H^*} S_b\right] h + \left[\frac{2g}{V^*} S_b\right] v = 0. \end{aligned}$$

2.3. **Characteristic form.** The characteristic coordinates are defined as follows:

$$\begin{aligned} \xi_1(t,x) &= v(t,x) + h(t,x)\sqrt{\frac{g}{H^*}}, \\ \xi_2(t,x) &= v(t,x) - h(t,x)\sqrt{\frac{g}{H^*}}, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} h(t,x) &= \frac{\xi_1(t,x) - \xi_2(t,x)}{2}\sqrt{\frac{H^*}{g}}, \\ v(t,x) &= \frac{\xi_1(t,x) + \xi_2(t,x)}{2}. \end{aligned}$$

With these definitions and notations, the linearised Saint-Venant equations are written in characteristic form:

$$\begin{cases} \partial_t \xi_1(t,x) + \lambda_1 \partial_x \xi_1(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0, \\ \\ \partial_t \xi_2(t,x) - \lambda_2 \partial_x \xi_2(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0. \end{cases} \tag{2}$$

with the characteristic velocities

$$\lambda_1 = V^* + \sqrt{gH^*}, \qquad -\lambda_2 = V^* - \sqrt{gH^*},$$

and the parameters

$$\gamma = gS_b\left(\frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right), \qquad \delta = gS_b\left(\frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right).$$

2.4. **Subcritical flow.** The steady-state flow is subcritical (or fluvial) if the following condition holds

$$gH^* - (V^*)^2 > 0. \tag{3}$$

Under this condition, the system is strictly hyperbolic with

$$-\lambda_2 = V^* - \sqrt{gH^*} < 0 < \lambda_1 = V^* + \sqrt{gH^*}.$$

It is worth noting that, under the subcritical flow condition (3), the parameters γ and δ are strictly positive such that $0 < \gamma < \delta$.

3. **Lyapunov stability analysis for a single pool.** In this section, we present a Lyapunov function which is used to analyse the asymptotic convergence of the classical solutions of the linearised model (2) under linear boundary conditions. This Lyapunov function has been previously introduced for analysing the stability of systems of **conservation** laws (see [5]). Here we shall show how it can be applied to the Saint-Venant **balance** laws. For the sake of simplicity, the arguments "t" and 'x" are sometimes omitted in the calculations.

We consider the system (2) with $t \in [0, +\infty)$ and $x \in [0, L]$, under linear boundary conditions of the form

$$\begin{pmatrix} \xi_1(t,0) \\ \xi_2(t,L) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \xi_1(t,L) \\ \xi_2(t,0) \end{pmatrix}, \qquad t \in [0, +\infty), \tag{4}$$

and an initial condition

$$[\xi_1(0,x) = \xi_1^o(x),\ \xi_2(0,x) = \xi_2^o(x)] \in L^2((0,L);\mathbb{R}^2). \tag{5}$$

The Cauchy problem (2)-(4)-(5) is well-posed (see e.g. [3, Section 2.1 and Section 2.3]). This means that for any initial condition $(\xi_1^o, \xi_2^o) \in L^2((0,L);\mathbb{R}^2)$ and for every $T > 0$, there exists $C(T) > 0$ such that a solution $(\xi_1(t,x), \xi_2(t,x)) \in C^0([0,+\infty); L^2((0,1);\mathbb{R}^2))$ exists, is unique and satisfies

$$\|(\xi_1(t,\cdot), \xi_2(t,\cdot))\|_{L^2((0,1);\mathbb{R}^2)} \leqslant C(T)\|(\xi_1^o, \xi_2^o)\|_{L^2((0,1);\mathbb{R}^2)},\ \forall t \in [0,T].$$

Our concern is to analyse the exponential stability of the system (2)-(4) according to the following definition.

Definition 3.1. The system (2)-(4) is exponentially stable (in L^2-norm) if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition $(\xi_1^o(x), \xi_2^o(x)) \in L^2((0,L);\mathbb{R}^2)$ the solution to the Cauchy problem (2)-(4)-(5) satisfies

$$\|(\xi_1(t,\cdot), \xi_2(t,\cdot))\|_{L^2((0,1);\mathbb{R}^2)} \leqslant Ce^{-\nu t}\|(\xi_1^o, \xi_2^o)\|_{L^2((0,1);\mathbb{R}^2)}.$$

For the stability analysis of the system (2)-(4), the following candidate Lyapunov function is considered:

$$V = \int_0^L \left(\xi_1^2 p_1 e^{-\mu x} + \xi_2^2 p_2 e^{\mu x}\right) dx, \qquad p_1, p_2, \mu > 0.$$

After some calculations, the time derivative of this Lyapunov function is

$$\dot{V} = \dot{V}_1 + \dot{V}_2$$

with

$$\dot{V}_1 = -\Big[\lambda_1 \xi_1^2 p_1 e^{-\mu x} - \lambda_2 \xi_2^2 p_2 e^{\mu x}\Big]_0^L, \tag{6a}$$

$$\dot{V}_2 = -\int_0^L (\xi_1 \;\; \xi_2) \begin{pmatrix} (\lambda_1\mu + 2\gamma)p_1 e^{-\mu x} & \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} \\ \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} & (\lambda_2\mu + 2\delta)p_2 e^{\mu x} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dx. \tag{6b}$$

It follows that the exponential stability is guaranteed if $\dot{V}_1$ and $\dot{V}_2$ are negative definite quadratic forms.

Using the boundary condition (4) with the notations $\xi_{1,2}(0) \triangleq \xi_{1,2}(t,0)$ and $\xi_{1,2}(L) \triangleq \xi_{1,2}(t,L)$, the stability condition relative to $\dot{V}_1$ becomes

$$\begin{aligned}
\dot{V}_1 &= -\Big[\lambda_1 \xi_1^2 p_1 e^{-\mu x} - \lambda_2 \xi_2^2 p_2 e^{\mu x}\Big]_0^L \\
&= -\Big[\lambda_1 p_1 \xi_1^2(L) e^{-\mu L} \\
&\quad + \lambda_2 p_2 \xi_2^2(0)\Big] + \lambda_1 p_1 \Big(k_{11}\xi_1(L) + k_{12}\xi_2(0)\Big)^2 + \lambda_2 p_2 \Big(k_{21}\xi_1(L) + k_{22}\xi_2(0)\Big)^2 e^{\mu L} \\
&< 0 \text{ for } (\xi_1(L), \xi_2(0)) \neq (0,0).
\end{aligned} \tag{7}$$

The stability condition relative to $\dot{V}_2$ is

$$\begin{pmatrix} (\lambda_1\mu + 2\gamma)p_1 e^{-\mu x} & \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} \\ \delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x} & (\lambda_2\mu + 2\delta)p_2 e^{\mu x} \end{pmatrix} > 0$$

$$\iff \boxed{\begin{array}{c} \textbf{(a)} \quad \lambda_1\mu + 2\gamma > 0, \qquad \lambda_2\mu + 2\delta > 0, \\ \\ \textbf{(b)} \quad p_1 p_2 (\lambda_1\mu + 2\gamma)(\lambda_2\mu + 2\delta) - (\delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x})^2 > 0. \end{array}} \tag{8}$$

The goal is now to determine conditions on the parameters k_{ij}, $p_1 > 0$, $p_2 > 0$ and $\mu > 0$ such that these sufficient stability conditions are satisfied.

Condition (8)-**(a)** is satisfied for any $\mu > 0$.

We are going to show that condition (8)-**(b)** is satisfied for sufficiently small $\mu > 0$ if the parameters p_1, p_2 are selected such that $\delta p_1 = \gamma p_2$. Indeed under this condition, the term $(\delta p_1 e^{-\mu x} + \gamma p_2 e^{\mu x})^2$ in condition (8)-**(b)** viewed as a function of x is maximum at $x = L$. For $x = L$, we then have

$$\begin{aligned}
&p_1 p_2 (\lambda_1\mu + 2\gamma)(\lambda_2\mu + 2\delta) - (\delta p_1 e^{-\mu L} + \gamma p_2 e^{\mu L})^2 \\
&= p_1 p_2 \Big[\mu^2 \lambda_1 \lambda_2 + 2\mu(\lambda_1\delta + \lambda_2\gamma)\Big] - (\delta p_1 e^{-\mu L} - \gamma p_2 e^{\mu L})^2 > 0
\end{aligned}$$

for $\mu > 0$ sufficiently small (because the function $F(\mu) \triangleq (\delta p_1 e^{-\mu L} - \gamma p_2 e^{\mu L})^2$ is quadratic in μ since $F(0) = 0$ and $F'(0) = 0$ if $\delta p_1 = \gamma p_2$).

We are now going to show that condition (7) is satisfied for sufficiently small $\mu > 0$ if

$$\left\| \begin{pmatrix} k_{11} & k_{12}\sqrt{\dfrac{\lambda_1\gamma}{\lambda_2\delta}} \\ k_{21}\sqrt{\dfrac{\lambda_2\delta}{\lambda_1\gamma}} & k_{22} \end{pmatrix} \right\| < 1. \tag{9}$$

Using $\delta p_1 = \gamma p_2$ and inequality (9), we have for $(\xi_1(L), \xi_2(0)) \neq (0,0)$

$$\lambda_1 p_1 \Big(k_{11}\xi_1(L) + k_{12}\xi_2(0)\Big)^2 + \lambda_2 p_2 \Big(k_{21}\xi_1(L) + k_{22}\xi_2(0)\Big)^2$$

$$= \left\| \begin{pmatrix} k_{11} & k_{12}\sqrt{\dfrac{\lambda_1\gamma}{\lambda_2\delta}} \\ k_{21}\sqrt{\dfrac{\lambda_2\delta}{\lambda_1\gamma}} & k_{22} \end{pmatrix} \begin{pmatrix} \xi_1(L)\sqrt{\lambda_1 p_1} \\ \xi_2(0)\sqrt{\lambda_2 p_2} \end{pmatrix} \right\|^2$$

$$< \left\| \begin{pmatrix} \xi_1(L)\sqrt{\lambda_1 p_1} \\ \xi_2(0)\sqrt{\lambda_2 p_2} \end{pmatrix} \right\|^2 = \Big[\lambda_1 p_1 \xi_1^2(L) + \lambda_2 p_2 \xi_2^2(0)\Big].$$

Comparing this inequality with (7), it follows that (7) is satisfied for μ sufficiently small. Hence, if p_1 and p_2 are selected such that $\delta p_1 = \gamma p_2$, if the parameters k_{ij} are selected such that inequality (9) holds, then, for sufficiently small μ, V is a Lyapunov function along the solutions of the linearised Saint-Venant equations and the steady-state is exponentially stable.

4. **Boundary feedback control for a channel with a cascade of n pools.** In navigable rivers or irrigation channels (see e.g. [2],[7]) the water is transported along the channel under the power of gravity through successive pools separated by automated gates that are used to regulate the water flow, as illustrated in Figures 1 and 2. We consider a channel with n pools the dynamics of which being described

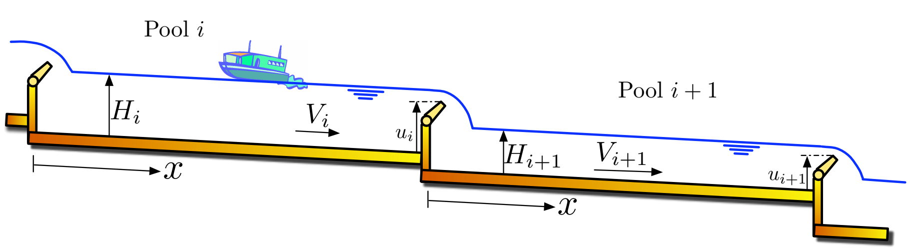

FIGURE 1. Lateral view of successive pools of an open-water channel with overflow gates.

by Saint-Venant equations

$$\partial_t \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \partial_x \begin{pmatrix} H_i V_i \\ \frac{1}{2}V_i^2 + gH_i \end{pmatrix} + \begin{pmatrix} 0 \\ g[C_i V_i^2 H_i^{-1} - S_i] \end{pmatrix} = \mathbf{0}, \qquad i = 1\ldots, n. \tag{10}$$

FIGURE 2. Automated control gates in the Sambre river (Belgium). The left gate is in operation. The right gate is lifted for maintenance. (©L.Moens)

In this model, for simplicity, we assume that all the pools have a rectangular section with the same width W. System (10) is subject to a set of $2n$ boundary conditions that are distributed into three subsets:

1) A first subset of $n-1$ conditions expresses the natural physical constraint of flow-rate conservation between the pools (the flow that exits pool i is (evidently) equal to the flow that enters pool $i+1$):
$$H_i(t,L)V_i(t,L) = H_{i+1}(t,0)V_{i+1}(t,0), \qquad i = 1,\dots,n-1. \tag{11}$$
2) A second subset of n boundary condition is made up of the equations that describe the gate operations. A standard gate model is given by the algebraic relation
$$H_i(t,L)V_i(t,L) = k_G\sqrt{\big[H_i(t,L) - u_i(t)\big]^3}, \qquad i = 1,\dots,n. \tag{12}$$
where k_G is a positive constant coefficient and $u_i(t)$ denotes the weir elevation which is a control input (see Fig.1).
3) The last boundary condition imposes the value of the canal inflow rate that we denote $Q_0(t)$:
$$WH_1(t,0)V_1(t,0) = Q_0(t). \tag{13}$$
Depending on the application, $Q_0(t)$ may be viewed as a control input (in irrigation channels) or as a disturbance input (in navigable rivers).

A steady-state (or equilibrium) is a constant state H_i^*, V_i^* ($i = 1,\dots,n$) that satisfies the relations
$$S_iH_i^* = C(V_i^*)^2, \qquad i = 1,\dots,n.$$
The subcritical flow condition is
$$gH_i^* - (V_i^*)^2 > 0, \qquad i = 1,\dots,n. \tag{14}$$

4.1. **Boundary control design.** From Section 2, we know that the characteristic state variables of system (10) are

$$\xi_i = (V_i - V_i^*) + (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}}, \quad \xi_{n+i} = (V_i - V_i^*) - (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}}, \quad i = 1, \dots, n. \tag{15}$$

Motivated by the stability analysis of the previous section, and in particular by the relation (4), we now assume that we want the boundary conditions to satisfy the following relations in characteristic coordinates:

$$\xi_{n+i}(t, L) = -k_i \xi_i(t, L), \qquad i = 1, \dots, n, \tag{16}$$

with k_i control tuning parameters. Then, eliminating ξ_i ($i = 1, \dots, 2n$) and V_i ($i = 1, \dots, n$) between (12), (15) and (16) we get the following expressions for boundary feedback control laws that realise the target boundary conditions (16):

$$u_i = H_i(t, L) - \left[\frac{H_i(t, L)}{k_G} \left(\frac{1 - k_i}{1 + k_i} (H_i(t, L) - H_i^*) \sqrt{\frac{g}{H_i^*}} + \sqrt{\frac{S_i H_i^*}{C}} \right) \right]^{2/3}. \tag{17}$$

It can be seen that these control laws have the form of a state feedback. In addition, it is remarkable that the implementation of the controls is particularly simple since only measurements of the water levels $H_i(t, L)$ at the gates are required. This means that the feedback implementation does not require neither level measurements inside the pools nor any velocity or flow rate measurements.

4.2. **Closed loop stability analysis.** We consider the closed loop system with a constant inflow rate $Q_0(t) = Q^*$. We are going to explicit sufficient conditions on the control tuning parameters k_i that guarantee the dissipativity of the boundary conditions and therefore the exponential stability of the steady-state.

As motivated in Section 2, the linearised Saint-Venant equations are written in characteristic form:

$$\begin{cases} \partial_t \xi_i(t, x) + \lambda_i \partial_x \xi_i(t, x) + \gamma_i \xi_i(t, x) + \delta_i \xi_{n+i}(t, x) = 0, \\ \partial_t \xi_{n+i}(t, x) - \lambda_{n+i} \partial_x \xi_{n+i}(t, x) + \gamma_i \xi_i(t, x) + \delta_i \xi_{n+i}(t, x) = 0, \end{cases} \quad i = 1, \dots, n \tag{18}$$

with the characteristic velocities

$$\lambda_i = V_i^* + \sqrt{g H_i^*}, \qquad -\lambda_{n+i} = V_i^* - \sqrt{g H_i^*}, \qquad i = 1, \dots, n$$

and the parameters

$$\gamma_i = g S_i \left(\frac{1}{V_i^*} - \frac{1}{2\sqrt{g H_i^*}} \right), \qquad \delta_i = g S_i \left(\frac{1}{V_i^*} + \frac{1}{2\sqrt{g H_i^*}} \right), \qquad i = 1, \dots, n$$

such that $0 < \lambda_{n+i} < \lambda_i$ and $0 < \gamma_i < \delta_i$. We introduce the notations $\boldsymbol{\xi}^+ \triangleq (\xi_1, \dots, \xi_n)^T$ and $\boldsymbol{\xi}^- \triangleq (\xi_{n+1}, \dots, \xi_{2n})^T$. The linearisation of the boundary conditions is expressed in the form

$$\begin{pmatrix} \boldsymbol{\xi}^+(t, 0) \\ \boldsymbol{\xi}^-(t, L)) \end{pmatrix} = \underbrace{\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} \boldsymbol{\xi}^+(t, L) \\ \boldsymbol{\xi}^-(t, 0) \end{pmatrix}. \tag{19}$$

The matrices K_{10} and K_{11} are directly given by conditions (16):

$$K_{10} = \mathrm{diag}\{-k_i, i = 1, \ldots, n\}, \qquad K_{11} = 0.$$

Straightforward calculations from (11) and (13) show that the matrix K_{01} is

$$K_{01} = \mathrm{diag}\{-\frac{\lambda_{n+i}}{\lambda_i}, i = 1, \ldots, n\}.$$

Moreover, K_{00} computed from (11) and (16) is the $n \times n$ matrix with entries

$$K_{00}[i+1, i] = \frac{(\lambda_i - k_i\lambda_{n+i})}{\lambda_{i+1}}\sqrt{\frac{H_i^*}{H_{i+1}^*}} \quad \text{and 0 elsewhere.}$$

Our main stability result is given in the following theorem.

Theorem 4.1. *If the control tuning parameters satisfy the dissipativity condition*

$$k_i^2 < \frac{\gamma_i \lambda_i}{\delta_i \lambda_{n+i}} \tag{20}$$

then the steady-state H_i^*, V_i^* ($i = 1, \ldots, n$) of the closed-loop system (18)-(19) is exponentially stable.

Proof. The following candidate Lyapunov function is defined:

$$V = \int_0^L \Big[(\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\xi}^+)e^{-\mu x} + (\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\xi}^-)e^{\mu x}\Big]\, dx \tag{21}$$

with $P_0 = \mathrm{diag}\{p_i > 0, i = 1, \ldots, n\}$, $P_1 = \mathrm{diag}\{p_{n+i} > 0, i = 1, \ldots, n\}$ and $\mu > 0$. The time derivative of V is

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \tag{22}$$

with

$$\dot{V}_1 = -\Big[\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\Lambda}^+ \boldsymbol{\xi}^+ e^{-\mu x}\Big]_0^L + \Big[\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\Lambda}^- \boldsymbol{\xi}^- e^{\mu x}\Big]_0^L,$$

$$\dot{V}_2 = -\sum_{i=1}^n \int_0^L (\xi_i \;\; \xi_{n+i}) \mathbf{G}_i \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} dx$$

and

$$\mathbf{G}_i = \begin{pmatrix} (\lambda_i \mu + 2\gamma_i) p_i e^{-\mu x} & \delta_i p_i e^{-\mu x} + \gamma_i p_{n+i} e^{\mu x} \\ \\ \delta_i p_i e^{-\mu x} + \gamma_i p_{n+i} e^{\mu x} & (\lambda_{n+i}\mu + 2\delta_i) p_{n+i} e^{\mu x} \end{pmatrix}.$$

In order to prove the theorem we will show that P_0, P_1 and μ can be selected such that $\dot{V}_1$ and $\dot{V}_2$ are negative definite quadratic forms.

First, it is a direct extension of our analysis of condition (8) in Section 2 that $\dot{V}_2$ is a negative definite quadratic form if the coefficients p_i are selected such that $\delta_i p_i = \gamma_i p_{n+i}$ ($i = 1, \ldots, n$) and μ is sufficiently small.

For the analysis of $\dot{V}_1$, we introduce the following notations:

$$\boldsymbol{\Lambda}^+ = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}, \qquad \boldsymbol{\Lambda}^- = \mathrm{diag}\{\lambda_{n+1}, \ldots, \lambda_{2n}\}.$$

Then we define the positive diagonal matrices D_0, D_1 and Δ such that $P_0\boldsymbol{\Lambda}^+ = D_0^2$, $P_1\boldsymbol{\Lambda}^- = D_1^2$ and $\Delta = \mathrm{diag}\{D_0, D_1\}$. Hence, using the condition $\delta_i p_i = \gamma_i p_{n+i}$ ($i = 1, \ldots, n$), the matrix $\Delta\mathbf{K}\Delta^{-1}$ can be shown to be

$$\Delta\mathbf{K}\Delta^{-1} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

with

$$A_{11} = 0, \qquad A_{01} = \operatorname{diag}\{-\sqrt{\frac{\gamma_i \lambda_{n+i}}{\delta_i \lambda_i}}, i = 1, \dots, n\},$$

$$A_{10} = \operatorname{diag}\{-k_i \sqrt{\frac{\delta_i \lambda_{n+i}}{\gamma_i \lambda_i}}, i = 1, \dots, n\},$$

while A_{00} is the $n \times n$ matrix with entries

$$A_{00}[i+1, i] = \frac{(\lambda_i - k_i \lambda_{n+i})}{\lambda_{i+1}} \sqrt{\frac{H_i^*}{H_{i+1}^*}} \sqrt{\frac{p_{i+1}\lambda_{i+1}}{p_i \lambda_i}} \quad \text{and} \quad 0 \text{ elsewhere}.$$

We now assume that the parameters p_i are selected such that $p_{i+1} = \varepsilon p_i$ $(i = 1, \dots, n-1)$. Then, for the special case $\varepsilon = 0$ we have $A_{00} = 0$ and it is easy to check that the singular values of the matrix $\Delta \mathbf{K} \Delta^{-1}$ reduce to the absolute values of the non-zero entries of the matrices A_{10} and A_{01}:

$$\sigma_i = |k_i| \sqrt{\frac{\delta_i \lambda_{n+i}}{\gamma_i \lambda_i}}, \qquad \sigma_{n+i} = \sqrt{\frac{\gamma_i \lambda_{n+i}}{\delta_i \lambda_i}}, \qquad i = 1, \dots, n.$$

It follows immediately that

- $\sigma_i < 1$, $(i = 1, \dots, n)$ because of the dissipativity condition (20);
- $\sigma_{n+i} < 1$, $(i = 1, \dots, n)$ because of the subcritical flow condition (14) which implies that $0 < \lambda_{n+i} < \lambda_i$ and $0 < \gamma_i < \delta_i$.

Since the norm of a matrix is its largest singular value, it follows by continuity that for $\varepsilon > 0$ sufficiently small we have

$$\|\Delta \mathbf{K} \Delta^{-1}\| < 1. \tag{23}$$

We now introduce the notations

$$\boldsymbol{\xi}_0^-(t) \triangleq \boldsymbol{\xi}^-(t, 0) \qquad \boldsymbol{\xi}_1^+(t) \triangleq \boldsymbol{\xi}^+(t, L).$$

Using the boundary condition (19), we have

$$\begin{aligned}
\dot{V}_1 &= -\left[\boldsymbol{\xi}^{+T} P_0 \boldsymbol{\Lambda}^+ \boldsymbol{\xi}^+ e^{-\mu x}\right]_0^L + \left[\boldsymbol{\xi}^{-T} P_1 \boldsymbol{\Lambda}^- \boldsymbol{\xi}^- e^{\mu x}\right]_0^L \\
&= -\left(\boldsymbol{\xi}_1^{+T} P_0 \boldsymbol{\Lambda}^+ \boldsymbol{\xi}_1^+ \, e^{-\mu L} + \boldsymbol{\xi}_0^{-T} P_1 \boldsymbol{\Lambda}^- \boldsymbol{\xi}_0^-\right) \\
&\quad + \left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T\right) P_0 \boldsymbol{\Lambda}^+ \left(K_{00} \boldsymbol{\xi}_1^+ + K_{01} \boldsymbol{\xi}_0^-\right) \\
&\quad + \left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T\right) P_1 \boldsymbol{\Lambda}^- \left(K_{10} \boldsymbol{\xi}_1^+ + K_{11} \boldsymbol{\xi}_0^-\right) e^{\mu L}.
\end{aligned}$$

We define $\mathbf{z}_0 \triangleq D_0 \boldsymbol{\xi}_0^-$, $\mathbf{z}_1 \triangleq D_1 \boldsymbol{\xi}_1^+$ and $\mathbf{z}^T \triangleq (\mathbf{z}_0^T, \mathbf{z}_1^T)$. Then, using inequality (23) we have

$$\begin{aligned}
&\left(\boldsymbol{\xi}_1^{+T} K_{00}^T + \boldsymbol{\xi}_0^{-T} K_{01}^T\right) P_0 \boldsymbol{\Lambda}^+ \left(K_{00} \boldsymbol{\xi}_1^+ + K_{01} \boldsymbol{\xi}_0^-\right) \\
+ &\left(\boldsymbol{\xi}_1^{+T} K_{10}^T + \boldsymbol{\xi}_0^{-T} K_{11}^T\right) P_1 \boldsymbol{\Lambda}^- \left(K_{10} \boldsymbol{\xi}_1^+ + K_{11} \boldsymbol{\xi}_0^-\right) \\
= &\|\Delta \mathbf{K} \Delta^{-1} \mathbf{z}\|^2 \; < \; \|\mathbf{z}\|^2 = \; \boldsymbol{\xi}_1^{+T} P_0 \boldsymbol{\Lambda}^+ \boldsymbol{\xi}_1^+ + \boldsymbol{\xi}_0^{-T} P_1 \boldsymbol{\Lambda}^- \boldsymbol{\xi}_0^-.
\end{aligned}$$

It follows that μ can be taken sufficiently small such that $\dot{V}_1$ is a negative definite quadratic form.

We may now summarize the proof as follows : if the parameters $p_i > 0$ are selected such that $\delta_i p_i = \gamma_i p_{n+i}$ $(i = 1, \dots, n)$ and $p_{i+1} = \varepsilon p_i$ $(i = 1, \dots, n-1)$, if $\varepsilon > 0$ and $\mu > 0$ are taken sufficiently small, then, under the subcritical flow condition (14) and the dissipativity condition (20), V is a Lyapunov function for the closed-loop system and the steady-state is exponentially stable. □

4.3. **Example.** Let us consider a channel made up of n identical pools with the following geometric and steady-state flow characteristics:

Length L_i=1000 m, Width W=80 m, Slope S_i=0.0002, Friction coefficient C_i=0.001 sec^2/m, Steady-state flow rate Q^*=400 m^3/sec.

From these values, we compute the steady-state water depth H_i^* and velocity V_i^* that satisfy the steady-state relations:

$$S_i H_i^* = C_i V_i^{*2}, \quad Q^* = W H_i^* V_i^* \quad \Longrightarrow \quad H_i^* = 5\text{m}, \quad V_i^* = 1\text{m/sec}.$$

We then have the following values for the parameters of the linearised flow model:

$\lambda_i = 8$ m/sec, $\lambda_{n+i} = 6$ m/sec, $\gamma_i = 0.0018$ sec^{-1}, $\delta_i = 0.0021$ sec^{-1}.

The convergence is guaranteed if $|k_i| < 1.069\ \forall i$.

5. **Conclusions.** In this paper we have given an explicit sufficient dissipative condition which guarantees the exponential stability, under subcritical flow conditions, of the linearised Saint-Venant equations for an open-channel with multiple sloping pools in cascade. For the case of a single pool, the issue is also discussed using a frequency domain approach in [6] where an implicit stability condition is derived with the help of the Nyquist stability criterion.

It must be stressed that the exponential stability of the linearised equations does not automatically imply the local exponential stability of the equilibrium for the **nonlinear** Saint-Venant equations. It can however be reasonably conjectured that such an implication can be proved by using an approach similar to that used in [4] for systems of conservation laws.

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