Automatica 48 (2012) 109–114

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws[☆]

Ababacar Diagne[a], Georges Bastin[b,1], Jean-Michel Coron[c]

[a] UFR de Sciences Appliquées et Technologies, Université Gaston Berger, BP 234, Saint-Louis, Sénégal
[b] Department of Mathematical Engineering, Université catholique de Louvain, 4, Avenue G. Lemaitre, 1348 Louvain-La-Neuve, Belgium
[c] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Institut Universitaire de France, 4, Place Jussieu, 75252 Paris Cedex, France

ARTICLE INFO

Article history:
Received 28 November 2010
Received in revised form
3 April 2011
Accepted 15 June 2011
Available online 17 October 2011

Keywords:
Hyperbolic systems
Lyapunov function
Saint–Venant equations
Stabilisation

ABSTRACT

Explicit boundary dissipative conditions are given for the exponential stability in L^2-norm of one-dimensional linear hyperbolic systems of balance laws $\partial_t \boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x \boldsymbol{\xi} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ over a finite interval, when the matrix $\mathbf{M}$ is marginally diagonally stable. The result is illustrated with an application to boundary feedback stabilisation of open channels represented by linearised Saint–Venant–Exner equations.

1. Introduction

Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g. Serre, 2001). Many physical systems having an engineering interest are described by systems of one-dimensional hyperbolic balance laws. Typical examples are the telegrapher equations for electrical lines, the shallow water (Saint–Venant) equations for open channels, the Euler equations for gas flow in pipelines or the Aw-Rascle equations for road traffic. In this paper, our concern is to analyse the exponential stability (in the sense of Lyapunov) of the steady-states of such systems. The analysis is developed for a general class of linear systems of one-dimensional hyperbolic balance laws. As a matter of illustration, an application to linearised Saint–Venant–Exner equations for open channels with a moving sediment bed is presented.

We are concerned with $n \times n$ linear hyperbolic systems of balance laws of the form:

$$\partial_t \boldsymbol{\xi} + \boldsymbol{\Lambda}\partial_x \boldsymbol{\xi} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0} \quad t \in [0, +\infty),\ x \in (0, L) \tag{1}$$

where $\boldsymbol{\xi} : [0, +\infty) \times [0, L] \to \mathbb{R}^n$, $\boldsymbol{\Lambda}$ and $\mathbf{M}$ are real $n \times n$ matrices. Without loss of generality, we may assume that $\boldsymbol{\Lambda}$ is diagonal with non-zero real diagonal entries such that

$$\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\},$$
$$\lambda_i > 0 \quad \forall i \in \{1, \dots, m\},$$
$$\lambda_i < 0 \quad \forall i \in \{m+1, \dots, n\}.$$

We introduce the notations

$$\boldsymbol{\xi}^+ = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \qquad \boldsymbol{\xi}^- = \begin{pmatrix} \xi_{m+1} \\ \vdots \\ \xi_n \end{pmatrix} \qquad \text{such that } \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix},$$

and

$$\begin{cases} \boldsymbol{\Lambda}^+ = \mathrm{diag}\{\lambda_1, \dots, \lambda_m\}, \\ \boldsymbol{\Lambda}^- = \mathrm{diag}\{|\lambda_{m+1}|, \dots, |\lambda_n|\}, \end{cases}$$

such that

$$\begin{cases} \boldsymbol{\Lambda} = \mathrm{diag}\{\boldsymbol{\Lambda}^+, -\boldsymbol{\Lambda}^-\}, \\ |\boldsymbol{\Lambda}| = \mathrm{diag}\left\{\boldsymbol{\Lambda}^+, \boldsymbol{\Lambda}^-\right\}. \end{cases}$$

With these notations, the linear hyperbolic system (1) is written as

$$\partial_t \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Lambda}^+ & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\Lambda}^- \end{pmatrix} \partial_x \begin{pmatrix} \boldsymbol{\xi}^+ \\ \boldsymbol{\xi}^- \end{pmatrix} - \mathbf{M}\boldsymbol{\xi} = \mathbf{0}. \tag{2}$$

Our concern is to analyse the exponential stability of this system under boundary conditions of the form

$$\begin{pmatrix} \boldsymbol{\xi}^+(t, 0) \\ \boldsymbol{\xi}^-(t, L) \end{pmatrix} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}^+(t, L) \\ \boldsymbol{\xi}^-(t, 0) \end{pmatrix} \tag{3}$$

☆ Research supported by the Belgian Programme on Interuniversity Attraction Poles (IAP V/22), by the "Agence Nationale de la Recherche" (ANR), Project C-QUID, number BLAN-3-139579, France, and by the "International Science Programme", Sweden. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Nicolas Petit under the direction of Editor Miroslav Krstic. This paper was prepared when the first author was a visiting Ph.D. student at Louvain University.

E-mail addresses: babacar16@yahoo.fr (A. Diagne), Georges.Bastin@uclouvain.be (G. Bastin), coron@ann.jussieu.fr (J.-M. Coron).
[1] Tel.: +32 10478038; fax: +32 10472180.

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.09.030

and an initial condition of the form

$$\boldsymbol{\xi}(0,x)=\boldsymbol{\xi}^o(x), \quad x\in(0,L). \tag{4}$$

The classical definition of a solution to the Cauchy problem (2)–(4) in $L^2((0,L);\mathbb{R}^n)$ is

Definition 1. Let $\boldsymbol{\xi}^0 \in L^2((0,L);\mathbb{R}^n)$. A map $\boldsymbol{\xi}:[0,+\infty)\times(0,L)\to\mathbb{R}^n$ is a solution of the Cauchy problem (2)–(4) if $\boldsymbol{\xi}\in C^0([0,+\infty);L^2((0,L);\mathbb{R}^n))$ is such that, for every $\varphi=(\varphi_+^T,\varphi_-^T)^T \in C^1([0,+\infty)\times[0,L];\mathbb{R}^n)$ with compact support and satisfying

$$\begin{pmatrix}\varphi_+(t,L)\\ \varphi_-(t,0)\end{pmatrix} = \begin{pmatrix}(\boldsymbol{\Lambda}^+)^{-1}K_{00}^T\boldsymbol{\Lambda}^+ & (\boldsymbol{\Lambda}^+)^{-1}K_{10}^T\boldsymbol{\Lambda}^-\\ (\boldsymbol{\Lambda}^-)^{-1}K_{01}^T\boldsymbol{\Lambda}^+ & (\boldsymbol{\Lambda}^-)^{-1}K_{11}^T\boldsymbol{\Lambda}^-\end{pmatrix}\begin{pmatrix}\varphi_+(t,0)\\ \varphi_-(t,L)\end{pmatrix}$$

we have

$$\int_0^{+\infty}\int_0^L(\varphi_t^T+\varphi_x^T\boldsymbol{\Lambda}+\varphi^T\mathbf{M})\boldsymbol{\xi}dxdt + \int_0^L\varphi^T(0,x)\boldsymbol{\xi}^0(x)dx=0.$$

With this definition, we have the following classical result (see e.g. Coron, 2007, Sections 2.1 and 2.3 for methods to get this result).

Proposition 1. *For every $\boldsymbol{\xi}^0\in L^2((0,L);\mathbb{R}^n)$, the Cauchy problem (2)–(4) has a unique solution. Moreover, for every $T>0$, there exists $C(T)>0$ such that, for every $\boldsymbol{\xi}^0\in L^2((0,L);\mathbb{R}^n)$, the solution to the Cauchy problem (2)–(4) satisfies*

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^n)}\leqslant C(T)\|\boldsymbol{\xi}^0\|_{L^2((0,L);\mathbb{R}^n)}, \quad \forall t\in[0,T].$$

We adopt the following definition for the exponential stability of the linear hyperbolic system (2)–(3).

Definition 2. The linear hyperbolic system (2)–(3) is exponentially stable if there exist $\nu>0$ and $C>0$ such that, for every $\boldsymbol{\xi}^0\in L^2((0,L);\mathbb{R}^n)$, the solution to the Cauchy problem (2)–(4) satisfies

$$\|\boldsymbol{\xi}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^n)}\leqslant Ce^{-\nu t}\|\boldsymbol{\xi}^0\|_{L^2((0,L);\mathbb{R}^n)}, \quad \forall t\in[0,+\infty).$$

The problem of analysing the exponential stability of the equilibrium $\boldsymbol{\xi}\equiv\mathbf{0}$ for nonlinear systems of *conservation* laws $\partial_t\boldsymbol{\xi}+\mathbf{C}(\boldsymbol{\xi})\partial_x\boldsymbol{\xi}=\mathbf{0}$ has been considered in the literature for more than 25 years. To our knowledge, first results were published in Greenberg and Li (1984) and Slemrod (1983) for the special case of 2×2 systems. A generalisation to $n\times n$ systems of conservation laws was given in Li (1994) and was recently extended to the case of conservation laws with a small perturbation source term in Prieur, Winkin, and Bastin (2008). All these results rely on the method of characteristics and establish the exponential convergence of the solutions in $C^1(0,L)$-norm under suitable dissipative boundary conditions.

In a different approach, a strict Lyapunov function introduced in Coron (1999) was used in Coron, Bastin, and d'Andréa-Novel (2008); Coron, d'Andréa-Novel, and Bastin (2007) in order to analyse the exponential stability of the equilibrium of nonlinear systems of conservation laws in $H^2(0,L)$-norm. This Lyapunov approach has also been used in Xu and Sallet (2002) to analyse the exponential stability of linear hyperbolic systems of *balance* laws of the form (1) in the special case where the matrix $\mathbf{M}$ is symmetric. The same kind of Lyapunov function is also considered in Gugat and Herty (2011) for the special case of gas pipelines represented by isentropic Euler equations.

In the present paper, our main contribution is to explain how this Lyapunov stability analysis can be further extended to the case of linear hyperbolic systems of the form (1). In Theorem 1 we first give a general implicit formulation of sufficient stability conditions. Then in Theorem 2, we show that, when the matrix $\mathbf{M}$ is diagonally marginally stable, an explicit boundary dissipativity condition holds for exponential stability in $L^2(0,L)$-norm. Finally, in Section 4, we present an application to the boundary feedback stabilisation of open channels represented by linearised Saint–Venant–Exner equations.

2. Lyapunov stability: general sufficient conditions

The system (2)–(4) is rewritten as

$$\partial_t\boldsymbol{\xi}+\boldsymbol{\Lambda}\partial_x\boldsymbol{\xi}-\mathbf{M}\boldsymbol{\xi}=\mathbf{0} \quad t\in[0,+\infty),\ x\in(0,L), \tag{5a}$$

$$\mathbf{K}_0\boldsymbol{\xi}(t,0)+\mathbf{K}_1\boldsymbol{\xi}(t,L)=\mathbf{0}, \quad t\in[0,+\infty), \tag{5b}$$

$$\boldsymbol{\xi}(0,x)=\boldsymbol{\xi}^o(x), \quad x\in(0,L) \tag{5c}$$

with

$$\mathbf{K}_0:=\begin{pmatrix}I & -K_{01}\\ \mathbf{0} & -K_{11}\end{pmatrix}, \qquad \mathbf{K}_1=\begin{pmatrix}-K_{00} & \mathbf{0}\\ -K_{10} & I\end{pmatrix}.$$

The following candidate Lyapunov function is introduced:

$$V=\int_0^L\boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\xi}dx. \tag{6}$$

The weighting matrix $\mathbf{P}(x)$ is defined as follows: $\mathbf{P}(x)\triangleq\operatorname{diag}\{p_ie^{-\sigma_i\mu x}, i=1,\dots,n\}$, with $\mu>0$, $p_i>0$ positive real numbers and $\sigma_i=\operatorname{sign}(\lambda_i)$.

The time derivative of V along the solutions of (5) is

$$\begin{aligned}\dot V &= \int_0^L\left(\partial_t\boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\xi}+\boldsymbol{\xi}^T\mathbf{P}(x)\partial_t\boldsymbol{\xi}\right)dx\\ &=\int_0^L\Big(-\partial_x\boldsymbol{\xi}^T\boldsymbol{\Lambda}\mathbf{P}(x)\boldsymbol{\xi}-\boldsymbol{\xi}^T\mathbf{P}(x)\boldsymbol{\Lambda}\partial_x\boldsymbol{\xi}\\ &\qquad+\boldsymbol{\xi}^T\mathbf{M}^T\mathbf{P}(x)\boldsymbol{\xi}+\boldsymbol{\xi}^T\mathbf{P}(x)\mathbf{M}\boldsymbol{\xi}\Big)dx.\end{aligned}$$

Then, integrating by parts, we obtain:

$$\begin{aligned}\dot V &= -\int_0^L\partial_x\left[\boldsymbol{\xi}^T\boldsymbol{\Lambda}\mathbf{P}(x)\boldsymbol{\xi}\right]dx\\ &\quad+\int_0^L\boldsymbol{\xi}^T\Big(-\mu|\boldsymbol{\Lambda}|\mathbf{P}(x)+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}\Big)\boldsymbol{\xi}\,dx\\ &=-\left[\boldsymbol{\xi}^T\boldsymbol{\Lambda}\mathbf{P}(x)\boldsymbol{\xi}\right]_0^L\\ &\quad+\int_0^L\boldsymbol{\xi}^T\Big(-\mu|\boldsymbol{\Lambda}|\mathbf{P}(x)+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}\Big)\boldsymbol{\xi}\,dx\\ &=-\begin{pmatrix}\boldsymbol{\xi}^T(t,L) & \boldsymbol{\xi}^T(t,0)\end{pmatrix}\begin{pmatrix}\boldsymbol{\Lambda}\mathbf{P}(L) & 0\\ 0 & -\boldsymbol{\Lambda}\mathbf{P}(0)\end{pmatrix}\begin{pmatrix}\boldsymbol{\xi}(t,L)\\ \boldsymbol{\xi}(t,0)\end{pmatrix}\\ &\quad+\int_0^L\boldsymbol{\xi}^T\Big(-\mu|\boldsymbol{\Lambda}|\mathbf{P}(x)+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}\Big)\boldsymbol{\xi}\,dx.\end{aligned}$$

We then have the following exponential stability result.

Theorem 1. *The system (5) is exponentially stable if there exist $\mu>0$ and $p_i>0$ $i=1,\dots,n$ such that*

C1. *The boundary quadratic form*

$$\begin{pmatrix}\boldsymbol{\xi}^T(t,L) & \boldsymbol{\xi}^T(t,0)\end{pmatrix}\begin{pmatrix}\boldsymbol{\Lambda}\mathbf{P}(L) & 0\\ 0 & -\boldsymbol{\Lambda}\mathbf{P}(0)\end{pmatrix}\begin{pmatrix}\boldsymbol{\xi}(t,L)\\ \boldsymbol{\xi}(t,0)\end{pmatrix}$$

is positive definite under the constraint of the linear boundary condition $\mathbf{K}_0\boldsymbol{\xi}(t,0)+\mathbf{K}_1\boldsymbol{\xi}(t,L)=\mathbf{0},\ \forall t\geqslant 0$ *along the solutions of system* (2)–(4);

C2. *The matrix* $-\mu|\boldsymbol{\Lambda}|\mathbf{P}(x)+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}$ *is negative definite* $\forall x\in(0,L)$.

Remark 1. Boundary conditions that satisfy condition C1 are called *Dissipative Boundary Conditions*. Condition C1 is satisfied if and only if the leading principal minors of order $>2n$ of the matrix

$$\begin{pmatrix}\mathbf{0} & \mathbf{K}_0 & \mathbf{K}_1\\ -\mathbf{K}_0^T & -\boldsymbol{\Lambda}\mathbf{P}(0) & \mathbf{0}\\ -\mathbf{K}_1^T & \mathbf{0} & \boldsymbol{\Lambda}\mathbf{P}(L)\end{pmatrix}$$

are strictly positive (see Väliaho, 1982, Theorem 2.1).

Remark 2. For $\mu>0$ sufficiently small, condition C2 is satisfied if there exist $p_i>0$ such that $\mathbf{M}^T\mathbf{P}(0)+\mathbf{P}(0)\mathbf{M}$ is a negative semi-definite matrix. A question that has attracted some attention in the literature concerns the conditions on a matrix $\mathbf{M}$ for which there exist a diagonal positive matrix $\mathbf{P}$ such that $\mathbf{M}^T\mathbf{P}+\mathbf{P}\mathbf{M}$ is negative definite (see e.g. Barker, Berman, & Plemmons, 1978 for an early reference and Shorten, Mason, & King, 2009 for a recent reference). When such a matrix $\mathbf{P}$ exists, the matrix $\mathbf{M}$ is said to be *diagonally stable* (because it is stable and the associated Lyapunov equation is satisfied with a diagonal $\mathbf{P}$). Here, with condition C2, we are rather concerned with a diagonally *marginally* stable matrix $\mathbf{M}$ which means that we require only that $\mathbf{M}^T\mathbf{P}+\mathbf{P}\mathbf{M}$ be negative semidefinite.

For general systems of the form (1), it is rather clear that more explicit stability conditions can be derived only on a case by case basis when the internal structure and the numerical values of the involved matrices $\boldsymbol{\Lambda}$, $\mathbf{M}$, $\mathbf{K}_0$, $\mathbf{K}_1$ are at least partially specified. In the next section, we investigate the special case of system (5) when $\mathbf{M}$ is diagonally marginally stable and we show that a fairly simple explicit dissipative boundary condition can be given in that case. This is of great practical interest since models with diagonally marginally stable $\mathbf{M}$ appear in many concrete physical and engineering applications as we illustrate in Section 4 with the example of Saint–Venant–Exner equations for open channels with non-constant bathymetry.

3. Dissipative boundary condition when M is diagonally marginally stable

In this section, we will present a variant of Theorem 1 with an explicit characterisation of a sufficient dissipative boundary condition which guarantees the system exponential stability in the case where $\mathbf{M}$ is diagonally marginally stable. We consider again the system written in the form (2)–(4) and we define the matrix

$$\mathbf{K}\triangleq\begin{pmatrix}K_{00} & K_{01}\\ K_{10} & K_{11}\end{pmatrix}.$$

Let $\mathcal{D}_p$ denote the set of diagonal $p\times p$ real matrices with strictly positive diagonal entries. We define the set $\mathcal{P}$ as follows:

$$\mathcal{P}\triangleq\left\{\mathrm{P}\in\mathcal{D}_n \text{ such that } \mathbf{M}^T\mathrm{P}+\mathrm{P}\mathbf{M} \text{ is negative semidefinite}\right\}.$$

With the above notations, the candidate Lyapunov function (6) is written

$$V=\int_0^L\left[(\boldsymbol{\xi}^{+T}P_0\boldsymbol{\xi}^{+})e^{-\mu x}+(\boldsymbol{\xi}^{-T}P_1\boldsymbol{\xi}^{-})e^{\mu x}\right]dx \tag{7}$$

with $P_0\in\mathcal{D}_m$, $P_1\in\mathcal{D}_{n-m}$ and $\mu>0$. We introduce the following norm for the matrix $\mathbf{K}$:

$$\rho(\mathbf{K})\triangleq\inf\left\{\|\Delta\mathbf{K}\Delta^{-1}\|,\ \Delta\in\mathcal{S}\right\}$$

where $\|\ \|$ denotes the usual matrix 2-norm and the set $\mathcal{S}$ is defined as follows:

$$\mathcal{S}\triangleq\left\{\Delta=\mathrm{diag}\{D_0,D_1\},\ D_0^2=P_0\Lambda^+,\right. \tag{8}$$

$$\left.D_1^2=P_1\Lambda^-,\ \mathrm{P}=\mathrm{diag}\{P_0,P_1\}\in\mathcal{P}\right\}. \tag{9}$$

We have the following theorem.

Theorem 2. *If* $\mathbf{M}$ *is diagonally marginally stable, if the boundary dissipative condition* $\rho(\mathbf{K})<1$ *is satisfied, then the linear hyperbolic system* (2)–(3) *is exponentially stable.*

Proof. The time derivative of the Lyapunov function V is

$$\dot V=\dot V_1+\dot V_2 \tag{10}$$

with

$$\dot V_1\triangleq-\left[\boldsymbol{\xi}^{+T}P_0\boldsymbol{\Lambda}^+\boldsymbol{\xi}^+e^{-\mu x}\right]_0^L+\left[\boldsymbol{\xi}^{-T}P_1\boldsymbol{\Lambda}^-\boldsymbol{\xi}^-e^{\mu x}\right]_0^L$$

$$\dot V_2\triangleq\int_0^L\boldsymbol{\xi}^T\left(-\mu\mathbf{P}(x)|\boldsymbol{\Lambda}|+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}\right)\boldsymbol{\xi}\,dx$$

and $\mathbf{P}(x)\triangleq\mathrm{diag}\left\{P_0e^{-\mu x},P_1e^{\mu x}\right\}$, $|\boldsymbol{\Lambda}|\triangleq\mathrm{diag}\left\{\boldsymbol{\Lambda}^+,\boldsymbol{\Lambda}^-\right\}$.

In order to prove that the boundary condition (3) is dissipative we will show that P_0, P_1 and μ can be selected such that $\dot V$ is a negative definite function. In order to prove that $\dot V_1$ is a negative definite quadratic form, we introduce the following notations:

$$\boldsymbol{\xi}_0^-(t)\triangleq\boldsymbol{\xi}^-(t,0)\qquad\boldsymbol{\xi}_1^+(t)\triangleq\boldsymbol{\xi}^+(t,L).$$

Using the boundary condition (3), we have

$$\begin{aligned}\dot V_1&=-\left[\boldsymbol{\xi}^{+T}P_0\boldsymbol{\Lambda}^+\boldsymbol{\xi}^+e^{-\mu x}\right]_0^L+\left[\boldsymbol{\xi}^{-T}P_1\boldsymbol{\Lambda}^-\boldsymbol{\xi}^-e^{\mu x}\right]_0^L\\&=-\left(\boldsymbol{\xi}_1^{+T}P_0\boldsymbol{\Lambda}^+\boldsymbol{\xi}_1^+\,e^{-\mu L}+\boldsymbol{\xi}_0^{-T}P_1\boldsymbol{\Lambda}^-\boldsymbol{\xi}_0^-\right)\\&\quad+\left(\boldsymbol{\xi}_1^{+T}K_{00}^T+\boldsymbol{\xi}_0^{-T}K_{01}^T\right)P_0\boldsymbol{\Lambda}^+\left(K_{00}\boldsymbol{\xi}_1^++K_{01}\boldsymbol{\xi}_0^-\right)\\&\quad+\left(\boldsymbol{\xi}_1^{+T}K_{10}^T+\boldsymbol{\xi}_0^{-T}K_{11}^T\right)P_1\boldsymbol{\Lambda}^-\left(K_{10}\boldsymbol{\xi}_1^++K_{11}\boldsymbol{\xi}_0^-\right)e^{\mu L}.\end{aligned}$$

Since $\mathbf{M}$ is diagonally marginally stable and $\rho(\mathbf{K})<1$ by assumption, we know that the set $\mathcal{P}$ is not empty and we can select matrices P_0 and P_1 such that

$$\mathrm{P}=\mathrm{diag}\{P_0,P_1\}\in\mathcal{P},\qquad D_0^2=P_0\Lambda^+,\qquad D_1^2=P_1\Lambda^-,$$

$$\Delta=\mathrm{diag}\{D_0,D_1\}\quad\text{and}\quad\|\Delta\mathbf{K}\Delta^{-1}\|<1. \tag{11}$$

We define

$$\mathbf{z}\triangleq\begin{pmatrix}D_0\boldsymbol{\xi}_1^+\\ D_1\boldsymbol{\xi}_0^-\end{pmatrix}.$$

Then, using inequality (11), we have

$$\begin{aligned}&\left(\boldsymbol{\xi}_1^{+T}K_{00}^T+\boldsymbol{\xi}_0^{-T}K_{01}^T\right)P_0\boldsymbol{\Lambda}^+\left(K_{00}\boldsymbol{\xi}_1^++K_{01}\boldsymbol{\xi}_0^-\right)\\&\quad+\left(\boldsymbol{\xi}_1^{+T}K_{10}^T+\boldsymbol{\xi}_0^{-T}K_{11}^T\right)P_1\boldsymbol{\Lambda}^-\left(K_{10}\boldsymbol{\xi}_1^++K_{11}\boldsymbol{\xi}_0^-\right)\\&=\|\Delta\mathbf{K}\Delta^{-1}\mathbf{z}\|^2\\&<\|\mathbf{z}\|^2=\boldsymbol{\xi}_1^{+T}P_0\boldsymbol{\Lambda}^+\boldsymbol{\xi}_1^++\boldsymbol{\xi}_0^{-T}P_1\boldsymbol{\Lambda}^-\boldsymbol{\xi}_0^-.\end{aligned}$$

It follows readily that μ can be taken sufficiently small such that $\dot V_1$ is a negative definite quadratic form with respect to $(\boldsymbol{\xi}_1^{+T}\ \boldsymbol{\xi}_0^{-T})\ \forall t\geqslant 0$ along the solutions of system (2)–(4). □

Moreover, since $\mathbf{M}^T\mathrm{P}+\mathrm{P}\mathbf{M}$ is negative semidefinite (because $\mathrm{P}\in\mathcal{P}$), $\mu>0$ can be taken sufficiently small such that $-\mu\mathbf{P}(x)|\boldsymbol{\Lambda}|+\mathbf{M}^T\mathbf{P}(x)+\mathbf{P}(x)\mathbf{M}$ is negative definite for all x in $[0,L]$. It follows that for μ sufficiently small there exist $\alpha>0$ such that

$$\dot V_2<-\alpha V\Longrightarrow\dot V=\dot V_1+\dot V_2<-\alpha V\quad\forall\boldsymbol{\xi}\neq 0.$$

Consequently the solutions of the system (2)–(4) exponentially converge to $\mathbf{0}$ in L^2-norm.

4. Application to the Saint–Venant–Exner model

In the previous section, we have shown that for systems with $\mathbf{M}$ diagonally marginally stable, the dissipative boundary condition $\rho(\mathbf{K}) < 1$ is a sufficient exponential stability condition. This is true in particular for hydraulic systems described by linearised shallow-water equations as long as the subcritical flow condition is satisfied as we shall illustrate in the present section for an open channel with variable bathymetry.

We consider a pool of a prismatic sloping open channel with a rectangular cross-section, a unit width and a moving bathymetry (because of sediment transportation). The state variables of the model are: the water depth $H(t,x)$, the water velocity $V(t,x)$ and the bathymetry $B(t,x)$ which is the depth of the sediment layer above the channel bottom. The dynamics of the system are described by the coupling of Saint–Venant and Exner equations (see e.g. Hudson & Sweby, 2003):

$$\begin{aligned}
&\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + H\frac{\partial V}{\partial x} = 0,\\
&\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + g\frac{\partial B}{\partial x} = gS_b - C_f\frac{V^2}{H},\\
&\frac{\partial B}{\partial t} + aV^2\frac{\partial V}{\partial x} = 0.
\end{aligned} \tag{12}$$

In these equations, g is the gravity constant, S_b is the bottom slope of the channel, C_f is a friction coefficient and a is a parameter that encompasses porosity and viscosity effects on the sediment dynamics.

4.1. Steady-state and linearisation

A *steady-state* is a constant state H^*, V^*, B^* which satisfies the relation

$$gS_bH^* = C_fV^{*2}.$$

In order to linearise the model, we define the deviations of the state $H(t,x)$, $V(t,x)$, $B(t,x)$ with respect to the steady-state:

$$\begin{aligned}
h(x,t) &= H(x,t) - H^*,\\
u(x,t) &= V(x,t) - V^*,\\
b(x,t) &= B(x,t) - B^*.
\end{aligned}$$

Then the linearised Saint–Venant–Exner model (12) around a steady-state is

$$\frac{\partial h}{\partial t} + V^*\frac{\partial h}{\partial x} + H^*\frac{\partial u}{\partial x} = 0, \tag{13a}$$

$$\frac{\partial u}{\partial t} + V^*\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial b}{\partial x} = C_f\frac{V^{*2}}{H^{*2}}h - 2C_f\frac{V^*}{H^*}u, \tag{13b}$$

$$\frac{\partial b}{\partial t} + aV^{*2}\frac{\partial u}{\partial x} = 0. \tag{13c}$$

4.2. Characteristic (Riemann) coordinates

In matrix form, the linearised model (13) can be written as

$$\frac{\partial W}{\partial t} + \mathbf{A}(W^*)\frac{\partial W}{\partial x} = \mathbf{B}(W^*)W \tag{14}$$

where

$$W = \begin{pmatrix} h\\ u\\ b\end{pmatrix}, \qquad \mathbf{A}(W^*) = \begin{pmatrix} V^* & H^* & 0\\ g & V^* & g\\ 0 & aV^{*2} & 0\end{pmatrix},$$

$$\mathbf{B}(W^*) = \begin{pmatrix} 0 & 0 & 0\\ C_f\dfrac{V^{*2}}{H^{*2}} & -2C_f\dfrac{V^*}{H^*} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Exact, but rather complicated expressions of the eigenvalues of $\mathbf{A}(W^*)$ can be obtained by using the *Cardano–Vieta* method, see Hudson and Sweby (2003). Once the eigenvalues λ_i of the matrix $\mathbf{A}(W^*)$ are obtained, the corresponding left eigenvectors can be computed as

$$L_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\begin{pmatrix} (V^* - \lambda_i)(V^* - \lambda_j) + gH^*\\ H^*\lambda_k\\ gH^*\end{pmatrix}, \quad k \neq i \neq j \in \{1,2,3\}.$$

We multiply (14) by L_k^T in order to rewrite the model in terms of the characteristic coordinates ψ_k ($k = 1,2,3$). Then we obtain

$$\frac{\partial \psi_k}{\partial t} + \lambda_k\frac{\partial \psi_k}{\partial x} = L_k^T\mathbf{B}W, \quad k = 1,2,3, \tag{15}$$

where

$$\psi_k = \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\big[\big((V^* - \lambda_i)(V^* - \lambda_j) + gH^*\big)h + H^*\lambda_k u + gH^*b\big].$$

Conversely, we can express h, u and b in terms of the characteristic coordinates:

$$\begin{aligned}
h &= \psi_1 + \psi_2 + \psi_3,\\
u &= \frac{1}{H^*}[(\lambda_1 - V^*)\psi_1 + (\lambda_2 - V^*)\psi_2 + (\lambda_3 - V^*)\psi_3],\\
b &= \frac{1}{gH^*}[((\lambda_1 - V^*)^2 - gH^*)\psi_1 + ((\lambda_2 - V^*)^2 - gH^*)\psi_2 + ((\lambda_3 - V^*)^2 - gH^*)\psi_3].
\end{aligned}$$

Using the new variables ψ_k, the RHS of (15) writes:

$$\begin{aligned}
L_k^T\mathbf{B}W &= \gamma_1 l_2^k h + \gamma_2 l_2^k u\\
&= \sum_{s=1}^{3}\left(\gamma_1 + \gamma_2\frac{\lambda_s - V^*}{H^*}\right) l_2^k\psi_s,
\end{aligned} \tag{16}$$

where

$$\gamma_1 = C_f\frac{V^{*2}}{H^{*2}}, \qquad \gamma_2 = -2C_f\frac{V^*}{H^*},$$

and l_2^k is the second component of L_k^T. Eq. (16) can be rewritten as:

$$L_k^T\mathbf{B}W = C_f\frac{V^*}{H^*}\frac{\lambda_k}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\sum_{s=1}^{3}\Big(3V^* - 2\lambda_s\Big)\psi_s, \quad k \neq i \neq j \in \{1,2,3\}.$$

For the sake of simplicity, we introduce the following notation θ_k:

$$\theta_k = C_f\frac{V^*}{H^*}\frac{\lambda_k}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}.$$

Then Eq. (15) writes:

$$\frac{\partial \xi_k}{\partial t} + \lambda_k\frac{\partial \xi_k}{\partial x} + \sum_{s=1}^{3}(2\lambda_s - 3V^*)\theta_s\xi_s = 0 \quad (k = 1,2,3) \tag{17}$$

where the characteristic coordinates are now defined as

$$\xi_k = \frac{1}{\theta_k}\psi_k.$$

From (17), the linearised model (15) in characteristic form may now be written as

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \boldsymbol{\xi}}{\partial x} - \mathbf{M}\boldsymbol{\xi} = 0,$$

where

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T, \qquad \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3),$$

$$\mathbf{M} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix},$$

with

$$\alpha_k = \left(3V^* - 2\lambda_k\right)\theta_k.$$

From Hudson and Sweby (2003), the three eigenvalues of the matrix $\mathbf{A}$ are such that

$$\lambda_1 < 0 < \lambda_2 \ll \lambda_3 \tag{18}$$

with λ_1 and λ_2 the characteristic velocities of the water flow and λ_2 the characteristic velocity of the sediment motion. Obviously the sediment motion is much slower than the water flow. On the basis of (18), we are now going to determine the sign of the coefficients α_k in $\mathbf{M}$.

For α_1, we have:

$$\alpha_1 = C_f \frac{V^*}{H^*}\left(3V^* - 2\lambda_1\right)\frac{\lambda_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}.$$

Since $\lambda_1 < 0$, we have $3V^* - 2\lambda_1 > 0$. Using 4.2, we infer that:

$$\lambda_1 - \lambda_2 < 0 \quad \text{and} \quad \lambda_1 - \lambda_3 < 0.$$

From the above inequalities, we conclude that $\alpha_1 < 0$.

For α_2, we have

$$\alpha_2 = C_f \frac{V^*}{H^*}\left(3V^* - 2\lambda_2\right)\frac{\lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}.$$

Since the sediment motion is much slower than the water flow, we may assume that $3V^* - 2\lambda_2 > 0$. Moreover from 4.2, we have also

$$\lambda_2 > 0, \quad \lambda_2 - \lambda_1 > 0 \quad \text{and} \quad \lambda_2 - \lambda_3 < 0.$$

From these inequalities, we conclude that $\alpha_2 < 0$.

Finally, for α_3, we have

$$\alpha_3 = C_f \frac{V^*}{H^*}\left(3V^* - 2\lambda_3\right)\frac{\lambda_3}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}.$$

From 4.2, we have

$$\lambda_3 > 0, \qquad \lambda_3 - \lambda_1 > 0 \quad \text{and} \quad \lambda_3 - \lambda_2 > 0.$$

Using the trace of $\mathbf{A}$, we have also

$$3V^* - 2\lambda_3 = 2\lambda_1 + 2\lambda_2 - V^*.$$

Since λ_2 is small, $3V^* - 2\lambda_3$ has the same sign as $2\lambda_1 - V^*$. Since $\lambda_1 < 0$ is negative, we obtain: $3V^* - 2\lambda_3 < 0$ and consequently $\alpha_3 < 0$.

Hence all the coefficients α_k in matrix $\mathbf{M}$ are strictly negative.

4.3. Lyapunov stability under boundary feedback control

We are now going to show how Theorem 2 may be applied to analyse the stability of an open channel under boundary feedback control.

We assume that the channel is provided with hydraulic control devices (pumps, valves, mobile spillways, sluice gates,...) which are located at both ends and allow to assign the values of the flow-rate. On-line measurements of the water levels at both ends $h(t, 0) + b(t, 0)$ and $h(t, L) + b(t, L)$ are assumed to be available for feedback control. Obviously, instead of the flow-rates, we may as well consider the velocities $u(t, 0)$ and $u(t, L)$ as being the control actions. Therefore we introduce the following boundary conditions:

$$u(t, 0) = -k_1 h(t, 0), \tag{19a}$$

$$u(t, L) = -k_2(h(t, L) + b(t, L)), \tag{19b}$$

$$b(t, 0) = 0. \tag{19c}$$

Conditions (19a)–(19b) are linear feedback static control laws with the tuning parameters k_1 and k_2. The third condition is supposed to be a physical constraint. In order to invoke Theorem 2, we have

(1) to find a matrix $\mathbf{P} = \mathrm{diag}\{p_1, p_2, p_3\}$ such that $\boldsymbol{\xi}^T\left(\mathbf{M}^T\mathbf{P} + \mathbf{PM}\right)\boldsymbol{\xi}$ is a negative semi definite quadratic form,
(2) to find the range of admissible values of the tuning parameters k_i such that the boundary conditions are dissipative.

For the matrix $\mathbf{P}$, a straightforward choice is $p_i = |\alpha_i|$ ($i = 1, 2, 3$) since then the quadratic form is

$$\boldsymbol{\xi}^T\left(\mathbf{M}^T\mathbf{P} + \mathbf{PM}\right)\boldsymbol{\xi} = -2\left(\sum_{i=1}^{3} |\alpha_i|\xi_i\right)^2.$$

In order to check the dissipativity condition $\rho(\mathbf{K}) < 1$, we have to compute the matrix $\mathbf{K}$ and the matrix Δ. It is easy to verify that, in the Riemann coordinates $\boldsymbol{\xi}$, the boundary conditions (19) can be written in the form (3) as follows:

$$\begin{pmatrix} \xi_1(t, L) \\ \xi_2(t, 0) \\ \xi_3(t, 0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \chi_2(k_2) & \chi_3(k_2) \\ \pi_2(k_1) & 0 & 0 \\ \pi_3(k_1) & 0 & 0 \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} \xi_1(t, 0) \\ \xi_2(t, L) \\ \xi_3(t, L) \end{pmatrix}$$

where π_i and χ_j are the following homographic transformations of the tuning parameters k_1 and k_2:

$$\pi_2(k_1) = \frac{a_{21} - c_{21}k_1}{a_{32} - c_{32}k_1}, \qquad \pi_3(k_1) = \frac{a_{13} - c_{13}k_1}{a_{32} - c_{32}k_1},$$

with

$$a_{ij} = (\lambda_j - \lambda_i)\left(1 + \frac{(\lambda_i - V^*)(\lambda_j - V^*)}{gH^*}\right)$$

$$c_{ij} = \frac{\lambda_k}{g}(\lambda_j - \lambda_i), \quad k \neq i \neq j \in \{1, 2, 3\}$$

and

$$\chi_j(k_2) = \left(\frac{\lambda_j - V^*}{\lambda_1 - V^*}\right)\left(\frac{g + (\lambda_j - V^*)k_2}{g + (\lambda_1 - V^*)k_2}\right) \quad j = 2, 3.$$

Moreover, we have, by definition, that $\mathbf{P} = \mathrm{diag}\{|\alpha_1|, |\alpha_2|, |\alpha_3|\}$ and $|\boldsymbol{\Lambda}| = \mathrm{diag}\{|\lambda_1|, \lambda_2, \lambda_3\}$. Consequently:

$$\Delta = \mathrm{diag}\left\{\sqrt{|\lambda_1|\,|\alpha_1|}, \sqrt{\lambda_2|\alpha_2|}, \sqrt{\lambda_3|\alpha_3|}\right\}$$

and

$$\Delta\mathbf{K}\Delta^{-1} = \begin{pmatrix} 0 & \chi_2(k_2)\sqrt{\dfrac{|\lambda_1|\,|\alpha_1|}{\lambda_2|\alpha_2|}} & \chi_3(k_2)\sqrt{\dfrac{|\lambda_1|\,|\alpha_1|}{\lambda_3|\alpha_3|}} \\ \pi_2(k_1)\sqrt{\dfrac{\lambda_2|\alpha_2|}{|\lambda_1|\,|\alpha_1|}} & 0 & 0 \\ \pi_3(k_1)\sqrt{\dfrac{\lambda_3|\alpha_3|}{|\lambda_1|\,|\alpha_1|}} & 0 & 0 \end{pmatrix}.$$

Then, it is a matter of tedious but fairly straightforward calculations to show that

$$\|\Delta\mathbf{K}\Delta^{-1}\| < 1$$

if and only if the tuning parameters k_1 and k_2 can be selected such that

$$\pi_2^2(k_1)\frac{\lambda_2|\alpha_2|}{|\lambda_1|\,|\alpha_1|}+\pi_3^2(k_1)\frac{\lambda_3|\alpha_3|}{|\lambda_1|\,|\alpha_1|}<1$$

and

$$\chi_2^2(k_2)\frac{|\lambda_1|\,|\alpha_1|}{\lambda_2|\alpha_2|}+\chi_3^2(k_2)\frac{|\lambda_1|\,|\alpha_1|}{\lambda_3|\alpha_3|}<1.$$

5. Conclusions

We have addressed the issue of stating sufficient boundary conditions for the exponential stability of linear hyperbolic systems of balance laws. In Theorem 1 we have first given a general implicit formulation of sufficient dissipative boundary conditions. Our analysis relies on the use of an explicit Lyapunov function. The weight $e^{\pm\mu x}$ is essential to get a strict Lyapunov function.

Then in Theorem 2, we have shown that the explicit dissipativity condition $\rho(\mathbf{K}) < 1$ gives a convergence in $L^2(0, L)$-norm for systems of balance laws with a diagonally marginally stable matrix **M**. This theorem has been applied to give tuning conditions for boundary feedback stabilisation of an open channel represented by the linearised Saint–Venant–Exner model. Obviously, the fact that the sediment motion is much slower than the water flow induces a separation of time scales that could also be used in order to design separate controllers for each motion.

The same Lyapunov function cannot be directly used to analyse the local stability of the steady-states in the *nonlinear case*. In order to extend the Lyapunov stability analysis to the nonlinear case, the Lyapunov function has to be augmented (as shown in detail in Coron et al. (2008, 2007)).

References

Barker, G. P., Berman, A., & Plemmons, R. J. (1978). Positive diagonal solutions to the Lyapunov equations. *Linear and Multilinear Algebra*, *5*(3), 249–256.

Coron, J.-M. (1999). On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain. *SIAM Journal on Control and Optimization*, *37*(6), 1874–1896.

Coron, J.-M. (2007). *Mathematical surveys and monographs*: *Vol. 136*. *Control and nonlinearity*. Providence, RI: American Mathematical Society.

Coron, J.-M., Bastin, G., & d'Andréa-Novel, B. (2008). Dissipative boundary conditions for one dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, *47*(3), 1460–1498.

Coron, J.-M., d'Andréa-Novel, B., & Bastin, G. (2007). A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, *52*(1), 2–11.

Greenberg, J. M., & Li, T. (1984). The effect of boundary damping for the quasilinear wave equations. *Journal of Differential Equations*, *52*, 66–75.

Gugat, M., & Herty, M. (2011). Existence of classical solutions and feedback stabilisation for the flow in gas networks. *ESAIM: Control, Optimisation and Calculus of Variations*, *17*(1), 28–51.

Hudson, J., & Sweby, P. K. (2003). Formulations for numerically approximating hyperbolic systems governing sediment transport. *Journal of Scientific Computing*, *19*, 225–252.

Li, T.-T. (1994). *Research in applied mathematics*, *Global classical solutions for quasilinear hyperbolic systems*. Masson and Wiley.

Prieur, C., Winkin, J., & Bastin, G. (2008). Robust boundary control of systems of conservation laws. *Mathematics of Control, Signal and Systems (MCSS)*, *20*, 173–197.

Serre, D. (2001). Systems of conservation laws: a challenge for the XXIst century. In B. Engquist, & W. Schmid (Eds.), *Mathematics unlimited* (pp. 1061–1080).

Shorten, R., Mason, O., & King, C. (2009). An alternative proof of the Barker, Berman, Plemmons (BBP) result on diagonal stability and extensions. *Linear Algebra and its Applications*, *430*, 34–40.

Slemrod, M. (1983). Boundary feedback stabilization for a quasilinear wave equation. In *Lecture notes in control and information sciences*: *Vol. 54*. *Control theory for distributed parameter systems* (pp. 221–237). Springer Verlag.

Väliaho, H. (1982). On the definity of quadratic forms subject to linear constraints. *Journal of Optimization Theory and Applications*, *38*(1), 143–145.

Xu, C. Z., & Sallet, G. (2002). Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, *7*, 421–442.

Ababacar Diagne received a Graduate Degree in Applied Mathematics and Computer Sciences from the University Gaston Berger of Saint Louis (Senegal). He is currently a Ph.D. Student jointly at University Gaston Berger of Saint Louis (Senegal) and Royal Institute of Technology (Sweden).

Georges Bastin received the electrical engineering degree and the Ph.D. degree, both from Université catholique de Louvain, Louvain-la-Neuve, Belgium. He his presently Professor in the Department of Mathematical Engineering at Université catholique de Louvain. His main research interests are in nonlinear control of compartmental systems and boundary control of hyperbolic systems with applications in biology and environmental systems.

Jean-Michel Coron received the Diplôme of engineer from Ecole Polytechnique in 1978 and from the Corps des Mines in 1981. He received the Thèse d'état in 1982. He has been Researcher at Ecole Nationale Supérieure des Mines de Paris, Associate Professor at the Ecole Polytechnique, Professor at Université Paris-Sud 11. He is currently a Professor at Université Pierre et Marie Curie (Paris-6) and Institut Universitaire de France. His research interests include nonlinear partial differential equations and nonlinear control theory.