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for positive linear systems

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POSITIVE LINEAR OBSERVERS FOR POSITIVE LINEAR SYSTEMS

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Abstract

A positive linear system is a linear system in which state variables, control and measurements are always nonnegative.

In many systems of practical importance, all the state variables are not available for measurements, although the entire state vector has to be known for control strategies for example. In this situation, one can construct an approximation of the full state vector based on available measurements. Such a device is known as an observer.

In particular, this study determines structural conditions on positive linear systems for the existence of convergent positive linear observers. Single output systems are first considered. The extension to multi output systems is then performed.

1 Introduction

In this paper, we consider continuous time **positive** linear systems :

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where
(i) A is a $n \times n$ Metzler matrix ($A \in R^{n \times n}$ and $a_{ij} \geq 0$, $\forall\, i \neq j$),
(ii) B is a $n \times m$ nonnegative matrix ($B \in R_+^{n \times m}$),
(iii) C is a $p \times n$ nonnegative matrix ($C \in R_+^{p \times n}$).

The linear system (1) is **positive** (see [3]) if and only if conditions (i), (ii) and (iii) are satisfied. This means that

- if the initial state $x(0)$ is nonnegative ($x(0) \in R_+^n$) and the control input $u(t)$ is nonnegative ($u(t) \in R_+^m, \forall t \geq 0$),
- then the state $x(t)$ and the output $y(t)$ are nonnegative along the system trajectories ($x(t) \in R_+^n, y(t) \in R_+^p, \forall t \geq 0$).

Positive systems arise naturally in practical applications where the state variables represent quantities that do not have physical meaning unless they are nonnegative. We are concerned with the use of Luenberger state observers for such systems with a view to process monitoring. A natural requirement is therefore that the observer should provide state estimates that are also nonnegative to have a physical meaning.

A **Luenberger Observer** [3] for system (1), is defined as

$$\dot{r}(t) = (A - EC)r(t) + Bu(t) + Ey(t) \tag{2}$$

where $r(t)$ plays the role of an online estimate of the state $x(t)$. The estimation error is denoted :

$$e(t) = (r(t) - x(t))$$

and, from (1) - (2), the error dynamics are written :

$$\dot{e}(t) = [A - EC]e(t)$$

The linear observer (2) is positive (i.e. $r(t) \in R_+^n, \forall t \geq 0$) and convergent (i.e. $e(t) \to 0$ as $t \to \infty$), if and only if the following conditions are satisfied :

a) the $n \times p$ matrix $E \in R_+^{n \times p}$,

b) $(A - EC)$ is a Metzler matrix,

c) $(A - EC)$ is an asymptotically stable matrix.

The problem considered in this paper is to explicitly define structural conditions for the existence of convergent positive linear observers for positive linear systems.

This problem was previously considered in [4] for the special case of compartmental systems (where A is a compartmental matrix). In this paper, we present more general conditions that are also valid for systems where A

may be a Metzler matrix which is not compartmental.

We have the following preliminary lemmas :

Lemma 1 *A Metzler matrix A has a real eigenvalue λ_{max} such that $Re(\lambda) < \lambda_{max}$ for all other eigenvalues of A.*

Proof see [3].

Lemma 2 *Given a $n \times n$ Metzler matrix A and a nonnegative matrix $C \in R_+^{p\times n}$, let $E_1 \in R_+^{n\times p}$ and $E_2 \in R_+^{n\times p}$ be two nonnegative matrices such that :*
$(A - E_1C)$ and $(A - E_2C)$ are $n \times n$ Metzler matrices and $(E_1 - E_2) \in R_+^{n\times p}$ then

$$\lambda_{max}(A - E_1C) \leq \lambda_{max}(A - E_2C)$$

Proof The proof is immediate by use of Corollary 8.1.19 in [2] adapted to Metzler matrices (which are a translation of positive matrices [3]). □

Given a nonnegative matrix E such that $(A - EC)$ is a Metzler matrix, in more intuitive terms, Lemma 2 means that the larger the matrix $E$is, the lower the maximal eigenvalue $\lambda_{max}(A - EC)$ is.

The next lemma presents a necessary stability condition for Metzler matrices :

Lemma 3 *A $n \times n$ Metzler matrix A is asymptotically stable only if all its diagonal elements are negative.*

Proof Theorem 2.4.14 of [1] can be adapted to Metzler matrices. It states that a Metzler matrix A is asymptotically stable if and only if all the principal minors of $-A$ are positive. An inherited necessary stability condition is that all diagonal elements of A must be negative. □

2 Single output positive linear systems

The following result presents a necessary structural condition for the existence of convergent positive linear observers for single output positive linear systems :

Lemma 4 *Given a Metzler matrix $A \in R^{n\times n}$ and a nonnegative matrix $C \in R_+^{1\times n}$, there exists a nonnegative matrix $E \in R_+^{n\times 1}$ such that $(A - EC)$ is an asymptotically stable Metzler matrix only if*

$$a_{ii}c_j < a_{ij}c_i, \ \ \forall i, \ \ \forall j \neq i, \ \ such\ that\ \ c_j \neq 0 \qquad (3)$$

Proof

1. $(A-EC)$ is Metzler implies that $(a_{ij} - e_ic_j) \geq 0, \forall j \neq i$ or
$$e_ic_j \leq a_{ij}, \forall j \neq i. \qquad (4)$$
2. $(A - EC)$ is asymptotically stable implies by Lemma 3 that $a_{ii} - e_ic_i < 0, \forall i$ or
$$e_ic_i > a_{ii}, \forall i. \qquad (5)$$

If $c_i \neq 0$ and $c_j \neq 0$, (4) and (5) together give

$$a_{ii}c_j < a_{ij}c_i, \forall i, \forall j \neq i$$

It can easily be verified that if $c_i = 0$ and $c_j \neq 0$, (5) reduces to $a_{ii} < 0$ and so expression (3) remains valid. □

Remark 1 The use of Theorem 2.4.14 of [1] evidences that the structural condition (3) is necessary and sufficient for second order systems.

The rationale behind the next theorem can be summarized as follows : for system (1), under the constraint that $(A - EC)$ is a Metzler matrix, take the matrix $(A - EC)$ with the highest possible values for the entries of E. If the maximal eigenvalue $\lambda_{max}(A - EC) \geq 0$, there is no convergent positive observer for that system. Otherwise one can use this matrix E to build a convergent positive observer.

Theorem 1 *Given a Metzler matrix $A \in R^{n\times n}$ and a nonnegative matrix $C \in R_+^{1\times n}$, let $E_0 \in R_+^{n\times 1}$ be a nonnegative matrix with entries e_i such that :*

$$e_i > a_{ii}/c_i \text{ if } c_j = 0, \forall j \neq i, \qquad (6)$$

$$e_i = \min_{j\neq i, c_j\neq 0}\{\frac{a_{ij}}{c_j}\}, \text{ otherwise .} \qquad (7)$$

Then there exists a convergent positive linear observer for the given system if and only if $\lambda_{max}(A - E_0C) < 0$.

Proof (⇐) First, one can verify that the choice of the elements of E_0 guarantees that $A - E_0C$ is a Metzler matrix. Since $\lambda_{max}(A - E_0C) < 0$, this matrix is asymptotically stable and can be used as a convergent positive observer for the system.
(⇒) Assume that $\lambda_{max}(A - E_0C) \geq 0$. If the matrix C has at least two non zero elements, then it can be shown that any matrix $E' \in R_+^{n\times 1}$ such that $A - E'C$ Metzler is such that $E_0 - E' \in R_+^{n\times 1}$ and by Lemma 2, $\lambda_{max}(A - E'C) \geq \lambda_{max}(A - E_0C)$.

If the matrix C has only one non zero element, there exists an infinity of matrices E_0 that can be built with the use of formula (6) and (7). For any matrix $E' \in R_+^{n\times 1}$ and $A - E'C$ Metzler

1. If $E_0 - E' \in R_+^{n\times 1}$, $\lambda_{max}(A - E'C) \geq \lambda_{max}(A - E_0C)$.
2. If $E' - E_0 \in R_+^{n\times 1}$, by hypothesis, $\forall j \neq i, c_j = 0$, one can only have $e'_i > e_i > a_{ii}/c_i$. In that case, by construction, the i^{th} column of $(A - E_0C)$ is identically zero except the diagonal term $a_{ii} - e_ic_i < 0$. This term is a negative eigenvalue $\lambda_i(A - E_0C) \neq \lambda_{max}(A - E_0C)$. By taking $e'_i > e_i$ this eigenvalue becomes more negative, but it has no influence on $\lambda_{max}(A - E'C) = \lambda_{max}(A - E_0C) \geq 0$.

□

This result provides an easy way to verify the existence of a convergent positive linear observer for a given positive linear system. An example will be given in Section 4.

Remark 2 If the matrix C has at least two non zero elements, the matrix E_0 built with (7) is unique. In that case, any nonnegative matrix $E \in R_+^{n\times 1}$ such that $E_0 - E \in R_+^{n\times 1}$ and $\lambda_{max}(A - EC) < 0$ is an acceptable matrix to build a convergent positive observer.

3 Multi output positive linear systems

In this section, the results presented above are extended to multi-output positive linear systems.

Lemma 5 *Given a Metzler matrix $A \in R^{n\times n}$ and a nonnegative matrix $C \in R_+^{p\times n}$, there exists a nonnegative matrix $E \in R_+^{n\times p}$ such that $(A - EC)$ is a stable Metzler matrix only if both following conditions hold simultaneously:*

$$\sum_{j=1}^{p} e_{ij}c_{jk} \leq a_{ik}, \quad \forall k \neq i, \tag{8}$$

$$\sum_{j=1}^{p} e_{ij}c_{ji} > a_{ii}, \quad \forall i, \tag{9}$$

Proof Condition (8) guarantees that $A-EC$ is a Metzler matrix. Condition (9) is a necessary stability condition (cf Lemma 3). □

Theorem 1 can be extended to the multi-output case in the following way :

Theorem 2 *Given a Metzler matrix $A \in R^{n\times n}$ and a nonnegative matrix $C \in R_+^{p\times n}$, define a nonnegative matrix $E_0 \in R_+^{n\times p}$ such that the elements of the i^{th} line of E_0 are solution of the following linear programming problem:*

$$\max_{e_{ij}} \sum_{j=1}^{p} e_{ij}c_{ji},$$

subject to

$$\sum_{j=1}^{p} e_{ij}c_{jk} \leq a_{ik}, \quad \forall \; k \neq i.$$

In case this optimization problem is unbounded ($c_{jk} = 0, \;\forall k \neq i$), then any value of e_{ij} such that

$$\sum_{j=1}^{p} e_{ij}c_{ji} > a_{ii}$$

is admissible.

Then there exists a convergent positive linear observer for the given system if and only if $\lambda_{max}(A - E_0C) < 0$.

Proof the proof is a straightforward extension of the proof of Theorem 1. □

4 Examples

The following example will illustrate that the necessary condition of Lemma 4 is not sufficient for a single output system with $n \geq 3$:

Example 1 for the linear system described by

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 10 & 2 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

the structural necessary condition (3) is fulfilled, but it is not possible to build a convergent positive linear observer. In fact, the matrix $E_0 = \begin{pmatrix} 2 & 4 & 2 \end{pmatrix}'$ as defined in Theorem 1 is such that $\lambda_{max}(A - E_0C) = 1$. Note that this system is detectable and for $E = \begin{pmatrix} 2 & 6 & 2 \end{pmatrix}'$, $A - EC$ is an asymptotically stable matrix but no more a Metzler matrix.

A multi-output example is now given :

Example 2 for the linear system described by

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 5 & 2 & 4 \\ 0 & 2 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

the structural conditions (8) and (9) are fulfilled. As specified in Theorem 2

$$E_0 = \begin{pmatrix} 0 & 3 \\ 4 & 5 \\ 2 & 0 \end{pmatrix}.$$

The Metzler matrix

$$A - E_0C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is asymptotically stable. Hence, E_0 can be used to build a convergent positive linear observer. The following class of matrices E :

$$E = \begin{pmatrix} 0 & 3 \\ 4 & 5 \\ \alpha & 0 \end{pmatrix}$$

where $1 < \alpha \leq 2$ are such that $E_0 - E \in R_+^{3\times 2}$ and $\lambda_{max}(A - EC) = 1 - \alpha < 0$. They can also be used to build a convergent positive linear observer.

5 Final comments

In this paper, necessary structural conditions on a positive linear system for the existence of a convergent positive linear observer have been given.

Furthermore, the main result of the paper provides an easy method to check for the existence of a convergent positive linear observer. The class of matrices that allow to build such an observer has been characterized.

Note also that given the positivity constraint of the problem, the results obtained here for positive observers can not be transposed by duality to solve the problem of positive stabilization by state feedback (See [1] for example).

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